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Bifurcation analysis of the Watt governor system

JORGE SOTOMAYOR[1], LUIS FERNANDO MELLO[2]
and DENIS DE CARVALHO BRAGA[3]

[1]Instituto de Matemática e Estatística, Universidade de São Paulo
Rua do Matão 1010, Cidade Universitária, 05508-090 São Paulo, SP, Brazil
[2]Instituto de Ciências Exatas, Universidade Federal de Itajubá
Avenida BPS 1303, Pinheirinho, 37500-903 Itajubá, MG, Brazil
[3]Instituto de Sistemas Elétricos e Energia, Universidade Federal de Itajubá
Avenida BPS 1303, Pinheirinho, 37500-903 Itajubá, MG, Brazil

E-mails: sotp@ime.usp.br / lfmelo@unifei.edu.br / braga_denis@yahoo.com.br

Abstract. This paper pursues the study carried out by the authors in *Stability and Hopf bifurcation in the Watt governor system* [14], focusing on the codimension one Hopf bifurcations in the centrifugal Watt governor differential system, as presented in Pontryagin's book *Ordinary Differential Equations*, [13]. Here are studied the codimension two and three Hopf bifurcations and the pertinent Lyapunov stability coefficients and bifurcation diagrams, illustrating the number, types and positions of bifurcating small amplitude periodic orbits, are determined. As a consequence it is found a region in the space of parameters where an attracting periodic orbit coexists with an attracting equilibrium.

Mathematical subject classification: 70K50, 70K20.

Key words: centrifugal governor, Hopf bifurcations, periodic orbit.

1 Introduction

The Watt centrifugal governor is a device that automatically controls the speed of an engine. Dating to 1788, it can be taken as the starting point for the theory of automatic control (see MacFarlane [10] and references therein). In this paper the system coupling the Watt-centrifugal-governor and the steam-engine will be

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called simply the Watt Governor System (WGS). See Section 2 for a description and illustration, in Fig. 1, of this system.

Landmarks for the study of the local stability analysis of the WGS are the works of Maxwell [11] and Vyshnegradskii [16]. A simplified version of the WGS local stability based on the work of Vyshnegradskii is presented by Pontryagin [13]. A local stability study generalized to a more general Watt governor design was carried out by Denny [4] and pursued by the authors in [14].

Enlightening historical comments about the Watt governor local mathematical stability and oscillatory analysis can be found in MacFarlane [10] and Denny [4]. There, as well as in [13], we learn that toward the mid XIX century, improvements in the engineering design led to less reliable operations in the WGS, leading to fluctuations and oscillations instead of the ideal stable constant speed output requirement. The first mathematical analysis of the stability conditions and subsequent indication of the modification in the design to avoid the problem was carried out by Maxwell [11] and, in a user friendly style likely to be better understood by engineers, by Vyshnegradskii [16].

From the mathematical point of view, the oscillatory, small amplitude, behavior in the WGS can be associated to a periodic orbit that appears from a Hopf bifurcation. This was established by Hassard et al. in [5] and Al-Humadi and Kazarinoff in [1]. Another procedure, based in the method of harmonic balance, has been suggested by Denny [4] to detect large amplitude oscillations.

In [14] we characterized the surface of Hopf bifurcations in a WGS, which is more general than that presented by Pontryagin [13], Al-Humadi and Kazarinoff [1] and Denny [4]. See Theorem 4.1 and Fig. 3 for a review of the critical curve on the surface where the first Lyapunov coefficient vanishes.

In the present paper, restricting ourselves to Pontryagin's system, we go deeper investigating the stability of the equilibrium along the above mentioned critical curve. To this end the second Lyapunov coefficient is calculated (Theorem 4.4) and it is established that it vanishes at a unique point (see Fig. 4 and 5). The third Lyapunov coefficient is calculated at this point (Theorem 4.5) and found to be positive. The pertinent bifurcation diagrams are established. See Fig. 6, 7 and 9. A conclusion derived from these diagrams, concerning the region – a solid "tongue" – in the space of parameters where an attracting periodic orbit coexists with an attracting equilibrium, is specifically commented in Section 5.

The extensive calculations involved in Theorems 4.4 and 4.5 have been corroborated with the software MATHEMATICA 5 [18] and the main steps have been posted in the site [17].

This paper is organized as follows. In Section 2 we introduce the WGS and review the Pontryagin differential equations [13]. The stability of the equilibrium points is also analyzed. This section is essentially a review of [13, 5, 1, 14]. The Hopf bifurcations in the WGS differential equations are studied in Sections 3 and 4. Expressions for the second and third Lyapunov coefficients, which fully clarify their sign, are obtained, pushing forward the method found in the works of Kuznetsov [8, 9]. With this data, the bifurcation diagrams are established. Concluding comments, synthesizing and interpreting the results achieved here, are presented in Section 5.

2 The Watt centrifugal governor system

2.1 Differential equations for the Watt governor system

According to Pontryagin [13], p. 217, the differential equations of the WGS illustrated in Fig. 1 are

$$\begin{aligned} \frac{d\,\varphi}{d\tau} &= \psi \\ \frac{d\,\psi}{d\tau} &= c^2\,\Omega^2\,\sin\varphi\,\cos\varphi - \frac{g}{l}\,\sin\varphi - \frac{b}{m}\,\psi \\ \frac{d\,\Omega}{d\tau} &= \frac{1}{I}\,(\mu\cos\varphi - F) \end{aligned} \tag{1}$$

where $\varphi \in \left(0, \frac{\pi}{2}\right)$ is the angle of deviation of the arms of the centrifugal governor from its vertical axis S_1, $\Omega \in [0, \infty)$ is the angular velocity of the rotation of the flywheel D, θ is the angular velocity of S_1, l is the length of the arms, m is the mass of each ball, H is a sleeve which supports the arms and slides along S_1, T is a set of transmission gears, V is the valve that determines the supply of steam to the engine, τ is the time, $\psi = d\varphi/d\tau$, g is the standard acceleration of gravity, $\theta = c\,\Omega$, $c > 0$ is a constant transmission ratio, $b > 0$ is a constant of the frictional force of the system, I is the moment of inertia of the flywheel, F is an equivalent torque of the load and $\mu > 0$ is a proportionality constant.

The reader is referred to Pontryagin [13] for the derivation of (1) from Newton's Second Law of Motion.

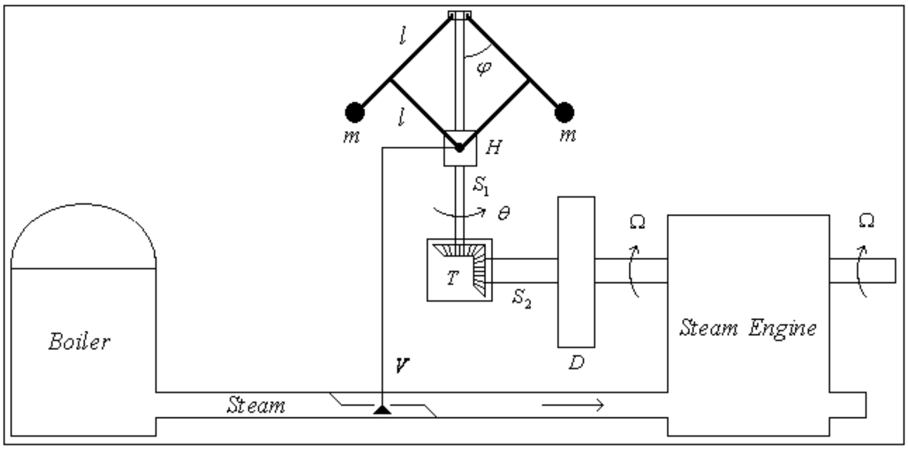

Figure 1 – Watt-centrifugal-governor-steam-engine system.

After the following change in the coordinates and time

$$x = \varphi, \quad y = \sqrt{\frac{l}{g}}\,\psi, \quad z = c\sqrt{\frac{l}{g}}\,\Omega, \quad \tau = \sqrt{\frac{l}{g}}\,t, \tag{2}$$

the differential equations (1) can be written as

$$\begin{aligned} x' = \frac{dx}{dt} &= y \\ y' = \frac{dy}{dt} &= z^2 \sin x \cos x - \sin x - \varepsilon\, y \\ z' = \frac{dz}{dt} &= \alpha\, (\cos x - \beta) \end{aligned} \tag{3}$$

where $\alpha > 0$, $0 < \beta < 1$ and $\varepsilon > 0$, given by

$$\varepsilon = \frac{b}{m}\sqrt{\frac{l}{g}}, \quad \alpha = \frac{c\, l\, \mu}{g\, I}, \quad \beta = \frac{F}{\mu}, \tag{4}$$

are the normalized variable parameters. Thus the differential equations (3) are in fact a three-parameter family of differential equations which can be rewritten as $\mathbf{x}' = f(\mathbf{x}, \mu)$, where

$$\begin{aligned} \mathbf{x} &= (x, y, z) \in \left(0, \tfrac{\pi}{2}\right) \times \mathbb{R} \times [0, \infty), \\ \mu &= (\beta, \alpha, \varepsilon) \in (0, 1) \times (0, \infty) \times (0, \infty) \end{aligned} \tag{5}$$

and

$$f(\mathbf{x}, \mu) = \left(y, z^2 \ \sin x \ \cos x - \sin x - \varepsilon \ y, \alpha \ (\cos x - \beta)\right). \tag{6}$$

2.2 Stability analysis of the equilibrium points

The differential equations (3) have one admissible equilibrium point

$$P_0 = (x_0, y_0, z_0) = \left(\arccos\beta, 0, \sqrt{\frac{1}{\beta}}\right). \tag{7}$$

The Jacobian matrix of f at P_0 has the form

$$Df\,(P_0) = \begin{pmatrix} 0 & 1 & 0 \\ -\dfrac{1-\beta^2}{\beta} & -\varepsilon & 2\sqrt{\beta(1-\beta^2)} \\ -\alpha\sqrt{1-\beta^2} & 0 & 0 \end{pmatrix} \tag{8}$$

and its characteristic polynomial is given by $p(\lambda)$, with

$$-p(\lambda) = \lambda^3 + \varepsilon\,\lambda^2 + \frac{1-\beta^2}{\beta}\,\lambda + 2\,\alpha\,\beta^{3/2}\,\frac{1-\beta^2}{\beta}. \tag{9}$$

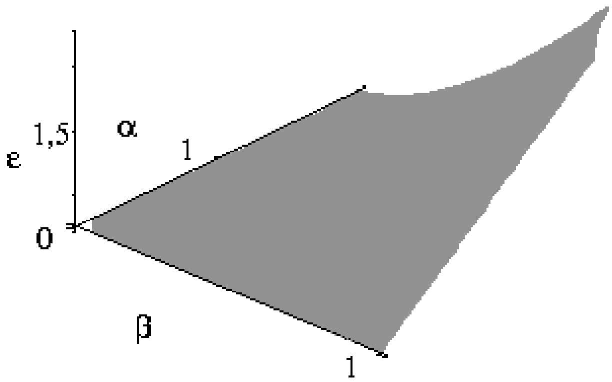

Figure 2 – Surface of critical parameters $\varepsilon_c = 2\,\alpha\,\beta^{3/2}$.

