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Periodic solutions for nonlinear telegraph equation via elliptic regularization

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Abstract. In this work we are concerned with the existence and uniqueness of T-periodic weak solutions for an initial-boundary value problem associated with nonlinear telegraph equations type in a domain $Q \subset \mathbb{R}^N$. Our arguments rely on elliptic regularization technics, tools from classical functional analysis as well as basic results from theory of monotone operators.

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1 Introduction and description of the elliptic regularization method

In this paper we deal with the existence of time-periodic solutions for the nonlinear telegraph equation

$$w'' + w' - \Delta w + w + |w'|^{p-2} w' = f, \quad (x,t) \in Q = \Omega \times]0,T[, \tag{1.1}$$

Ω being a bounded domain in $\mathbb{R}^N$ with a sufficiently regular boundary $\partial\Omega$.

All derivatives are in the sense of distributions, and by ξ' it denotes $\dfrac{\partial \xi}{\partial t}$. The function f we will be assumed as regular as necessary.

We shall use, throughout this paper, the same terminology of the functional spaces used, for instance, in the books of Lions [6]. In particular, we denote by

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$V = H_0^1(\Omega)$ and $H = L^2(\Omega)$. The Hilbert space V has inner product $((.\,,.))$ and norm $\|.\|$ given by $((u, v)) = \int_\Omega \nabla u.\nabla v dx$, $\|u\|^2 = \int_\Omega |\nabla u|^2 dx$. For the Hilbert space H we represent its inner product and norm, respectively, by $(.\,,.)$ and $|.|$, defined by $(u, v) = \int_\Omega uv dx$, $|u|^2 = \int_\Omega |u|^2\, dx$.

The telegraph equation appears when we look for a mathematical model for the electrical flow in a metallic cable. From the laws of electricity we deduce a system of partial differential equations where the unknown are the intensity of current i and the voltage u, cf. Courant-Hilbert [4], p. 192–193, among others.

By algebraic calculations we eliminate i and we get the partial differential equation:

$$\frac{\partial^2 u}{\partial t^2} - C^2\frac{\partial^2 u}{\partial x^2} + (\alpha + \beta)\frac{\partial u}{\partial t} + \alpha\beta u = 0,$$

called Telegraph Equation. In this case the coefficients C, α, β are constants.

Motivated by this model, Prodi [10] investigated the existence of periodic solution in t for the equations

$$\frac{\partial^2 u}{\partial t^2} - \Delta u + u + \frac{\partial u}{\partial t} + |\frac{\partial u}{\partial t}|\frac{\partial u}{\partial t} = f,$$

in a bounded open set Ω of R^N with Dirichlet zero conditions on the boundary.

The problem posed by Prodi [10] was further developed by Lions [6] with the aid of elliptic regularization associated to the theory of monotonous operator, cf. Browder [3].

More precisely, Lions [6] investigate periodic solutions of the problem

$$\left|\begin{array}{l} w'' - \Delta w + \gamma(w') = f \text{ in } Q = \Omega\times]0, T[, \\ w = 0 \text{ on } \Sigma = \partial\Omega\times]0, T[, \\ w(0) = w(T), \quad w'(0) = w'(T) \text{ in } \Omega, \end{array}\right. \tag{1.2}$$

with $\gamma(w') = |w'|^{p-2}w'$.

Because of this important physical background, the existence of time-periodic solutions of the telegraph equations with boundary condition for space variable x has been studied by many authors, see [7, 8, 9, 11] and the references therein.

We consider the existence of the solutions $w(x, t)$ of Eq. (1.1), which satisfy the time-periodic (or T-periodic) condition

$$w(0) = w(x, T), \quad w'(x, 0) = w'(T), \quad x \in \Omega, \tag{1.3}$$

subject to the Dirichlet condition

$$w(x,t)=0, \quad (x,t)\in \partial\Omega \times]0,T[. \tag{1.4}$$

Based on physical considerations, we restrict our analysis to the two dimensional space and standard hypothesis on f is assumed. Arguments within this paper are inspired by the work by Lions [6].

However, the classical energy method approach cannot be employed straightly, giving raise to a new mathematical difficulty. In fact, multiplying both sides of the equation (1.1) by w' and integrating on Q, we have, using the periodicity condition, that

$$\int_Q |w'(x,t)|^2 dxdt + \int_Q |w'(x,t)|^p dxdt = \int_Q f(x,t)w'(x,t)dxdt.$$

In this way we obtain only estimates for

$$\int_Q |w'(x,t)|^2 dxdt \quad \text{and} \quad \int_Q |w'(x,t)|^p dxdt,$$

which is not sufficient to obtain solution for (1.1).

In view of this, as in Lions [6], we use an approach due to Prodi [10] which relies heavily on the following set of ideas: we investigate solutions for (1.1) of the type

$$\left|\begin{array}{l} w=u+u_0, \\ u_0 \text{ independent of } t \\ \displaystyle\int_0^T u(t)\,dt=0, \quad \text{the average of } u \text{ is zero.} \end{array}\right. \tag{1.5}$$

Substituting w given by (1.5) in (1.1), we obtain

$$u''+u'-\Delta u+u+|u'|^{p-2}u' = f+\Delta u_0 - u_0, \tag{1.6}$$

which contains a new unknown u_0, independent of t by definition.