Theorem 2.1. *For all*

$$\varepsilon > 2\,\alpha\,\beta^{3/2} \tag{10}$$

the WGS differential equations (3) *have an asymptotically stable equilibrium point at P_0. If $0 < \varepsilon < 2\,\alpha\,\beta^{3/2}$ then P_0 is unstable.*

The proof of this theorem can essentially be found in Pontryagin [13]; it has also been established in a more general setting in [14].

The surface of critical parameters $\mu_0 = (\beta, \alpha, \varepsilon_c)$ such that $\varepsilon_c = \varepsilon(\beta, \alpha) = 2\,\alpha\,\beta^{3/2}$ is illustrated in Fig. 2. In the Section 4 we will analyze the stability of P_0 as $\varepsilon_c = 2\,\alpha\,\beta^{3/2}$. The change in the stability at the equilibrium P_0 as the parameters cross the critical surface produces a Hopf bifurcation in the WGS, whose analysis has been carried out by [1], [5] and, in a more general setting, by [14].

From (4), ε represents the friction coefficient of the system. The case $\varepsilon = 0$ maybe of theoretical interest due to its connection with conservative systems. However, as made explicit in *Vyshnegradskii's Rules*, friction is an essential ingredient to attain stability. This point is neatly presented in Pontryagin [13], of which Figure 2 is a geometric, dimensionless, synthesis.

3 Lyapunov coefficients

The beginning of this section is a review of the method found in [8], pp. 177-181, and in [9] for the calculation of the first and second Lyapunov coefficients. The calculation of the third Lyapunov coefficient has not been found by the authors in the current literature. The extensive calculations and the long expressions for these coefficients have been corroborated with the software MATHEMATICA 5 [18].

Consider the differential equations

$$\mathbf{x}' = f(\mathbf{x}, \mu), \tag{11}$$

where $\mathbf{x} \in \mathbb{R}^n$ and $\mu \in \mathbb{R}^m$ are respectively vectors representing phase variables and control parameters. Assume that f is of class C^∞ in $\mathbb{R}^n \times \mathbb{R}^m$. Suppose (11) has an equilibrium point $\mathbf{x} = \mathbf{x_0}$ at $\mu = \mu_\mathbf{0}$ and, denoting the variable $\mathbf{x} - \mathbf{x_0}$ also by $\mathbf{x}$, write

$$F(\mathbf{x}) = f(\mathbf{x}, \mu_\mathbf{0}) \tag{12}$$

as

$$\begin{aligned} F(\mathbf{x}) &= A\mathbf{x} + \frac{1}{2}\, B(\mathbf{x},\mathbf{x}) + \frac{1}{6}\, C(\mathbf{x},\mathbf{x},\mathbf{x}) + \frac{1}{24}\, D(\mathbf{x},\mathbf{x},\mathbf{x},\mathbf{x}) \\ &\quad + \frac{1}{120}\, E(\mathbf{x},\mathbf{x},\mathbf{x},\mathbf{x},\mathbf{x}) + \frac{1}{720}\, K(\mathbf{x},\mathbf{x},\mathbf{x},\mathbf{x},\mathbf{x},\mathbf{x}) \\ &\quad + \frac{1}{5040}\, L(\mathbf{x},\mathbf{x},\mathbf{x},\mathbf{x},\mathbf{x},\mathbf{x},\mathbf{x}) + O(||\mathbf{x}||^8), \end{aligned} \tag{13}$$

where $A = f_{\mathbf{x}}(0, \mu_{\mathbf{0}})$ and

$$B_i(\mathbf{x},\mathbf{y}) = \sum_{j,k=1}^{n} \left.\frac{\partial^2 F_i(\xi)}{\partial \xi_j\, \partial \xi_k}\right|_{\xi=0} x_j\, y_k, \tag{14}$$

$$C_i(\mathbf{x},\mathbf{y},\mathbf{z}) = \sum_{j,k,l=1}^{n} \left.\frac{\partial^3 F_i(\xi)}{\partial \xi_j\, \partial \xi_k\, \partial \xi_l}\right|_{\xi=0} x_j\, y_k\, z_l, \tag{15}$$

$$D_i(\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{u}) = \sum_{j,k,l,r=1}^{n} \left.\frac{\partial^4 F_i(\xi)}{\partial \xi_j\, \partial \xi_k\, \partial \xi_l\, \partial \xi_r}\right|_{\xi=0} x_j\, y_k\, z_l\, u_r, \tag{16}$$

$$E_i(\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{u},\mathbf{v}) = \sum_{j,k,l,r,p=1}^{n} \left.\frac{\partial^5 F_i(\xi)}{\partial \xi_j\, \partial \xi_k\, \partial \xi_l\, \partial \xi_r\, \partial \xi_p}\right|_{\xi=0} x_j\, y_k\, z_l\, u_r\, v_p, \tag{17}$$

$$K_i(\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{u},\mathbf{v},\mathbf{s}) = \sum_{j,\ldots,q=1}^{n} \left.\frac{\partial^6 F_i(\xi)}{\partial \xi_j\, \partial \xi_k\, \partial \xi_l\, \partial \xi_r\, \partial \xi_p\, \partial \xi_q}\right|_{\xi=0} x_j\, y_k\, z_l\, u_r\, v_p\, s_q, \tag{18}$$

$$L_i(\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{u},\mathbf{v},\mathbf{s},\mathbf{t}) = \sum_{j,\ldots,h=1}^{n} \left.\frac{\partial^7 F_i(\xi)}{\partial \xi_j \partial \xi_k \partial \xi_l \partial \xi_r \partial \xi_p \partial \xi_q \partial \xi_h}\right|_{\xi=0} x_j\, y_k\, z_l\, u_r\, v_p\, s_q\, t_h, \tag{19}$$

for $i = 1, \ldots, n$.

Suppose $(\mathbf{x_0}, \mu_{\mathbf{0}})$ is an equilibrium point of (11) where the Jacobian matrix A has a pair of purely imaginary eigenvalues $\lambda_{2,3} = \pm i\omega_0$, $\omega_0 > 0$, and admits no other eigenvalue with zero real part. Let T^c be the generalized eigenspace of A corresponding to $\lambda_{2,3}$. By this is meant that it is the largest subspace invariant by A on which the eigenvalues are $\lambda_{2,3}$.

Let $p, q \in \mathbb{C}^n$ be vectors such that

$$Aq = i\omega_0\, q, \quad A^\top p = -i\omega_0\, p, \quad \langle p, q\rangle = \sum_{i=1}^{n} \bar{p}_i\, q_i \;\; = 1, \tag{20}$$

where $A^\top$ is the transposed matrix. Any vector $y \in T^c$ can be represented as $y = wq + \bar{w}\bar{q}$, where $w = \langle p, y\rangle \in \mathbb{C}$. The two dimensional center manifold can be parametrized by w, $\bar{w}$, by means of an immersion of the form $\mathbf{x} = H(w, \bar{w})$, where $H : \mathbb{C}^2 \to \mathbb{R}^n$ has a Taylor expansion of the form

$$H(w, \bar{w}) = wq + \bar{w}\bar{q} + \sum_{2 \le j+k \le 7} \frac{1}{j!k!}\, h_{jk} w^j \bar{w}^k + O(|w|^8), \tag{21}$$

with $h_{jk} \in \mathbb{C}^n$ and $h_{jk} = \bar{h}_{kj}$. Substituting this expression into (11) we obtain the following differential equation

$$H_w w' + H_{\bar{w}} \bar{w}' = F(H(w, \bar{w})), \tag{22}$$

where F is given by (12).

The complex vectors h_{ij} are to be determined so that system (22), on the chart w for a central manifold, writes as follows

$$w' = i\omega_0 w + \frac{1}{2}\, G_{21} w|w|^2 + \frac{1}{12}\, G_{32} w|w|^4 + \frac{1}{144}\, G_{43} w|w|^6 + O(|w|^8),$$

with $G_{jk} \in \mathbb{C}$.