To eliminate u_0 in (1.6) we consider the derivative of (1.6) with respect to t obtaining

$$\left|\begin{array}{l} \dfrac{d}{dt}(u''+u'-\Delta u+u+|u'|^{p-2}u') = \dfrac{df}{dt} \\ \displaystyle\int_0^T u(t)dt=0 \\ u(0)=u(T), \quad u'(0)=u'(T). \end{array}\right. \tag{1.7}$$

Suppose that we have found u by (1.7). Observe that by $(1.7)_1$,

$$\frac{d}{dt}(u'' + u' - \Delta u + u + |u'|^{p-2}u' - f) = 0.$$

Thus u is solution of

$$u'' + u' - \Delta u + u + |u'|^{p-2}u' - f = g_0, \tag{1.8}$$

g_0 independent of t, in which g_0 is a known function.

Then u_0 is obtained as the solution of the Dirichlet problem:

$$\left| \begin{array}{l} -\Delta u_0 + u_0 = -g_0 \\ u_0 = 0 \text{ on } \partial\Omega. \end{array} \right. \tag{1.9}$$

Therefore, $w = u + u_0$ is the T – periodic solution of (1.1). We are going to resolve problem (1.7) by using elliptic regularization.

Observe that Lions [6] investigate the problem (1.2) by elliptic regularization, reducing the problem to the theory of monotonous operators, cf. Lions [6].

In this work we consider the time – periodic problem (1.1), (1.3) and (1.4) and solve it by elliptic regularization as an application of the monotony type results, cf. Browder [3]. Thus our proof is a simpler alternative to the earlier approaches existing in the current literature.

In fact, we consider the periodic problem

$$\left| \begin{array}{l} w'' + w' - \Delta w + w + |w'|^{p-2}w' = f \ \text{ in } \ Q = \Omega\times]0,T[, \\ w = 0 \text{ on } \ \partial\Omega\times]0,T[, \\ w(x,0) = w(x,T), \quad w'(x,0) = w'(x,T) \quad \text{ in } \Omega. \end{array} \right. \tag{1.10}$$

Thus for $w = u + u_0$, the function u is determined by (1.7).

We begin the functional space

$$\begin{aligned} W = \Big\{ & v;\ v \in L^2(0,T;V),\ v' \in L^2(0,T;V) \cap L^p(Q), \\ & v'' \in L^2(0,T;H),\ \int_0^T v(s)ds = 0,\ v(0) = v(T),\ v'(0) = v'(T) \Big\}. \end{aligned} \tag{1.11}$$

The Banach structure of W is defined by

$$\|v\|_W = \|v\|_{L^2(0,T;V)} + \|v'\|_{L^2(0,T;V)} + \|v'\|_{L^p(0,T;L^p(\Omega))} + \|v''\|_{L^2(0,T;H)}.$$

In the sequel by $\langle . , . \rangle$ we will represent the duality pairing between X and X', X' being the topological dual of the space X, and by c (sometimes $c_1, c_2, \ldots$) we denote various positive constants.

Motivated by (1.7) we define the bilinear form $b(u, v)$ for $u, v \in W$ by

$$b(u, v) = \int_0^T \left[(u'' + u' + u, v') + \langle Au, v' \rangle + \langle \gamma(u'), v' \rangle \right] dt,$$

where $A = -\Delta$ and $\gamma(u') = |u'|^{p-2}u'$.

Then the weak formulation of (1.7) is to find $u \in W$ such that

$$b(u, v) = \int_0^T \big(f, v'\big)\, dt, \tag{1.12}$$

for all $v \in W$.

Let us point out that the main difficulty in applying standard techniques from classical functional is due to the fact that the bilinear form $b(u, v)$ is not coercive. To resolve this issue, we perform an elliptic regularization on $b(u, v)$, following the ideas of Lions [6]. Subsequently we apply Theorem 2.1, p. 171 of Lions [6] to finally establish existence and uniqueness of solution to elliptic problem (1.12).

2 Main result

As we said in the Section 1, the method developed in this article is a variant of the elliptic regularization method introduced in Lions [6] in the context of the telegraph equation.

Indeed, following the same type of reasoning cf. Lions [6], to obtain the elliptic regularization, given $\mu > 0$ and $u, v \in W$ we define

$$\begin{aligned} \pi_\mu(u, v) \;\;&=\;\; \mu \int_0^T \left[(u'', v'') + (u', v') + (Au', v') \right] dt \\ &\quad + \int_0^T \big(u'' + u' + Au + u + \gamma(u'), v'\big)\, dt, \end{aligned} \tag{2.1}$$

where $A = -\Delta$ and $\gamma(u') = |u'|^{p-2}u'$.

It is easy to see, cf. Lemma 2.2, that the application $v \to \pi_\mu(u, v)$ is continuous on W. This allows to build a linear operator $\mathcal{B}_\mu : W \longrightarrow W'$, $\langle \mathcal{B}_\mu(u), v\rangle = \pi_\mu(u, v)$.

As we shall see, the linear operator $\mathcal{B}_\mu$ satisfies the following properties:

(a) $\mathcal{B}_\mu$ is a strictly monotonous operator; $\langle \mathcal{B}_\mu(v) - \mathcal{B}_\mu(z), v - z\rangle > 0$ for all $v, z \in W$, $v \neq z$;

(b) $\mathcal{B}_\mu$ is a hemicontinuous operator; $\lambda \to \langle \mathcal{B}_\mu(v + \lambda z), w\rangle$ is continuous in $\mathbb{R}$;

(c) $\mathcal{B}_\mu(S)$ is bounded in W' for all bounded set S in W;

(d) $\mathcal{B}_\mu$ is coercive; $\dfrac{\langle \mathcal{B}_\mu(v), v\rangle}{|v|_W} \to \infty$ as $|v|_W \to \infty$.

In view of these properties and as consequence of Theorem 2.1, p. 171 of Lions [6], the existence and uniqueness of a function $u_\mu \in W$ such that

$$\pi_\mu(u_\mu, v) = \int_0^T (f, v')\, dt, \qquad \text{for all } v \in W, \tag{2.2}$$

follows immediately.

The Eq. (2.2) is called of elliptic regularization of problem (1.7).

Our main result is as follows

Theorem 2.1. *Suppose $f \in L^{p'}(0, T; L^{p'}(\Omega))$, with $\dfrac{1}{p} + \dfrac{1}{p'} = 1$ and $p > 2$. Then there exists only one real function $w = w(x, t)$, $(x, t) \in Q$, $w \in W$, such that*

$$w = u + u_0,\ u_0 \in H_0^1(\Omega) \tag{2.3}$$

$$u \in L^2(0, T; V) \tag{2.4}$$

$$u' \in L^p(0, T; L^p(\Omega)) \tag{2.5}$$

and w satisfying (1.1) *in the sense of* $L^2(0, T; V') + L^{p'}(0, T; L^{p'}(\Omega))$.

Now, we begin by stating some lemmas that will be used in the proof of the Theorem 2.1.

Lemma 2.1. *If* $\int_0^T u(x,t)dt = 0$ *then*

$$\int_0^T \|u\|_V^2 dt \leq C \int_0^T \|u'\|_V^2 dt \quad and \quad \int_0^T \|u\|_{L^p(\Omega)}^p dt \leq C \int_0^T \|u'\|_{L^p(\Omega)}^p dt,$$

for u *derivable with respect to* t *in* $[0,T]$ *and* $u \in L^2(0,T;V)$, $u' \in L^2(0,T; V) \cap L^p(0,T;L^p(\Omega))$.

Proof. The proof of Lemma 2.1 can be obtained with slight modifications from Lions [6] or Medeiros [8].

Lemma 2.2. *The form* $v \to \pi_\mu(u,v)$ *defined in* (2.1) *is continuous on* W.

Proof. By Cauchy-Schwarz inequality and Young's inequality we have

$$|\pi_\mu(u,v)| \leq c_\mu \|u\|_W \|v\|_{W'}, \tag{2.6}$$

where c_μ is a constant positive that depend of μ. Then the result follows. □

Lemma 2.3. *The operator* $\mathcal{B}_\mu : W \longrightarrow W'$, $\langle \mathcal{B}_\mu(u), v\rangle = \pi_\mu(u,v)$ *is hemicontinuous, bounded, coercive and strictly monotonous from* $W \to W'$.

Proof. It follows of (2.6) that $\mathcal{B}_\mu(u)$ is bounded. From Lemma 2.1 and equality $\int_0^T \big(\gamma(u'), u'\big)\, dt = \|u'\|_{L^p(Q)}^p$, we obtain

$$\langle \mathcal{B}_\mu(v), v\rangle \geq c_\eta \|v\|_W^2,$$

because $\int_0^T u(x,t)dt = 0$. Thus $\mathcal{B}_\mu$ is W-coercive. The hemicontinuity of the operator $v \to |v|^{p-2}v$ allow us to conclude that the operator $\mathcal{B}_\mu$ is hemicontinuous. Finally, the proof that the operator $\mathcal{B}_\mu$ is strictly monotonous follows as in Lions [6], p. 494. □

Proof of Theorem 2.1. The arguments above show that there exists a unique solution $u_\mu \in W$ of the elliptic problem (2.2).

Explicitly the Eq. (2.2) has the form:

$$\begin{aligned}\mu\int_0^T \left[(u''_\mu, v'') + (u'_\mu, v') + ((u'_\mu, v'))\right] dt + \int_0^T (u''_\mu + u'_\mu + u_\mu, v')\, dt \\ + \int_0^T ((u_\mu, v'))\, dt + \int_0^T \langle\gamma(u'_\mu), v'\rangle\, dt = \int_0^T \left(f, v'\right)\, dt.\end{aligned} \tag{2.7}$$

We need let μ goes to zero in order to obtain $u_\mu \rightharpoonup u$ for the solution. Then we need estimates for u_μ.

In fact, setting $v = u_\mu$ in (2.7) and observing that u_μ and u'_μ are periodic since they belongs to W, we obtain

$$\begin{aligned}\mu\int_0^T \left(|u''_\mu|^2 + |u'_\mu|^2 + \|u'_\mu\|^2\right) dt + \int_0^T |u'_\mu|^2 dt + \int_0^T \|u'_\mu\|^p_{L^p(\Omega)} dt \\ \leq \frac{1}{\epsilon p'}\int_0^T ||f|^{p'}_{L^{p'}(\Omega)}\, dt + \frac{\epsilon}{p}\int_0^T |u'_\mu|^p_{L^p(\Omega)}\, dt.\end{aligned} \tag{2.8}$$

This implies that

$$(u'_\mu) \text{ is bounded in } L^2(0, T; H) \text{ when } \mu \to 0 \tag{2.9}$$

$$(u'_\mu) \text{ is bounded in } L^p(0, T; L^p(\Omega)) \text{ when } \mu \to 0 \tag{2.10}$$

$$\mu\int_0^T \left(|u''_\mu|^2 + |u'_\mu|^2 + \|u'_\mu\|^2\right) dt \leq c_1 \tag{2.11}$$