Solving for the vectors h_{ij} the system of linear equations defined by the coefficients of the quadratic terms of (22), taking into account the coefficients of F in the expressions (13) and (14), one has

$$h_{11} = -A^{-1} B(q, \bar{q}), \tag{23}$$

$$h_{20} = (2i\omega_0 I_n - A)^{-1} B(q, q), \tag{24}$$

where I_n is the unit $n \times n$ matrix. Pursuing the calculation to cubic terms, from the coefficients of the terms w^3 in (22) follows that

$$h_{30} = (3i\omega_0 I_n - A)^{-1} \left[3B(q, h_{20}) + C(q, q, q)\right]. \tag{25}$$

From the coefficients of the terms $w^2\bar{w}$ in (22) one obtains a singular system for h_{21}

$$(i\omega_0 I_n - A) h_{21} = C(q, q, \bar{q}) + B(\bar{q}, h_{20}) + 2B(q, h_{11}) - G_{21} q, \tag{26}$$

which has a solution if and only if

$$\langle p, C(q, q, \bar{q}) + B(\bar{q}, h_{20}) + 2B(q, h_{11}) - G_{21} q \rangle = 0.$$

Therefore

$$G_{21} = \langle p, C(q, q, \bar{q}) + B(\bar{q}, (2i\omega_0 I_n - A)^{-1} B(q, q)) - 2B(q, A^{-1} B(q, \bar{q})) \rangle. \tag{27}$$

The *first Lyapunov coefficient* l_1 is defined by

$$l_1 = \frac{1}{2}\, Re\; G_{21}. \tag{28}$$

The complex vector h_{21} can be found by solving the nonsingular $(n+1)$-dimensional system

$$\begin{aligned} &\begin{pmatrix} i\omega_0 I_n - A & q \\ \bar{p} & 0 \end{pmatrix} \begin{pmatrix} h_{21} \\ s \end{pmatrix} = \\ &\begin{pmatrix} C(q,q,\bar{q}) + B(\bar{q}, h_{20}) + 2B(q, h_{11}) - G_{21} q \\ 0 \end{pmatrix}, \end{aligned} \tag{29}$$

with the condition $\langle p, h_{21}\rangle = 0$.

For the sake of completeness, in Remark 3.1 we prove that the system (29) is nonsingular and that if (v, s) is a solution of (29) with the condition $\langle p, v\rangle = 0$ then v is a solution of (26).

Remark 3.1. Write $\mathbb{R}^n = T^c \oplus T^{su}$, where T^c and T^{su} are invariant by A. It can be proved that $y \in T^{su}$ if and only if $\langle p, y\rangle = 0$. Define

$$a = C(q,q,\bar{q}) + B(\bar{q}, h_{20}) + 2B(q, h_{11}) - G_{21} q.$$

Let (v, s) be a solution of the homogeneous equation obtained from (29). Equivalently

$$(i\omega_0 I_n - A)v + sq = 0, \;\; \langle p, v\rangle = 0. \tag{30}$$

From the second equation of (30), it follows that $v \in T^{su}$, and thus $(i\omega_0 I_n - A)v \in T^{su}$. Therefore $\langle p, (i\omega_0 I_n - A)v\rangle = 0$. Taking the inner product of p with the first equation of (30) one has $\langle p, (i\omega_0 I_n - A)v + sq\rangle = 0$, which can be written as $\langle p, (i\omega_0 I_n - A)v\rangle + s\langle p, q\rangle = 0$. Since $\langle p, q\rangle = 1$ and $\langle p, (i\omega_0 I_n - A)v\rangle = 0$ it follows that $s = 0$. Substituting $s = 0$ into the first equation of (30) one has $(i\omega_0 I_n - A)v = 0$. This implies that

$$v = \alpha q, \;\alpha \in \mathbb{C}. \tag{31}$$

But $0 = \langle p, v\rangle = \langle p, \alpha q\rangle = \alpha\langle p, q\rangle = \alpha$. Substituting $\alpha = 0$ into (31) it follows that $v = 0$. Therefore $(v, s) = (0, 0)$.

Let (v, s) be a solution of (29). Equivalently

$$(i\omega_0 I_n - A)v + sq = a, \;\langle p, v\rangle = 0. \tag{32}$$

From the second equation of (32), it follows that $v \in T^{su}$ and thus $(i\omega_0 I_n - A)v \in T^{su}$. Therefore $\langle p, (i\omega_0 I_n - A)v\rangle = 0$. Taking the inner product of p with the first equation of (32) one has $\langle p, (i\omega_0 I_n - A)v + sq\rangle = \langle p, a\rangle$, which can be written as

$$\langle p, (i\omega_0 I_n - A)v\rangle + s\langle p, q\rangle = \langle p, a\rangle.$$

As $\langle p, a\rangle = 0$, $\langle p, q\rangle = 1$ and $\langle p, (i\omega_0 I_n - A)v\rangle = 0$ it follows that $s = 0$. Substituting $s = 0$ into the first equation of (32) results $(i\omega_0 I_n - A)v = a$. Therefore v is a solution of (26).

The procedure above will be adapted below in connection with the determination of h_{32} and h_{43}.

From the coefficients of the terms w^4, $w^3\bar{w}$ and $w^2\bar{w}^2$ in (22), one has respectively

$$\begin{aligned} h_{40} = {} & (4i\omega_0 I_n - A)^{-1}\big[3B(h_{20}, h_{20}) + 4B(q, h_{30}) \\ & +\ 6C(q, q, h_{20}) + D(q, q, q, q)\big], \end{aligned} \tag{33}$$

$$\begin{aligned} h_{31} = {} & (2i\omega_0 I_n - A)^{-1}\big[3B(q, h_{21}) + B(\bar{q}, h_{30}) + 3B(h_{20}, h_{11}) \\ & +\ 3C(q, q, h_{11}) + 3C(q, \bar{q}, h_{20}) + D(q, q, q, \bar{q}) - 3G_{21}h_{20}\big], \end{aligned} \tag{34}$$

$$\begin{aligned} h_{22} = {} & -A^{-1}\big[D(q, q, \bar{q}, \bar{q}) + 4C(q, \bar{q}, h_{11}) + C(\bar{q}, \bar{q}, h_{20}) + C(q, q, \bar{h}_{20}) \\ & +\ 2B(h_{11}, h_{11}) + 2B(q, \bar{h}_{21}) + 2B(\bar{q}, h_{21}) + B(\bar{h}_{20}, h_{20})\big], \end{aligned} \tag{35}$$

where the term $-2h_{11}(G_{21} + \bar{G}_{21})$ has been omitted in the last equation, since $G_{21} + \bar{G}_{21} = 0$ as $l_1 = 0$.

Defining $\mathcal{H}_{32}$ as

$$\begin{aligned} \mathcal{H}_{32} = {} & 6B(h_{11}, h_{21}) + B(\bar{h}_{20}, h_{30}) + 3B(\bar{h}_{21}, h_{20}) + 3B(q, h_{22}) \\ & + 2B(\bar{q}, h_{31}) + 6C(q, h_{11}, h_{11}) + 3C(q, \bar{h}_{20}, h_{20}) \\ & + 3C(q, q, \bar{h}_{21}) + 6C(q, \bar{q}, h_{21}) + 6C(\bar{q}, h_{20}, h_{11}) \\ & + C(\bar{q}, \bar{q}, h_{30}) + D(q, q, q, \bar{h}_{20}) + 6D(q, q, \bar{q}, h_{11}) \\ & + 3D(q, \bar{q}, \bar{q}, h_{20}) + E(q, q, q, \bar{q}, \bar{q}) - 6G_{21}h_{21} - 3\bar{G}_{21}h_{21}, \end{aligned} \tag{36}$$

and from the coefficients of the terms $w^3\bar{w}^2$ in (22), one has a singular system for h_{32}

$$(i\omega_0 I_n - A)h_{32} = \mathcal{H}_{32} - G_{32}q, \tag{37}$$

which has solution if and only if

$$\langle p, \mathcal{H}_{32} - G_{32}q\rangle = 0, \tag{38}$$

where the terms $-6G_{21}h_{21} - 3\bar{G}_{21}h_{21}$ in the last line of (36) actually does not enter in last equation, since $\langle p, h_{21}\rangle = 0$.

The *second Lyapunov coefficient* is defined by

$$l_2 = \frac{1}{12}\, Re\, G_{32}, \tag{39}$$

where, from (38), $G_{32} = \langle p, \mathcal{H}_{32}\rangle$.

The complex vector h_{32} can be found solving the nonsingular $(n+1)$-dimensional system

$$\begin{pmatrix} i\omega_0 I_n - A & q \\ \bar{p} & 0 \end{pmatrix} \begin{pmatrix} h_{32} \\ s \end{pmatrix} = \begin{pmatrix} \mathcal{H}_{32} - G_{32} q \\ 0 \end{pmatrix}, \tag{40}$$

with the condition $\langle p, h_{32}\rangle = 0$.

From the coefficients of the terms $w^4\bar{w}$, $w^4\bar{w}^2$ and $w^3\bar{w}^3$ in (22), one has respectively

$$\begin{aligned} h_{41} = (3i\omega_0 I_n - A)^{-1}\big[&4B(h_{11}, h_{30}) + 6B(h_{20}, h_{21}) + 4B(q, h_{31}) \\ &+ B(\bar{q}, h_{40}) + 12C(q, h_{11}, h_{20}) + 6C(q, q, h_{21}) + 4C(q, \bar{q}, h_{30}) \\ &+ 3C(\bar{q}, h_{20}, h_{20}) + 4D(q, q, q, h_{11}) + 6D(q, q, \bar{q}, h_{20}) \\ &+ E(q, q, q, q, \bar{q}) - 6G_{21}h_{30}\big], \end{aligned} \tag{41}$$

$$\begin{aligned} h_{42} = (2i\omega_0 I_n - A)^{-1}\big[&8B(h_{11}, h_{31}) + 6B(h_{20}, h_{22}) + B(\bar{h}_{20}, h_{40}) \\ &+ 6B(h_{21}, h_{21}) + 4B(\bar{h}_{21}, h_{30}) + 4B(q, h_{32}) + 2B(\bar{q}, h_{41}) \\ &+ 12C(h_{11}, h_{11}, h_{20}) + 3C(h_{20}, h_{20}, \bar{h}_{20}) + 24C(q, h_{11}, h_{21}) \\ &+ 12C(q, h_{20}, \bar{h}_{21}) + 4C(q, \bar{h}_{20}, h_{30}) + 6C(q, q, h_{22}) + 8C(q, \bar{q}, h_{31}) \\ &+ 8C(\bar{q}, h_{11}, h_{30}) + 12C(\bar{q}, h_{20}, h_{21}) + C(\bar{q}, \bar{q}, h_{40}) \\ &+ 12D(q, q, h_{11}, h_{11}) + 6D(q, q, h_{20}, \bar{h}_{20}) + 4D(q, q, q, \bar{h}_{21}) \\ &+ 12D(q, q, \bar{q}, h_{21}) + 24D(q, \bar{q}, h_{11}, h_{20}) + 4D(q, \bar{q}, \bar{q}, h_{30}) \\ &+ 3D(\bar{q}, \bar{q}, h_{20}, h_{20}) + E(q, q, q, q, \bar{h}_{20}) + 8E(q, q, q, \bar{q}, h_{11}) \\ &+ 6E(q, q, \bar{q}, \bar{q}, h_{20}) + K(q, q, q, q, \bar{q}, \bar{q}) \\ &- 4(G_{32}h_{20} + 3G_{21}h_{31} + \bar{G}_{21}h_{31})\big], \end{aligned} \tag{42}$$

$$\begin{aligned} h_{33} = -A^{-1}\big[&9B(h_{11}, h_{22}) + 3B(h_{20}, \bar{h}_{31}) + 3B(\bar{h}_{20}, h_{31}) + 9B(h_{21}, \bar{h}_{21}) \\ &+ B(\bar{h}_{30}, h_{30}) + 3B(q, \bar{h}_{32}) + 3B(\bar{q}, h_{32}) + 6C(h_{11}, h_{11}, h_{11}) \\ &+ 9C(h_{11}, \bar{h}_{20}, h_{20}) + 18C(q, h_{11}, \bar{h}_{21}) + 3C(q, h_{20}, \bar{h}_{30}) \\ &+ 9C(q, \bar{h}_{20}, h_{21}) + 3C(q, q, \bar{h}_{31}) + 9C(q, \bar{q}, h_{22}) \\ &+ 18C(\bar{q}, h_{11}, h_{21}) + 9C(\bar{q}, h_{20}, \bar{h}_{21}) + 3C(\bar{q}, \bar{h}_{20}, h_{30}) \\ &+ 3C(\bar{q}, \bar{q}, h_{31}) + 9D(q, q, \bar{h}_{20}, h_{11}) + D(q, q, q, \bar{h}_{30}) \\ &+ 9D(q, q, \bar{q}, \bar{h}_{21}) + 18D(q, \bar{q}, h_{11}, h_{11}) + 9D(q, \bar{q}, \bar{h}_{20}, h_{20}) \\ &+ 9D(q, \bar{q}, \bar{q}, h_{21}) + 9D(\bar{q}, \bar{q}, h_{11}, h_{20}) + 3E(q, q, q, \bar{q}, \bar{h}_{20}) \\ &+ 9E(q, q, \bar{q}, \bar{q}, h_{11}) + 3E(q, \bar{q}, \bar{q}, \bar{q}, h_{20}) + K(q, q, q, \bar{q}, \bar{q}, \bar{q}) \\ &- 3(G_{32} + \bar{G}_{32})h_{11} - 9(G_{21} + \bar{G}_{21})h_{22}\big]. \end{aligned} \tag{43}$$

Defining $\mathcal{H}_{43}$ as

$$
\begin{aligned}
\mathcal{H}_{43} = {} & 12B(h_{11}, h_{32}) + 6B(h_{20}, \bar{h}_{32}) + 3B(\bar{h}_{20}, h_{41}) \\
& + 18B(h_{21}, h_{22}) + 12B(\bar{h}_{21}, h_{31}) + 4B(h_{30}, \bar{h}_{31}) + B(\bar{h}_{30}, h_{40}) \\
& + 4B(q, h_{33}) + 3B(\bar{q}, h_{42}) + 36C(h_{11}, h_{11}, h_{21}) + 36C(h_{11}, h_{20}, \bar{h}_{21}) \\
& + 12C(h_{11}, \bar{h}_{20}, h_{30}) + 3C(h_{20}, h_{20}, \bar{h}_{30}) + 18C(h_{20}, \bar{h}_{20}, h_{21}) \\
& + 36C(q, h_{11}, h_{22}) + 12C(q, h_{20}, \bar{h}_{31}) + 12C(q, \bar{h}_{20}, h_{31}) \\
& + 36C(q, h_{21}, \bar{h}_{21}) + 4C(q, h_{30}, \bar{h}_{30}) + 6C(q, q, \bar{h}_{32}) \\
& + 12C(q, \bar{q}, h_{32}) + 24C(\bar{q}, h_{11}, h_{31}) + 18C(\bar{q}, h_{20}, h_{22}) \\
& + 3C(\bar{q}, \bar{h}_{20}, h_{40}) + 18C(\bar{q}, h_{21}, h_{21}) + 12C(\bar{q}, \bar{h}_{21}, h_{30}) \\
& + 3C(\bar{q}, \bar{q}, h_{41}) + 24D(q, h_{11}, h_{11}, h_{11}) + 36D(q, h_{11}, h_{20}, \bar{h}_{20}) \\
& + 36D(q, q, h_{11}, \bar{h}_{21}) + 6D(q, q, h_{20}, \bar{h}_{30}) + 18D(q, q, \bar{h}_{20}, h_{21}) \\
& + 4D(q, q, q, \bar{h}_{31}) + 18D(q, q, \bar{q}, h_{22}) + 72D(q, \bar{q}, h_{11}, h_{21}) \\
& + 36D(q, \bar{q}, h_{20}, \bar{h}_{21}) + 12D(q, \bar{q}, \bar{h}_{20}, h_{30}) + 12D(q, \bar{q}, \bar{q}, h_{31}) \\
& + 36D(\bar{q}, h_{11}, h_{11}, h_{20}) + 9D(\bar{q}, h_{20}, h_{20}, \bar{h}_{20}) + 12D(\bar{q}, \bar{q}, h_{11}, h_{30}) \\
& + 18D(\bar{q}, \bar{q}, h_{20}, h_{21}) + D(\bar{q}, \bar{q}, \bar{q}, h_{40}) + 12E(q, q, q, h_{11}, \bar{h}_{20}) \\
& + E(q, q, q, q, \bar{h}_{30}) + 12E(q, q, q, \bar{q}, \bar{h}_{21}) + 36E(q, q, \bar{q}, h_{11}, h_{11}) \\
& + 18E(q, q, \bar{q}, h_{20}, \bar{h}_{20}) + 18E(q, q, \bar{q}, \bar{q}, h_{21}) + 36E(q, \bar{q}, \bar{q}, h_{11}, h_{20}) \\
& + 4E(q, \bar{q}, \bar{q}, \bar{q}, h_{30}) + 3E(\bar{q}, \bar{q}, \bar{q}, h_{20}, h_{20}) + 3K(q, q, q, q, \bar{q}, \bar{h}_{20}) \\
& + 12K(q, q, q, \bar{q}, \bar{q}, h_{11}) + 6K(q, q, \bar{q}, \bar{q}, \bar{q}, h_{20}) + L(q, q, q, q, \bar{q}, \bar{q}, \bar{q}) \\
& - 6(2G_{32}h_{21} + \bar{G}_{32}h_{21} + 3G_{21}h_{32} + 2\bar{G}_{21}h_{32}),
\end{aligned}
\tag{44}
$$

and from the coefficients of the terms $w^4\bar{w}^3$, one has a singular system for h_{43}

$$
(i\omega_0 I_n - A)h_{43} = \mathcal{H}_{43} - G_{43}q \tag{45}
$$

which has solution if and only if

$$
\langle p, \mathcal{H}_{43} - G_{43}q \rangle = 0, \tag{46}
$$

where the terms $-6(2G_{32}h_{21} + \bar{G}_{32}h_{21} + 3G_{21}h_{32} + 2\bar{G}_{21}h_{32})$ appearing in the last line of equation (44) actually do not enter in the last equation, since $\langle p, h_{21} \rangle = 0$ and $\langle p, h_{32} \rangle = 0$.

The *third Lyapunov coefficient* is defined by

$$
l_3 = \frac{1}{144}\, Re\, G_{43}, \tag{47}
$$

where, from (46), $G_{43} = \langle p, \mathcal{H}_{43} \rangle$.

The expressions for the vectors $h_{50}, h_{60}, h_{51}, h_{70}, h_{61}, h_{52}$ have been omitted since they are not important here.

Remark 3.2. Other equivalent definitions and algorithmic procedures to write the expressions for the Lyapunov coefficients l_j, $j = 1, 2, 3$, for two dimensional systems can be found in Andronov et al. [2] and Gasull et al. [6], among others. These procedures apply also to the three dimensional systems of this work, if properly restricted to the center manifold. The authors found, however, that the method outlined above, due to Kuznetsov [8, 9], requiring no explicit formal evaluation of the center manifold, is better adapted to the needs of this work.

A *Hopf point* $(\mathbf{x_0}, \mu_\mathbf{0})$ is an equilibrium point of (11) where the Jacobian matrix $A = f_\mathbf{x}(\mathbf{x_0}, \mu_\mathbf{0})$ has a pair of purely imaginary eigenvalues $\lambda_{2,3} = \pm i\omega_0$, $\omega_0 > 0$, and admits no other critical eigenvalues – i.e. located on the imaginary axis. At a Hopf point a two dimensional center manifold is well-defined, it is invariant under the flow generated by (11) and can be continued with arbitrary high class of differentiability to nearby parameter values. In fact, what is well defined is the ∞-jet – or infinite Taylor series – of the center manifold, as well as that of its continuation, any two of them having contact in the arbitrary high order of their differentiability class.