Since $\displaystyle\int_0^T u_\mu dt = 0$, we have by Lemma 2.1 that

$$(u_\mu) \text{ is bounded in } L^p(0, T; L^p(\Omega)) \tag{2.12}$$

$$\mu\int_0^T \|u_\mu\|^2 dt \leq c_2. \tag{2.13}$$

Setting

$$v(t) = \int_0^t u_\mu(\sigma)\, d\sigma - \frac{1}{T}\int_0^T (T-\sigma)u_\mu(\sigma)\, d\sigma, \tag{2.14}$$

it implies

$$\left|\begin{array}{l}\displaystyle\int_0^T v(t)\,dt = 0, \quad \forall v \in W\\ v' = u_\mu.\end{array}\right. \tag{2.15}$$

In fact, integrating both sides of the equation (2.14) on $[0, T]$, we obtain

$$\int_0^T v(t)\,dt = \int_0^T \int_0^t u_\mu(\sigma)\,d\sigma dt - \int_0^T \frac{1}{T}\int_0^T (T-\sigma)u_\mu(\sigma)\,d\sigma dt.$$

On the other hand,

$$\int_0^T \frac{1}{T}\int_0^T (T-\sigma)u_\mu(\sigma)\,d\sigma dt = \int_0^T \frac{1}{T}\,dt \int_0^T (T-\sigma)u_\mu(\sigma)\,d\sigma$$

$$= (T-\sigma)\int_0^\sigma u_\mu(s)\,ds\Big|_0^T + \int_0^T \int_0^\sigma u_\mu(s)\,dsd\sigma = \int_0^T \int_0^\sigma u_\mu(s)\,dsd\sigma.$$

Therefore, we reach our aim (2.15).

Thus, taking into account (2.14) in (2.2) we get

$$\begin{gathered}\mu \int_0^T \left[(u''_\mu, u'_\mu) + (u'_\mu, u_\mu) + (Au'_\mu, u_\mu)\right] dt\\ + \int_0^T \left[(u''_\mu, u_\mu) + (u'_\mu, u_\mu) + (Au_\mu, u_\mu) + (u_\mu, u_\mu) + (\gamma(u'_\mu), u_\mu)\right] dt\\ = \int_0^T (f, u_\mu)\,dt.\end{gathered} \tag{2.16}$$

By using periodicity of $u_\mu, u'_\mu \in W$, we obtain

$$\int_0^T (u''_\mu, u'_\mu)\,dt = \int_0^T (u'_\mu, u_\mu)\,dt = \int_0^T (Au'_\mu, u_\mu)\,dt = 0. \tag{2.17}$$

On the other hand,

$$\begin{aligned}\int_0^T (u''_\mu, u_\mu)\,dt &= (u'_\mu(T), u_\mu(T)) - (u'_\mu(0), u_\mu(0))\\ &\quad - \int_0^T (u'_\mu, u'_\mu)\,dt = -\int_0^T |u'_\mu|^2 dt.\end{aligned} \tag{2.18}$$

From (2.17), (2.18) and estimate (2.9), we have

$$\left|\int_0^T (u''_\mu, u_\mu)\,dt\right| \le c_2 \quad \text{when } \mu \to 0. \tag{2.19}$$

Also, from (2.10) and (2.12) we obtain

$$\begin{aligned}&\int_0^T |u_\mu|^2 dt + \int_0^T (\gamma(u'_\mu), u_\mu)\, dt \\ &\leq \int_0^T |u_\mu|^2 dt + \|\gamma(u'_\mu)\|_{L^{p'}(0,T;L^{p'}(\Omega))} \|u_\mu\|_{L^p(0,T;L^p(\Omega))} \leq c_3.\end{aligned} \tag{2.20}$$

Combining (2.17), (2.19) and (2.20) with (2.16) we deduce

$$\int_0^T \|u_\mu\|^2 dt \leq c_4. \tag{2.21}$$

It follows from (2.21) and (2.10) that there exists a subsequence from (u_μ), still denoted by (u_μ), such that

$$u_\mu \longrightarrow u \text{ weak in } L^2(0, T; V) \tag{2.22}$$

$$u'_\mu \longrightarrow u' \text{ weak in } L^p(0, T; L^p(\Omega)) \tag{2.23}$$

$$\gamma(u'_\mu) \longrightarrow \chi \text{ weak in } L^{p'}(0, T; L^{p'}(\Omega)). \tag{2.24}$$

Our next goal is tho show that u verifies $(1.7)_2 - (1.7)_3$.

Indeed, it follows from (2.22) and (2.23) that $u_\mu \in C^0([0, T]; H)$ and

$$\lim_{\mu\to 0} \int_0^T (u'_\mu, \varphi)\, dt = \int_0^T (u', \varphi)\, dt, \ \forall\varphi \in L^2(0, T; H) \tag{2.25}$$

$$\lim_{\mu\to 0} \int_0^T (u_\mu, \varphi)\, dt = \int_0^T (u, \varphi)\, dt, \ \forall\varphi \in L^2(0, T; V) \tag{2.26}$$

Setting $\varphi = \theta v$ into (2.25) with $\theta \in C^1([0, T]; \mathbb{R}),\ \theta(0) = \theta(T)$ and $v \in V$, we have

$$\int_0^T (u'_\mu, \theta v)\, dt \longrightarrow \int_0^T (u', \theta v)\, dt \tag{2.27}$$

$$\int_0^T (u_\mu, \theta' v)\, dt \longrightarrow \int_0^T (u, \theta' v)\, dt. \tag{2.28}$$