A Hopf point is called *transversal* if the parameter dependent complex eigenvalues cross the imaginary axis with non-zero derivative. In a neighborhood of a transversal Hopf point – H1 point, for concision – with $l_1 \neq 0$ the dynamic behavior of the system (11), reduced to the family of parameter-dependent continuations of the center manifold, is orbitally topologically equivalent to the following complex normal form

$$w' = (\eta + i\omega)w + l_1 w|w|^2,$$

$w \in \mathbb{C}$, η, ω and l_1 are real functions having derivatives of arbitrary high order, which are continuations of 0, ω_0 and the first Lyapunov coefficient at the H1 point. See [8]. As $l_1 < 0$ ($l_1 > 0$) one family of stable (unstable) periodic orbits can be found on this family of manifolds, shrinking to an equilibrium point at the H1 point.

A *Hopf point of codimension 2* is a Hopf point where l_1 vanishes. It is called *transversal* if $\eta = 0$ and $l_1 = 0$ have transversal intersections, where $\eta = \eta(\mu)$ is the real part of the critical eigenvalues. In a neighborhood of a transversal Hopf point of codimension 2 – H2 point, for concision – with $l_2 \neq 0$ the dynamic behavior of the system (11), reduced to the family of parameter-dependent continuations of the center manifold, is orbitally topologically equivalent to

$$w' = (\eta + i\omega_0)w + \tau w|w|^2 + l_2 w|w|^4,$$

where η and τ are unfolding parameters. See [8]. The bifurcation diagrams for $l_2 \neq 0$ can be found in [8], p. 313, and in [15].

A *Hopf point of codimension 3* is a Hopf point of codimension 2 where l_2 vanishes. A Hopf point of codimension 3 is called *transversal* if $\eta = 0$, $l_1 = 0$ and $l_2 = 0$ have transversal intersections. In a neighborhood of a transversal Hopf point of codimension 3 – H3 point, for concision – with $l_3 \neq 0$ the dynamic behavior of the system (11), reduced to the family of parameter-dependent continuations of the center manifold, is orbitally topologically equivalent to

$$w' = (\eta + i\omega_0)w + \tau w|w|^2 + \nu w|w|^4 + l_3 w|w|^6,$$

where η, τ and ν are unfolding parameters. The bifurcation diagram for $l_3 \neq 0$ can be found in Takens [15].

Theorem 3.3. *Suppose that the system*

$$\mathbf{x}' = f(\mathbf{x}, \mu), \quad \mathbf{x} = (x, y, z), \quad \mu = (\beta, \alpha, \varepsilon)$$

has the equilibrium $\mathbf{x} = \mathbf{0}$ *for* $\mu = 0$ *with eigenvalues*

$$\lambda_{2,3}(\mu) = \eta(\mu) \pm i\omega(\mu),$$

where $\omega(0) = \omega_0 > 0$*. For* $\mu = 0$ *the following conditions hold*

$$\eta(0) = 0, \quad l_1(0) = 0, \quad l_2(0) = 0,$$

where $l_1(\mu)$ *and* $l_2(\mu)$ *are the first and second Lyapunov coefficients, respectively. Assume that the following genericity conditions are satisfied*

1. $l_3(0) \neq 0$*, where* $l_3(0)$ *is the third Lyapunov coefficient;*
2. *the map* $\mu \to (\eta(\mu), l_1(\mu), l_2(\mu))$ *is regular at* $\mu = 0$*.*

Then, by the introduction of a complex variable, the above system reduced to the family of parameter-dependent continuations of the center manifold, is orbitally topologically equivalent to

$$w' = (\eta + i\omega_0)w + \tau w|w|^2 + \nu w|w|^4 + l_3 w|w|^6$$

where η*,* τ *and* ν *are unfolding parameters.*

Remark 3.4. The proof of this theorem given by Takens for C^∞ families of vector fields, using the Malgrange-Mather Preparation Theorem [7], is also valid in the present case of arbitrarily high, but finite, class of differentiability, using the appropriate extensions of the Preparation Theorem. See Bakhtin [3] and Milman [12], among others.

4 Hopf bifurcations

The stability of the equilibrium point P_0 given in (7) as $\varepsilon_c = \varepsilon(\beta, \alpha) = 2\,\alpha\,\beta^{3/2}$ is analyzed here. According to (13) and the subsequent expressions (14), (15), (16), (17), (18) and (19), for B_i to L_i, one has

$$A = \begin{pmatrix} 0 & 1 & 0 \\ -\omega_0^2 & -\varepsilon_c & 2\,\beta\,\omega_0 \\ -\alpha\,\sqrt{\beta}\,\omega_0 & 0 & 0 \end{pmatrix}, \tag{48}$$

where

$$\omega_0 = \sqrt{\frac{1-\beta^2}{\beta}}, \tag{49}$$

and referring to the expressions in equations (11) and (12)

$$F(\mathbf{x}) - A\mathbf{x} = (0,\, F_2(\mathbf{x}),\, F_3(\mathbf{x}))\,, \tag{50}$$

where

$$\begin{aligned} F_2(\mathbf{x}) = & -\frac{3}{2}\,\omega_0\,\sqrt{\beta}\,x^2 + \omega_0\,\beta^{3/2}\,z^2 + \frac{2(2\beta^2-1)}{\sqrt{\beta}}\,x\,z + \frac{4-7\beta^2}{6\beta}\,x^3 \\ & - 4\,\omega_0\,\beta\,x^2\,z + (2\beta^2-1)\,x\,z^2 + \frac{5}{8}\sqrt{\beta}\omega_0 x^4 + \frac{4}{3}\beta^{-1/2}(1-2\beta^2)x^3 z \\ & - 2\beta^{3/2}\omega_0 x^2 z^2 + \frac{31\beta^2-16}{120\beta}x^5 + \frac{4}{3}\beta\omega_0 x^4 z + \frac{2-4\beta^2}{3}x^3 z^2 \\ & - \frac{7}{80}\sqrt{\beta}\omega_0 x^6 + \frac{4}{15}\beta^{-1/2}(2\beta^2-1)x^5 z + \frac{2}{3}\beta^{3/2}\omega_0 x^4 z^2 \\ & + \frac{64-127\beta^2}{5040\beta}x^7 - \frac{8}{45}\beta\omega_0 x^6 z + \frac{4\beta^2-2}{15}x^5 z^2 + O(||\mathbf{x}||^8), \end{aligned}$$

$$\begin{aligned} F_3(\mathbf{x}) = & -\frac{1}{2}\,\alpha\,\beta\,x^2 + \frac{1}{6}\,\alpha\,\sqrt{\beta}\,\omega_0 x^3 + \frac{1}{24}\,\alpha\,\beta\,x^4 - \frac{1}{120}\,\alpha\,\sqrt{\beta}\,\omega_0 x^5 \\ & - \frac{1}{720}\,\alpha\,\beta\,x^6 + \frac{1}{5040}\,\alpha\,\sqrt{\beta}\,\omega_0 x^7 + O(||\mathbf{x}||^8). \end{aligned}$$

From equations (13), (14), (15), (16), (17), (18), (19) and (50) one has

$$B(\mathbf{x},\mathbf{y}) = (0,\, B_2(\mathbf{x},\mathbf{y}),\, -\alpha\,\beta\,x_1\,y_1)\,, \tag{51}$$

where

$$\begin{aligned} B_2(\mathbf{x},\mathbf{y}) = & -3\,\omega_0\,\sqrt{\beta}\,x_1\,y_1 + 2\,\omega_0\,\beta^{3/2}\,x_3\,y_3 \\ & + \frac{2\left(2\,\beta^2-1\right)}{\sqrt{\beta}}\left(x_1\,y_3 + x_3\,y_1\right), \end{aligned}$$

$$C(\mathbf{x},\mathbf{y},\mathbf{z}) = \left(0,\, C_2(\mathbf{x},\mathbf{y},\mathbf{z}),\, \alpha\,\sqrt{\beta}\,\omega_0\, x_1\, y_1\, z_1\right), \tag{52}$$

where

$$\begin{aligned} C_2(\mathbf{x},\mathbf{y},\mathbf{z}) \;=\; & \frac{4-7\beta^2}{\beta}\, x_1\, y_1\, z_1 - 8\omega_0\,\beta\,\left(x_1\, y_1\, z_3 + x_1\, y_3\, z_1 + x_3\, y_1\, z_1\right) \\ & +\; 2\,\left(2\beta^2-1\right)\left(x_1\, y_3\, z_3 + x_3\, y_1\, z_3 + x_3\, y_3\, z_1\right), \end{aligned}$$

$$D(\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{u}) = \left(0,\, D_2(\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{u}),\, \alpha\beta x_1 y_1 z_1 u_1\right), \tag{53}$$

where

$$\begin{aligned} D_2(\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{u}) \;=\; & 15\omega_0\beta^{1/2} x_1 y_1 z_1 u_1 + 8\left(\frac{1-2\beta^2}{\beta^{1/2}}\right)\left(x_1 y_1 z_1 u_3\right. \\ & +\; x_1 y_1 z_3 u_1 +\; x_1 y_3 z_1 u_1 + x_3 y_1 z_1 u_1\left.\right) \\ & -\; 8\omega_0\beta^{3/2}\left(x_1 y_1 z_3 u_3 + x_1 y_3 z_1 u_3 +\; x_1 y_3 z_3 u_1\right. \\ & +\; x_3 y_3 z_1 u_1 + x_3 y_1 z_1 u_3 + x_3 y_1 z_3 u_1\left.\right), \end{aligned}$$

$$E(\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{u},\mathbf{v}) = \left(0,\, E_2(\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{u},\mathbf{v}),\, -\alpha\omega_0\beta^{1/2} x_1 y_1 z_1 u_1 v_1\right), \tag{54}$$