Again, by using periodicity of u_μ and u'_μ we obtain

$$\int_0^T \frac{d}{dt}(u_\mu, \theta v)\, dt = (u_\mu(T), \theta(T)v) - (u_\mu(0), \theta(0)v) = 0.$$

Thus

$$\int_0^T (u'_\mu, \theta v)\, dt + \int_0^T (u_\mu, \theta' v)\, dt = 0.$$

Since

$$\int_0^T (u', \theta v)\, dt + \int_0^T (u, \theta' v)\, dt = 0,$$

as $\mu \to 0$, we obtain

$$\int_0^T \frac{d}{dt}(u, \theta v)\, dt = 0.$$

This implies that

$$(u(T), \theta(T)v) - (u(0), \theta(0)v) = 0,$$

that is,

$$u(T) = u(0). \tag{2.29}$$

The proof that $u'(0) = u'(T)$ will be given later. Now, we go to prove that $\int_0^T u(t)\, dt = 0$.

Taking the scalar product on H of $\int_0^T u_\mu(\sigma)\, d\sigma = 0$ with $\varphi(t)$, $\varphi \in L^2(0, T; H)$, we find

$$\left(\int_0^T u_\mu(\sigma)\, d\sigma, \varphi(t) \right) = 0.$$

Thus

$$\int_0^T \big(u_\mu(\sigma), \varphi(t)\big)\, d\sigma = 0.$$

Therefore,

$$\int_0^T (u(\sigma), \varphi(t))\, d\sigma = \left(\int_0^T u(\sigma)\, d\sigma, \varphi(t) \right) = 0, \quad \forall \varphi(t) \in H, \tag{2.30}$$

as $\mu \to 0$.

It follows from (2.30) that

$$\int_0^T u(t)\, dt = 0. \tag{2.31}$$

From (2.9), (2.10), (2.11) and (2.13), we deduce

$$u'_\mu \longrightarrow u' \text{ weak in } L^2(0, T; H), \tag{2.32}$$

$$u'_\mu \longrightarrow u' \text{ weak in } L^p(0, T; L^p(\Omega)), \tag{2.33}$$

$$\sqrt{\mu} u''_\mu \longrightarrow \chi_1 \text{ weak in } L^2(0, T; H), \tag{2.34}$$

$$\sqrt{\mu} u'_\mu \longrightarrow \chi_2 \text{ weak in } L^2(0, T; H), \tag{2.35}$$

$$\sqrt{\mu} u'_\mu \longrightarrow \chi_3 \text{ weak in } L^2(0, T; V). \tag{2.36}$$

It follows from (2.34) that

$$\lim_{\mu \to 0} \sqrt{\mu} \int_0^T (u''_\mu, \varphi)\, dt = \int_0^T (\chi_1, \varphi)\, dt \quad \forall \varphi \in L^2(0, T; H). \tag{2.37}$$

Hence, taking $\varphi = v''$, $v \in W$, in (2.37), we find

$$\lim_{\mu \to 0} \sqrt{\mu} \int_0^T (u''_\mu, v'')\, dt = \int_0^T (\chi_1, v'')\, dt.$$

Therefore

$$\lim_{\mu \to 0} \mu \int_0^T (u''_\mu, v'')\, dt = \lim_{\mu \to 0} \sqrt{\mu} \left(\sqrt{\mu} \int_0^T (u''_\mu, v'') dt \right) = 0. \tag{2.38}$$

By analogy, we prove that

$$\lim_{\mu \to 0} \mu \int_0^T (u'_\mu, v')\, dt = \lim_{\mu \to 0} \mu \int_0^T (A u'_\mu, v')\, dt = 0. \tag{2.39}$$

By using periodicity of u_μ, $v \in W$, we obtain

$$\int_0^T \frac{d}{dt} (u'_\mu, v')\, dt = 0.$$

This implies that

$$\int_0^T (u''_\mu, v')\, dt = - \int_0^T (u'_\mu, v'')\, dt. \tag{2.40}$$

It follows of (2.9) that

$$\int_0^T (u'_\mu, \varphi) dt \longrightarrow \int_0^T (u', \varphi)\, dt \; \forall \varphi \in L^2(0, T; H). \tag{2.41}$$

Taking $\varphi = v'' \in L^2(0, T; H)$ in (2.41) we obtain

$$\lim_{\mu \to 0} \int_0^T (u'_\mu, v'')\, dt = \int_0^T (u', v'')\, dt. \tag{2.42}$$

From (2.2), we can write

$$\begin{aligned} &\mu \int_0^T \big[(u''_\mu, v'') + (u'_\mu, v') + (Au'_\mu, v')\big]\, dt \\ &+ \int_0^T \big[(u''_\mu, v') + (u'_\mu, v') + (Au_\mu, v') + (u_\mu, v') + (\gamma(u'_\mu), v')\big]\, dt \\ &= \int_0^T (f, v')\, dt. \end{aligned} \tag{2.43}$$

From (2.9), (2.10), (2.22), (2.38), (2.39), (2.40) and (2.42), we can pass to the limit in (2.43) when $\mu \to 0$ and obtain

$$\begin{aligned} &\int_0^T \big[(-u', v'') + (u', v') + (Au, v') + (u, v') + (\chi, v')\big]\, dt \\ &= \int_0^T (f, v')\, dt, \quad \forall v \in W. \end{aligned} \tag{2.44}$$

Let (ρ_ν) be a regularizing sequence of even periodic functions in t, with period T.