where

$$\begin{aligned} E_2(\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{u},\mathbf{v}) \;=\; & \frac{31\beta^2-16}{\beta}\, x_1 y_1 z_1 u_1 v_1 + 32\omega_0\beta\,\left(x_1 y_1 z_1 u_1 v_3\right. \\ & +\; x_1 y_1 z_1 u_3 v_1 + x_1 y_1 z_3 u_1 v_1 + x_1 y_3 z_1 u_1 v_1 \\ & +\; x_3 y_1 z_1 u_1 v_1\left.\right) + 8\left(1-2\beta^2\right)\left(x_1 y_1 z_1 u_3 v_3\right. \\ & +\; x_1 y_1 z_3 u_1 v_3 + x_1 y_1 z_3 u_3 v_1 + x_1 y_3 z_1 u_1 v_3 \\ & +\; x_1 y_3 z_1 u_3 v_1 + x_1 y_3 z_3 u_1 v_1 + x_3 y_3 z_1 u_1 v_1 \\ & +\; x_3 y_1 z_3 u_1 v_1 + x_3 y_1 z_1 u_3 v_1 + x_3 y_1 z_1 u_1 v_3\left.\right), \end{aligned}$$

$$K(\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{u},\mathbf{v},\mathbf{s}) \;=\; (0,\, K_2(\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{u},\mathbf{v},\mathbf{s}),\, -\alpha\beta x_1 y_1 z_1 u_1 v_1 s_1)\,, \tag{55}$$

where

$$\begin{aligned} K_2(\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{u},\mathbf{v},\mathbf{s}) \;=\; & -\,63\omega_0\beta^{1/2} x_1 y_1 z_1 u_1 v_1 s_1 \\ & +\; 32\left(\frac{2\beta^2-1}{\beta^{1/2}}\right)\left(x_1 y_1 z_1 u_1 v_1 s_3 +\; x_1 y_1 z_1 u_1 v_3 s_1\right. \\ & +\; x_1 y_1 z_1 u_3 v_1 s_1 + x_1 y_1 z_3 u_1 v_1 s_1 + x_1 y_3 z_1 u_1 v_1 s_1 \\ & +\; x_3 y_1 z_1 u_1 v_1 s_1\left.\right) + 32\omega_0\beta^{3/2}\left(x_1 y_1 z_1 u_1 v_3 s_3\right. \\ & +\; x_1 y_1 z_1 u_3 v_1 s_3 + x_1 y_1 z_3 u_1 v_1 s_3 + x_1 y_3 z_1 u_1 v_1 s_3 \\ & +\; x_3 y_1 z_1 u_1 v_1 s_3 + x_1 y_1 z_1 u_3 v_3 s_1 + x_1 y_1 z_3 u_1 v_3 s_1 \end{aligned}$$

$$
\begin{aligned}
&+\ x_1 y_3 z_1 u_1 v_3 s_1 + x_3 y_1 z_1 u_1 v_3 s_1 + x_1 y_1 z_3 u_3 v_1 s_1 \\
&+ x_1 y_3 z_1 u_3 v_1 s_1 + x_3 y_1 z_1 u_3 v_1 s_1 + x_1 y_3 z_3 u_1 v_1 s_1 \\
&+\ x_3 y_1 z_3 u_1 v_1 s_1 + x_3 y_3 z_1 u_1 v_1 s_1\big),
\end{aligned}
$$

$$
L(\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{u},\mathbf{v},\mathbf{s},\mathbf{t}) \;=\; \big(0,\, L_2(\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{u},\mathbf{v},\mathbf{s},\mathbf{t}),\, \omega_0 \beta^{1/2} x_1 y_1 z_1 u_1 v_1 s_1 t_1\big), \tag{56}
$$

where

$$
\begin{aligned}
L_2(\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{u},\mathbf{v},\mathbf{s},\mathbf{t}) \;=\; &\big(64\omega_0^2 - 63\beta\big) x_1 y_1 z_1 u_1 v_1 s_1 t_1 - 128\omega_0 \beta \big(x_1 y_1 z_1 u_1 v_1 s_1 t_3 \\
&+\ x_1 y_1 z_1 u_1 v_1 s_3 t_1 + x_1 y_1 z_1 u_1 v_3 s_1 t_1 + x_1 y_1 z_1 u_3 v_1 s_1 t_1 \\
&+\ x_1 y_1 z_3 u_1 v_1 s_1 t_1 + x_1 y_3 z_1 u_1 v_1 s_1 t_1 + x_3 y_1 z_1 u_1 v_1 s_1 t_1\big) \\
&+\ 32\beta\big(\beta - \omega_0^2\big)\big(x_1 y_1 z_1 u_1 v_1 s_3 t_3 + x_1 y_1 z_1 u_1 v_3 s_1 t_3 \\
&+\ x_1 y_1 z_1 u_3 v_1 s_1 t_3 + x_1 y_1 z_3 u_1 v_1 s_1 t_3 + x_1 y_3 z_1 u_1 v_1 s_1 t_3 \\
&+\ x_3 y_1 z_1 u_1 v_1 s_1 t_3 + x_1 y_1 z_1 u_1 v_3 s_3 t_1 + x_1 y_1 z_1 u_3 v_1 s_3 t_1 \\
&+\ x_1 y_1 z_3 u_1 v_1 s_3 t_1 + x_1 y_3 z_1 u_1 v_1 s_3 t_1 + x_3 y_1 z_1 u_1 v_1 s_3 t_1 \\
&+\ x_1 y_1 z_1 u_3 v_3 s_1 t_1 + x_1 y_1 z_3 u_1 v_3 s_1 t_1 + x_1 y_3 z_1 u_1 v_3 s_1 t_1 \\
&+\ x_3 y_1 z_1 u_1 v_3 s_1 t_1 + x_1 y_1 z_3 u_3 v_1 s_1 t_1 + x_1 y_3 z_1 u_3 v_1 s_1 t_1 \\
&+\ x_3 y_1 z_1 u_3 v_1 s_1 t_1 + x_1 y_3 z_3 u_1 v_1 s_1 t_1 + x_3 y_1 z_3 u_1 v_1 s_1 t_1 \\
&+\ x_3 y_3 z_1 u_1 v_1 s_1 t_1\big).
\end{aligned}
$$

The eigenvalues of A (equation (48)) are

$$
\lambda_1 = -\varepsilon_c = -2\alpha\beta^{3/2}, \;\; \lambda_2 = i\,\omega_0, \;\; \lambda_3 = -i\,\omega_0. \tag{57}
$$

and from (20) one has

$$
q \;=\; \left(-i,\; \omega_0,\; \frac{\varepsilon_c}{2\beta}\right) \qquad \text{and} \tag{58}
$$

$$
p \;=\; \left(-\frac{i}{2},\; \frac{\omega_0 - i\varepsilon_c}{2\left(\omega_0^2 + \varepsilon_c^2\right)},\; \frac{\beta\,(\varepsilon_c + i\omega_0)}{\omega_0^2 + \varepsilon_c^2}\right). \tag{59}
$$

Theorem 4.1. *Consider the three-parameter family of differential equations* (3). *The first Lyapunov coefficient is given by*

$$
l_1(\beta,\alpha,\varepsilon_c) \;=\; -\,\frac{1}{2}\left(\frac{\alpha\beta^{3/2}(1-\beta^2)\left(3 + (\alpha^2-5)\beta^2 + \alpha^4\beta^6\right)}{\left(1-\beta^2+\alpha^2\beta^4\right)\left(1-\beta^2+4\alpha^2\beta^4\right)}\right). \tag{60}
$$

If

$$
g(\beta,\alpha) = 3 + (\alpha^2-5)\beta^2 + \alpha^4\beta^6 \tag{61}
$$

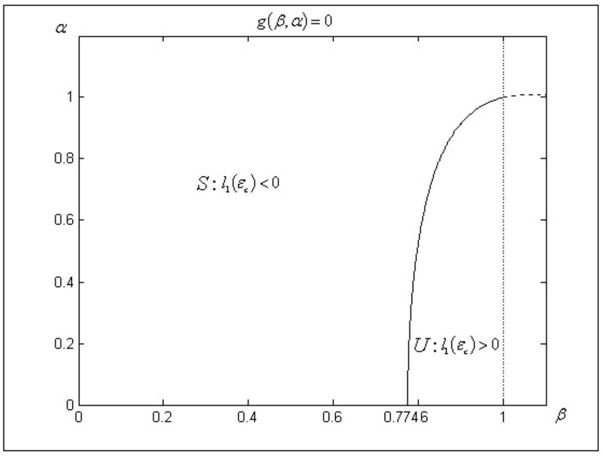

Figure 3 – Signs of the first Lyapunov coefficient.

is different from zero then the three-parameter family of differential equations (3) *has a transversal Hopf point at* P_0 *for* $\varepsilon_c = \varepsilon(\beta, \alpha) = 2\,\alpha\,\beta^{3/2}$.

If $(\beta, \alpha, \varepsilon_c) \in S \cup U$ *(see Fig.* 3*) then the three-parameter family of differential equations* (3) *has a H1 point at* P_0. *If* $(\beta, \alpha, \varepsilon_c) \in S$ *then the H1 point at* P_0 *is asymptotically stable and for each* $\varepsilon < \varepsilon_c$, *but close to* ε_c, *there exists a stable periodic orbit near the unstable equilibrium point* P_0. *If* $(\beta, \alpha, \varepsilon_c) \in U$ *then the H1 point at* P_0 *is unstable and for each* $\varepsilon > \varepsilon_c$, *but close to* ε_c, *there exists an unstable periodic orbit near the asymptotically stable equilibrium point* P_0.

This theorem summarizes Proposition 3.2 and Theorems 3.5, 3.6 and 3.7 established in [14]. Equation (61) gives a simple expression for the sign of the first Lyapunov coefficient (60). Its graph is illustrated in Fig. 3, where the signs of the first Lyapunov coefficient are also represented. The curve $l_1 = 0$ divides the surface of critical parameters into two connected components denoted by S and U where $l_1 < 0$ and $l_1 > 0$ respectively.

We have the following theorem.

Theorem 4.2. *Consider the three-parameter family of differential equations* (3) *restricted to* $\varepsilon = \varepsilon_c$. *The second Lyapunov coefficient is given by*

$$l_2(\beta, \alpha, \varepsilon_c) = \frac{\alpha\,\beta^{3/2}\,h(\beta, \alpha, \varepsilon_c)}{36\left(1 - \beta^2 + \alpha^2\beta^4\right)^3\left(9 - 9\beta^2 + 4\alpha^2\beta^4\right)\left(1 - \beta^2 + 4\alpha^2\beta^4\right)^3}, \quad (62)$$

where

$$
\begin{aligned}
h(\beta,\alpha,\varepsilon_c) \;=\; &- \; 162 - 54(-9+37\alpha^2)\beta^2 - 9(-126+61\alpha^2+60\alpha^4)\beta^4 \\
&- \; 18(405-3212\alpha^2+1128\alpha^4)\beta^6 \\
&+ \; (13770-210843\alpha^2+113612\alpha^4-5533\alpha^6)\beta^8 \\
&- \; 6(2133-57687\alpha^2+38218\alpha^4+5186\alpha^6)\beta^{10} \\
&+ \; (5994-301275\alpha^2+215340\alpha^4+284264\alpha^6-16022\alpha^8)\beta^{12} \\
&+ \; 2(-567+67878\alpha^2-45196\alpha^4-379430\alpha^6+9347\alpha^8)\beta^{14} \\
&+ \; \alpha^2(-25029+9540\alpha^2+990831\alpha^4+155856\alpha^6-21205\alpha^8)\beta^{16} \\
&+ \; 4\alpha^4(513-\alpha^2(163340+120616\alpha^2-16768\alpha^4))\beta^{18} \\
&- \; 2\alpha^6(-86887-258835\alpha^2+30173\alpha^4+7208\alpha^6)\beta^{20} \\
&+ \; 2\alpha^8(-96867-8956\alpha^2+23208\alpha^4)\beta^{22} \\
&+ \; \alpha^{10}(33671-58288\alpha^2-4880\alpha^4)\beta^{24} \\
&+ \; 16\alpha^{12}(1603+718\alpha^2)\beta^{26} - 16\alpha^{14}(453+40\alpha^2)\beta^{28} \\
&+ \; 640\alpha^{16}\beta^{30}.
\end{aligned}
$$

Proof. Define the following functions

$$T_1 = \mathrm{Re}\,\langle p, E(q,q,q,\bar q,\bar q)\rangle,\;\; T_2 = \mathrm{Re}\,\langle p, D(q,q,q,\bar h_{20})\rangle,\;\; T_3 = \mathrm{Re}\,\langle p, D(q,\bar q,\bar q,h_{20})\rangle,$$

$$T_4 = \mathrm{Re}\,\langle p, D(q,q,\bar q,h_{11})\rangle,\;\; T_5 = \mathrm{Re}\,\langle p, C(\bar q,\bar q,h_{30})\rangle,\;\; T_6 = \mathrm{Re}\,\langle p, C(q,q,\bar h_{21})\rangle,$$

$$T_7 = \mathrm{Re}\,\langle p, C(q,\bar q,h_{21})\rangle,\;\; T_8 = \mathrm{Re}\,\langle p, C(q,\bar h_{20},h_{20})\rangle,\;\; T_9 = \mathrm{Re}\,\langle p, C(q,h_{11},h_{11})\rangle,$$

$$T_{10} = \mathrm{Re}\,\langle p, C(\bar q,h_{20},h_{11})\rangle,\;\; T_{11} = \mathrm{Re}\,\langle p, B(\bar q,h_{31})\rangle,\;\; T_{12} = \mathrm{Re}\,\langle p, B(q,h_{22})\rangle,$$

$$T_{13} = \mathrm{Re}\,\langle p, B(\bar h_{20},h_{30})\rangle,\;\; T_{14} = \mathrm{Re}\,\langle p, B(\bar h_{21},h_{20})\rangle,\;\; T_{15} = \mathrm{Re}\,\langle p, B(h_{11},h_{21})\rangle.$$

From (39) one has

$$
\begin{aligned}
Re\ G_{32} \;=\; & T_1 + T_2 + 3T_3 + 6T_4 + T_5 + 3T_6 + 6T_7 + 3T_8 + 6T_9 + 6T_{10} \\
& + 2T_{11} + 3T_{12} + T_{13} + 3T_{14} + 6T_{15}.
\end{aligned}
$$

The theorem follows by expanding the expressions in definition of the second Lyapunov coefficient (39). It relies on extensive calculation involving the vector q (58), the vector p (59), the functions B, C, D and E, listed equations (51), (52), (53) and (54), respectively, the long complex vectors h_{11}, h_{20}, h_{30}, h_{21}, h_{31} and h_{22}, and the above functions T_1 to T_{15}.

The calculations in this proof, corroborated by Computer Algebra, have been posted in [17]. Here, the complex vectors h_{11}, h_{20}, h_{30}, h_{21}, h_{31} and h_{22} have particularly long expressions. They have listed in MATHEMATICA 5 files. □

Theorem 4.3. *For the system* (3) *there is unique point* $Q = (\beta, \alpha, \varepsilon_c)$*, with coordinates*

$$\beta = 0.86828033997971281542..., \ \ \alpha = 0.85050048430685017856...$$

and

$$\varepsilon_c = 1.37624106484659953171...$$

where the curves $l_1 = 0$ *and* $l_2 = 0$ *on the critical surface intersect and there do it transversally.*

Computer assisted proof. The point Q is the intersection of the curves $l_1 = 0$ and $l_2 = 0$ on the Hopf critical surface. It is defined and obtained by the solution of the equations

$$g(\beta, \alpha) = 0,$$

given in (61), and

$$h(\beta, \alpha) = h(\beta, \alpha, \varepsilon_c) = 0, \tag{63}$$

where $h(\beta, \alpha, \varepsilon_c)$ is given by (62). The existence and uniqueness of Q with the above coordinates has been established numerically with the software MATHEMATICA 5.

Figure 4 presents a geometric synthesis interpreting the long calculations involved in this proof. The sign of $h(\beta, \alpha)$ gives the sign of the second Lyapunov coefficient (62). The graph of $h(\beta, \alpha) = 0$, where the signs of the first and second Lyapunov coefficients are also illustrated. As follows, $l_2 < 0$ on the open arc of the curve $l_1 = 0$, denoted by C_1. On this arc a typical reference point R is depicted. Also $l_2 > 0$ on the open arc of the curve $l_1 = 0$, denoted by C_2. This arc contains the typical reference point, denoted by T. See also Fig. 5.

The bifurcation diagrams of the system (3) at the points T and R are illustrated in Fig. 6 and 7, as a consequence of [8] and [15].

The main steps of the calculations that provide the numerical evidence for this theorem have been posted in [17]. □

Theorem 4.4. *If* $(\beta, \alpha, \varepsilon_c) \in C_1 \cup C_2$ *then the three-parameter family of differential equations* (3) *has a transversal Hopf point of codimension 2 at* P_0*. If* $(\beta, \alpha, \varepsilon_c) \in C_2$ *then the H2 point at* P_0 *is unstable and the bifurcation diagram is drawn in Fig.* 6*. If* $(\beta, \alpha, \varepsilon_c) \in C_1$ *then the H2 point at* P_0 *is asymptotically stable and the bifurcation diagram is illustrated in Fig.* 7*.*

This theorem is a synthesis of the discussion in the last part in the proof of Theorem 4.3.

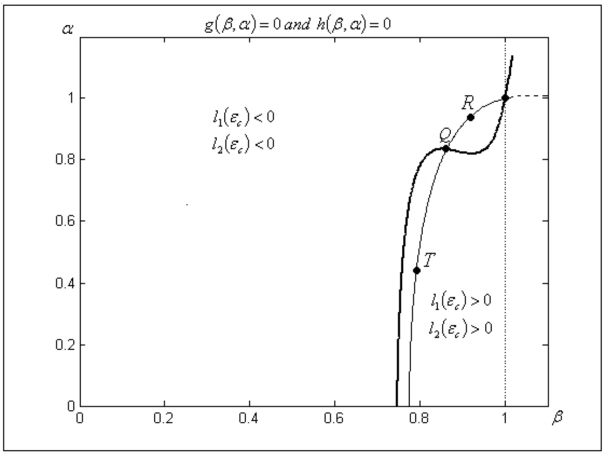

Figure 4 – Signs of the first and second Lyapunov coefficients.

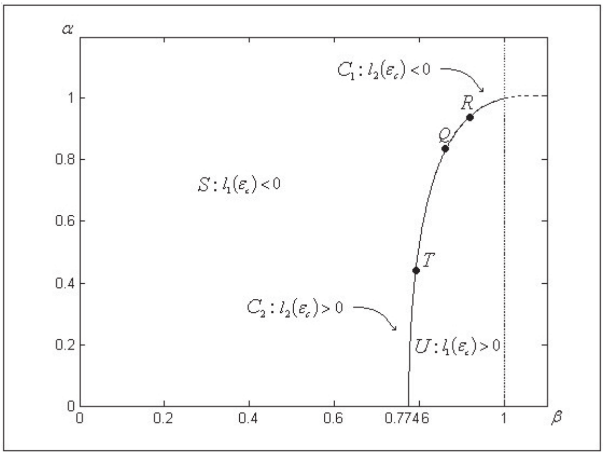

Figure 5 – Signs of l_2 on the curve $l_1 = 0$.