Denote by $\widetilde{v} = u * \rho_\nu * \rho_\nu$, where $*$ is the convolution operator. Integrating by parts, we find $u' * \rho_\nu * \rho_\nu = u * \rho'_\nu * \rho_\nu$.

Observe by (2.12) and (2.21) that $\widetilde{v} \in C^\infty(\mathbb{R}; V)$, $\widetilde{v}' \in C^\infty(\mathbb{R}; L^p(\Omega))$, $\widetilde{v}'' \in C^\infty(\mathbb{R}; H)$, v and $\widetilde{v}'$ periodic in t.

As in Brézis [2], p. 67, we to show that

$$\int_0^T (u', \widetilde{v}'')\, dt = 0. \tag{2.45}$$

In fact, we have

$$\begin{aligned} \int_0^T \frac{d}{dt}(u', u' * \rho_\nu * \rho_\nu)\, dt &= \int_0^T (u'', u' * \rho_\nu * \rho_\nu) + \int_0^T (u', u'' * \rho_\nu * \rho_\nu)\, dt \\ &= 2\int_0^T (u', u' * \rho'_\nu * \rho_\nu)\, dt = 2\int_0^T (u', \widehat{v}'')\, dt. \end{aligned}$$

As

$$\int_0^T (u', u' * \rho_\nu' * \rho_\nu)\, dt = \int_0^T \frac{1}{2}\frac{d}{dt}(u', u' * \rho_\nu * \rho_\nu)\, dt = 0,$$

due to periodicity of u' and ρ_ν, it follows (2.45).

Similarly, we show that

$$\int_0^T (u', \widetilde{v}')\, dt = 0. \tag{2.46}$$

$$\int_0^T (Au, \widetilde{v}')\, dt = 0. \tag{2.47}$$

$$\int_0^T (u, \widetilde{v}')\, dt = 0. \tag{2.48}$$

From (2.44) to (2.48) we obtain

$$\int_0^T (\chi, u')\, dt = \int_0^T (f, u')\, dt. \tag{2.49}$$

Now, let us prove that $\chi = \gamma(u')$.

In fact, from (2.2) and (2.1) we get

$$\begin{aligned}\mu \int_0^T [|u_\mu''|^2 + |u_\mu'|^2 + \|u_\mu'\|^2]\, dt + \int_0^T [|u_\mu'|^2 + (\gamma(u_\mu'), u_\mu')]\, dt \\ = \int_0^T (f, u_\mu')\, dt.\end{aligned} \tag{2.50}$$

We define

$$\begin{aligned} X_\mu \;=\; & \int_0^T (\gamma(u_\mu') - \gamma(\varphi), u_\mu' - \varphi)\, dt \\ & + \mu \int_0^T [|u_\mu''|^2 + |u_\mu'|^2 + \|u_\mu'\|^2]\, dt \\ & + \int_0^T [|u_\mu'|^2\, dt,\ \forall \varphi \in L^p(0, T; L^p(\Omega)) \end{aligned} \tag{2.51}$$

It follows from (2.50) and (2.51) that

$$X_\mu = \int_0^T (f, u_\mu')\, dt - \int_0^T (\gamma(\varphi), u_\mu' - \varphi) dt - \int_0^T (\gamma(u_\mu'), \varphi)\, dt. \tag{2.52}$$

From the convergences above, we get

$$X_\mu \longrightarrow X = \int_0^T (f, u')\,dt - \int_0^T (\gamma(\varphi), u' - \varphi)\,dt - \int_0^T (\chi, \varphi)\,dt. \quad (2.53)$$

Taking into account (2.53) into (2.49) yields

$$X = \int_0^T (\chi, u')\,dt - \int_0^T (\gamma(\varphi), u' - \varphi)\,dt - \int_0^T (\chi, \varphi)\,dt. \quad (2.54)$$

Combining (2.53) and (2.54), we obtain

$$X = \int_0^T (\chi - \gamma(\varphi), u' - \varphi)\,dt. \quad (2.55)$$

Since $X_\mu \geq 0$, for all $\varphi \in L^p(0, T; L^p(\Omega))$, then $X \geq 0$.

Thus,

$$\int_0^T (\chi - \gamma(\varphi), u' - \varphi)\,dt \geq 0, \quad \forall \varphi \in L^p(0, T; L^p(\Omega)). \quad (2.56)$$

Since $\gamma : L^p(0, T; L^p(\Omega)) \longrightarrow L^{p'}(0, T; L^{p'}(\Omega))$, $\gamma(u') = |u'|^{p-2}u'$, is hemicontinuous operator, the inequality above implies $\chi = \gamma(u')$. It is sufficient to set $\varphi(t) = u'(t) - \lambda w(t)$, $\lambda > 0$, $w \in L^p(0, T; L^p(\Omega))$ arbitrarily and let $\lambda \to 0$.

We consider $\psi \in C^\infty([0, T]; V \cap L^p(\Omega))$ satisfying

$$\left| \begin{array}{l} \displaystyle\int_0^T \psi\,dt = 0, \\ \psi(0) = \psi(T). \end{array} \right. \quad (2.57)$$

Setting

$$v(t) = \int_0^T \psi\,d\sigma - \frac{1}{T}\int_0^T (T - \sigma)\psi(\sigma)\,d\sigma \quad (2.58)$$

in (2.44), yields

$$\begin{aligned} \int_0^T &\big[(-u', \psi') + (u', \psi) + (Au, \psi) \\ &+ (u, \psi) + (\gamma(u'), \psi) - (f, \psi)\big]dt = 0, \end{aligned} \quad (2.59)$$

because $v'(t) = \psi(t)$, $v''(t) = \psi'(t)$.

In particular, choosing $\psi = \theta' v$, with $\theta \in \mathcal{D}]0, T[$ and $v \in V \cap L^p(\Omega)$, in (2.59) we get

$$\int_0^T \big[(-u', \theta'' v) + (u', \theta' v) + (Au, \theta' v) + (u, \theta' v) \\ + (\gamma(u'), \theta' v) - (f, \theta' v)\big] dt = 0, \quad \forall \theta \in \mathcal{D}]0, T[, \ \ v \in V \cap L^p(\Omega), \tag{2.60}$$

or equivalently,

$$\int_0^T \big(u'' + u' + Au + u + \gamma(u') - f, v\big)\theta' dt = 0, \tag{2.61}$$

for all $v \in V \cap L^4(\Omega)$ and $\theta \in \mathcal{D}]0, T[$.

Hence,

$$\frac{d}{dt}[(u'' + u' + Au + u + \gamma(u') - f, v)] = 0, \quad \forall v \in V \cap L^p(\Omega).$$

Consequently, there exists a function g_0 independent of t such that

$$u'' + u' + Au + u + \gamma(u') - f = g_0, \text{ independent of } t. \tag{2.62}$$

We verify that

$$u''(\varphi) = \int_0^T u''(t)\varphi(t)\, dt = -\int_0^T u'(t)\varphi'(t)\, dt \in L^p(\Omega) \tag{2.63}$$

$$Au(\varphi) = \int_0^T (Au(t))\varphi(t)\, dt \in V' \tag{2.64}$$

$$\gamma(u')(\varphi) = \int_0^T \gamma(u')\varphi\, dt \in L^{p'}(\Omega) \tag{2.65}$$

$$u'(\varphi) = \int_0^T u'(t)\varphi(t)\, dt \in L^p(\Omega) \tag{2.66}$$

$$u(\varphi) = \int_0^T u(t)\varphi(t)\, dt \in L^2(\Omega) \tag{2.67}$$

$$f(\varphi) = \int_0^T f(t)\varphi(t)\, dt \in L^{p'}(\Omega), \tag{2.68}$$

for all $\varphi \in \mathcal{D}]0, T[$, because $u' \in L^p(0, T; L^p(\Omega))$.

Thus, from (2.63) to (2.68) and (2.62), we can write

$$g_0 \int_0^T \varphi(t)\, dt \in V' + L^{p'}(\Omega).$$

Therefore

$$g_0 \in V' + L^{p'}(\Omega). \tag{2.69}$$

It follows from (2.62) that

$$\begin{aligned} u'' &= f + g_0 - u' - Au - u - \gamma(u') \\ &\in L^2(0, T; V') + L^{p'}(0, T; L^{p'}(\Omega). \end{aligned} \tag{2.70}$$

Hence, we deduce from (2.62) that,

$$\int_0^T \big(u'' + u' + Au + u + \gamma(u') - f - g_0, \psi\big)\, dt = 0, \tag{2.71}$$

with ψ given in (2.57).

Thus

$$\begin{aligned} &\int_0^T (u'' + u' + Au + u + \gamma(u') - f - g_0, \psi)\, dt \\ &= \int_0^T \frac{d}{dt}\big(u'(t), \psi)\big)\, dt + \int_0^T \big[(-u'(t), \psi') + (u'(t), \psi) \\ &+ (Au(t), \psi) + (u, \psi) + (\gamma(u'), \psi) - (f, \psi) - (g_0, \psi)\big]\, dt \\ &= (u'(T), \psi(T)) - (u'(0), \psi(0). \end{aligned} \tag{2.72}$$

Substituting (2.72) into (2.57) we obtain

$$u'(0) = u'(T). \tag{2.73}$$

Note that $u'(0)$ and $u'(T)$ make sense because u' an u'' belongs to $L^p(0, T; L^p(\Omega))$ and $L^2(0, T; V') + L^{p'}(0, T; L^{p'}(\Omega))$, respectively.

Let u_0 be defined by

$$\left| \begin{array}{l} -\Delta u_0 + u_0 = -g_0, \\ u_0 = 0 \text{ on } \partial\Omega. \end{array} \right. \tag{2.74}$$

We recall that because $n \leq 2$ and $p > 2$, we have

$$H_0^1(\Omega) \hookrightarrow L^p(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow L^{p'}(\Omega) \hookrightarrow H^{-1}(\Omega) = V',$$

where each space is dense in the following one and the injections are continuous.

This and (2.69) implies that $g_0 \in H^{-1}(\Omega) = V'$.

Finally, we apply the Lax-Milgram Theorem to find a unique solution $u_0 \in H_0^1(\Omega)$ of the Dirichlet problem (2.74).

Thus, $w = u + u_0 \in L^2(0, T; V)$ with $w' \in L^p(0, T; L^p(\Omega))$ satisfies

$$\left| \begin{array}{l} w'' + w' - \Delta w + w + |w'|^{p-2}w' = f \\ \quad \text{in } L^2(0, T; V') + L^{p'}(0, T; L^{p'}(\Omega)), \\ w(0) = w(T) \\ w'(0) = w'(T), \end{array} \right.$$

that is, w is a T-periodic weak solutions of problem (1.1).