Theorem 4.5. *For the parameter values at the point Q determined in Theorem* 4.3*, the three-parameter family of differential equations* (3) *has a tranversal Hopf point of codimension 3 at P_0 which is asymptotically unstable since $l_3(Q) > 0$. The bifurcation diagram of system* (3) *at the point Q is illustrated in Figs.* 8 *and* 9*.*

Computer assisted proof. For the point Q take five decimal round-off coordinates $\beta = 0.86828$, $\alpha = 0.85050$ and $\varepsilon_c = 1.37624$. For these values of the parameters

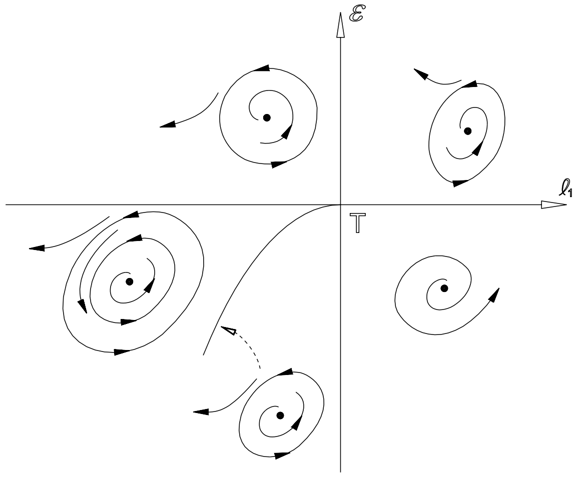

Figure 6 – Bifurcation diagram of the system (3) at point T.

one has

$$
\begin{aligned}
p &= \big(-i/2, 0.12224 - 0.31601i, 0.54878 + 0.21228i\big),\\
q &= \big(-i, 0.53237, 0.79250\big),\\
h_{11} &= \big(-1.75030, 0, 0.48792\big),\\
h_{20} &= \big(-2.24198 - 0.11191i, 0.11916 - 2.38715i, 0.04434 - 1.58196i\big),\\
h_{30} &= \big(-2.68329 + 5.27951i, -8.43202 - 4.28554i, -4.24045 - 0.86409i\big),\\
G_{21} &= -2.90053i,
\end{aligned}
\tag{64}
$$

$$
\begin{aligned}
h_{21} &= \big(1.20918 + 0.65492i, -3.24920 + 0.64374i, 1.26042 + 1.11353i\big),\\
h_{40} &= \big(9.27690 + 25.24802i, -53.76550 + 19.75510i, -9.11345 + 11.36572i\big),\\
h_{31} &= \big(-25.72175 - 5.12199i, 4.47976 - 7.87822i, 6.22842 - 15.97687i\big),\\
h_{22} &= \big(-15.72589, 0, 10.92671\big),\\
G_{32} &= -34.93331i,
\end{aligned}
\tag{65}
$$

$$
\begin{aligned}
h_{32} &= \big(27.17768 + 53.16361i, -57.53733 + 3.94677i, 52.73722 + 27.89259i\big),\\
h_{41} &= \big(-35.5370 + 180.2333i, -195.9736 - 10.0589i, -125.3480 - 33.7428i\big),\\
h_{42} &= \big(-778.4924 - 466.4510i, 362.1612 + 81.2385i, 390.2364 - 503.3807i\big),\\
h_{33} &= \big(-536.09324, 0, 835.33555\big),\\
G_{43} &= 56.23254 - 2424.27069i.
\end{aligned}
\tag{66}
$$

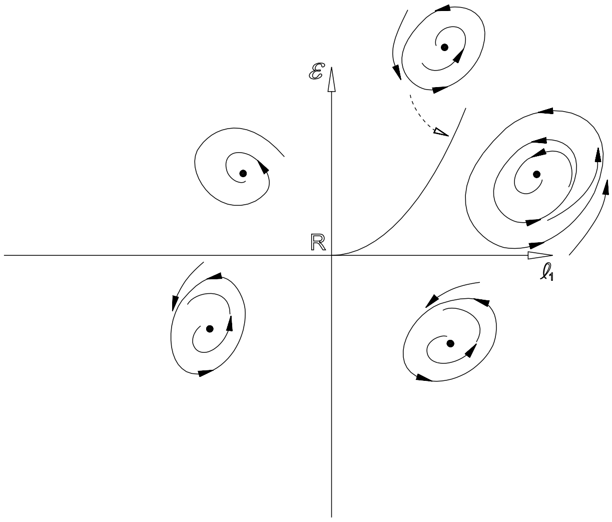

Figure 7 – Bifurcation diagram of the system (3) at point R.

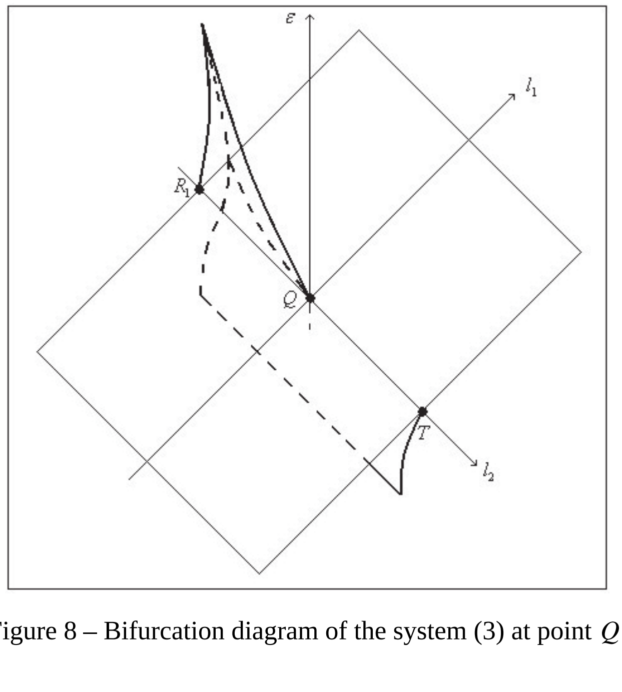

Figure 8 – Bifurcation diagram of the system (3) at point Q.

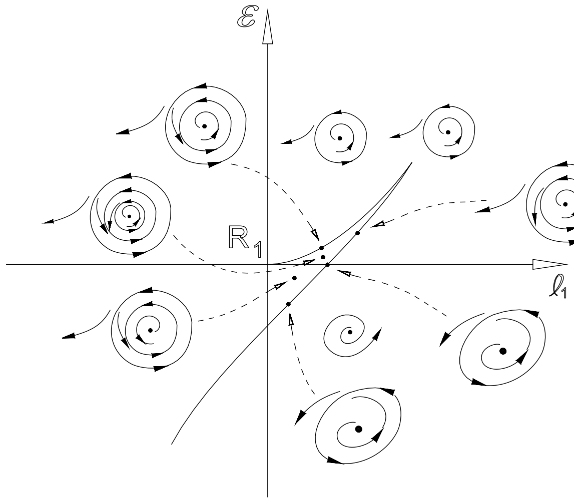

Figure 9 – Bifurcation diagram of the system (3) at point R_1.

From (28), (39), (47), (64), (65) and (66) one has

$$l_1(Q) = 0,\ \ l_2(Q) = 0,\ \ l_3(Q) = \frac{1}{144}\, Re\, G_{43} = 0.39050.$$

The calculations above have also been corroborated with 20 decimals round-off precision performed using the software MATHEMATICA 5 [18]. See [17].

The gradients of the functions l_1, given in (60), and l_2, given in (62), at the point Q are, respectively

$$(0.80095, -0.31847),\ \ (-0.38861, -0.85118).$$

The transversality condition at Q is equivalent to the non-vanishing of the determinant of the matrix whose columns are the above gradient vectors, which is evaluated gives -0.80552. The transversality condition being satisfied, the bifurcation diagrams in Figs. 8 and 9, follow from the work of Takens [15], taking into consideration the orientation and signs established in Theorems 4.3 and 4.4. □

5 Concluding comments

The historical relevance of the Watt governor study as well as its importance for present day theoretical and technological aspects of Automatic Control has been widely discussed by Denny [4] and others. See also [10, 14].

This paper starts reviewing the stability analysis due to Maxwell and Vyshnegradskii, which accounts for the characterization, in the space of parameters, of the structural as well as Lyapunov stability of the equilibrium of the Watt Centrifugal Governor System, WGS. It continues with recounting the extension of the analysis to the first order, codimension one stable points, happening on the complement of a curve in the critical surface where the eigenvalue criterium of Lyapunov holds, as studied in [5], [1] and by the authors [14], based on the calculation of the first Lyapunov coefficient. Here the bifurcation analysis at the equilibrium point of the WGS is pushed forward to the calculation of the second and third Lyapunov coefficients which make possible the determination of the Lyapunov as well as higher order structural stability at the equilibrium point. See also [8, 9], [6] and [2].

The calculations of these coefficients, being extensive, rely on Computer Algebra and Numerical evaluations carried out with the software MATHEMATICA 5 [18]. In the site [17] have been posted the main steps of the calculations in the form of notebooks for MATHEMATICA 5.

With the analytic and numeric data provided in the analysis performed here, the bifurcation diagrams are established along the points of the curve where the first Lyapunov coefficient vanishes. Pictures 8 and 9 provide a qualitative synthesis of the dynamical conclusions achieved here at the parameter values where the WGS achieves most complex equilibrium point. A reformulation of these conclusions follow:

There is a "solid tongue" where two stable regimes coexist: one is an equilibrium and the other is a small amplitude periodic orbit, i.e. an oscillation.

For parameters inside the "tongue", this conclusion suggests, a *hysteresis* explanation for the phenomenon of "hunting" observed in the performance of WGS in an early stage of the research on its stability conditions. Which attractor represents the actual state of the system will depend on the path along which the parameters evolve to reach their actual values of the parameters under consideration. See Denny [4] for historical comments, where he refers to the term "hunting" to mean an oscillation around an equilibrium going near but not reaching it.

Finally, we would like to stress that although this work ultimately focuses the specific three dimensional, three parameter system of differential equations given by (1), the method of analysis and calculations explained in Section 3 can be adapted to the study of other systems with three or more phase variables and depending on three or more parameters.

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http://www.ici.unifei.edu.br/luisfernando/wgs.

[18] Software MATHEMATICA 5: http://www.wolfram.com/.