Uniqueness. Let us consider w_1 and w_2 be two functions satisfying Theorem 2.1 and let $\xi = w_1 - w_2$.

We subtract the equations $(1.1)_1$ corresponding to w_1 and w_2 and we obtain

$$\xi'' + \xi' + A\xi + \xi + \gamma(w_1') - \gamma(w_2') = 0. \tag{2.75}$$

Denoting by (ρ_μ) the regularizing sequence defined above, by a similar argument used in the proof of existence of solutions for Theorem 2.1 we obtain

$$\xi' * \rho_\mu * \rho_\mu = \xi * \rho_\mu' * \rho_\mu. \tag{2.76}$$

Hence, by using (2.3) and (2.4), we can write

$$\xi = \psi + \xi_0, \quad \text{with } \xi_0 \in V \quad \text{and} \quad \psi \in L^2(0, T; V). \tag{2.77}$$

Also, from (2.76) we get

$$\xi' * \rho_\mu * \rho_\mu = \xi * \rho_\mu' * \rho_\mu = \psi' * \rho_\mu * \rho_\mu. \tag{2.78}$$

Thus, we have by (2.5) that $\psi' \in L^p(0, T; L^p(\Omega))$. Therefore $\xi' * \rho_\mu * \rho_\mu$ is periodic and

$$\xi' * \rho_\mu * \rho_\mu \in C^\infty([0, T]; L^p(\Omega)). \tag{2.79}$$

Then by (2.70) we can write

$$\xi'' \in L^2(0, T; V') + L^{p'}(0, T; L^{p'}(\Omega)).$$

This and (2.79) show that $\int_0^T (\xi'', \xi' * \rho_\mu * \rho_\mu)\, dt$ make sense and

$$\int_0^T (\xi'', \xi' * \rho_\mu * \rho_\mu)\, dt = 0. \tag{2.80}$$

Indeed,

$$\begin{aligned} \int_0^T \frac{d}{dt}(\xi', \xi' * \rho_\mu * \rho_\mu)\, dt &= \int_0^T (\xi'', \xi' * \rho_\mu * \rho_\mu)\, dt \\ + \int_0^T (\xi', \xi'' * \rho_\mu * \rho_\mu)\, dt &= \int_0^T (\xi'', \xi' * \rho_\mu * \rho_\mu)\, dt \\ + \int_0^T (\xi'', \xi' * \rho_\mu * \rho_\mu)\, dt. \end{aligned} \tag{2.81}$$

Therefore,

$$\int_0^T (\xi'', \xi' * \rho_\mu * \rho_\mu)\, dt = \frac{1}{2} \int_0^T \frac{d}{dt}(\xi', \xi' * \rho_\mu * \rho_\mu)\, dt = 0, \tag{2.82}$$

because ξ' and ρ_μ are periodic.

Similarly

$$\int_0^T (A\xi, \xi' * \rho_\mu * \rho_\mu)\, dt = 0 \tag{2.83}$$

$$\int_0^T (\xi', \xi' * \rho_\mu * \rho_\mu)\, dt = 0 \tag{2.84}$$

$$\int_0^T (\xi, \xi' * \rho_\mu * \rho_\mu)\, dt = 0. \tag{2.85}$$

Consequently, it follows from (2.75), (2.82), (2.83), (2.84) and (2.85) that

$$\int_0^T (\gamma(w_1') - \gamma(w_2'), \xi' * \rho_\mu * \rho_\mu)\, dt = 0. \tag{2.86}$$

Hence using (2.86), letting μ tend to zero, we have

$$\int_0^T (\gamma(w_1') - \gamma(w_2'), w_1' - w_2')\, dt = 0, \tag{2.87}$$

that is, $w_1' = w_2'$.

This implies that

$$\xi = w_1 - w_2 = \theta, \quad \theta \text{ independent of } t.$$

Integrating the last equality on $[0, T]$ and observing that $w_i = u_i + u_{0_i}$ yields

$$\int_0^T (w_1 - w_2)\, dt = \theta \int_0^T dt = \theta T = T(u_{0_1} - u_{0_2}),$$

because $\int_0^T u_i\, dt = 0$. Thus $\theta \in V$.

It follows from (2.83) that

$$\begin{aligned} \int_0^T (A\xi, \xi' * \rho_\mu * \rho_\mu)\, dt &= \int_0^T (A(w_1 - w_2), \xi' * \rho_\mu * \rho_\mu)\, dt \\ &= \int_0^T (A\theta, \theta * \rho_\mu' * \rho_\mu)\, dt = 0. \end{aligned}$$

This implies that, when $\mu \longrightarrow 0$

$$\int_0^T (A\theta, \theta) = 0, \quad \forall \theta \in V.$$

Therefore

$$A\theta = 0, \quad \forall \theta \in V. \tag{2.88}$$

Employing Green's Theorem, we find

$$(A\theta, \theta) = \int_\Omega -\Delta\theta\, \theta\, dx = \int_\Omega (\nabla\theta)^2 dx - \int_\Gamma \theta \frac{\partial \theta}{\partial \nu}\, d\Gamma = \|\theta\|^2. \tag{2.89}$$

Taking into account (2.89) into (2.88) yields $\theta = 0$, which proves the uniqueness of solutions of problem (1.2). Thus, the proof of Theorem 2.1 is complete. $\square$

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