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Free Vibration of Laminated Composites Beams Using Strain Gradient Notation Finite Element Models

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This works presents free vibration analysis of laminated composite beam problems using Timoshenko beam finite elements formulated in strain gradient notation. The formulation in physically interpretable notation identifies precisely the one parasitic shear term present in the in-plane, two-node, six-degree-of-freedom Timoshenko beam element. The spurious term can be eliminated *a-priori* of implementation and analysis. An assessment of the deleterious effects of parasitic shear in the computation of natural frequencies and mode shapes of laminated composite beams is performed via convergence studies. Beams with different boundary conditions and lamination schemes are analyzed. Results from models containing parasitic shear and from models corrected for it are compared. It is seen that parasitic shear affects significantly the convergence characteristics of the model as it retards convergence. It is also observed that parasitic shear affects the shapes of vibration modes. After elimination of parasitic shear, convergence of natural frequencies is attained quite rapidly. Such result is most pronounced when computing the fundamental frequency. Further, convergent results are compared to results presented in the literature for accuracy assessment, and very good agreement is found.

Keywords: *Free vibration, Laminated composites, Timoshenko beam, Finite element method, Strain gradient notation, Parasitic shear.*

1. Introduction

Laminated composite structures are largely employed in engineering practice mostly where high strength- and stiffness-to-weight ratios are key design factors. Such applications involve, for instance, the aerospace, the aeronautical and the automotive structures among others. In most of these applications, dynamic loads are predominant, and, thus, dynamic responses must be accurately evaluated for successful design and safety of these structures. As analytical solutions are difficult to obtain and sometimes are limited and only applicable to simple structures with specific laminae stacking sequence, engineers resort to numerical methods such that more complex and general structures can be analyzed. Therefore, researchers constantly seek for numerical methods that can provide accurate and stable dynamic responses of laminated composite structures with the lowest computational effort possible. Finite element methods are among the most widely used numerical methods for dynamic analyses of laminated composite structures. The first step in establishing a finite element method as suitable for dynamic analysis is to determine its accuracy in computing natural frequencies and modes of vibration, which provide the vibration characteristics of a structure. They are obtained by the solution of the free vibration problem cast in eigen form. As in Rao[1], free vibration occurs when an elastic system oscillates harmonically after an initial disturbance is applied to it in the absence of external forces. This work investigates on the free vibration of laminated composite beam problems modeled by Timoshenko beam elements formulated using strain gradient notation[2]. As shear occurs between laminae of a laminate, Timoshenko beam theory[3] is suitably employed. Therefore, first-order shear deformation theory (FSDT) is the theory employed here to model laminated composites. Strain gradient notation is a physically interpretable notation which explicitly relates physical quantities such as displacements and strains to the kinematic quantities of the continuum that can be represented by a given model. It also allows for modeling capabilities and deficiencies to be clearly identified. The Timoshenko beam element model employed here has been formulated earlier for static analysis of laminated composites by the senior author and co-workers[4]. Here, its formulation is extended to allow also for free vibration analysis. The new routines are implemented into the in-house code LAMFEM developed in FORTRAN language.

One of the modeling deficiencies that can plague finite elements is parasitic shear. Such error occurs in shear strain polynomials of given finite elements as a result of

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the displacement assumptions made during formulation. Parasitic shear causes the well documented phenomenon of shear locking. Mostly, shear locking is associated to retarding convergence, i.e., a quantitative error. In laminated composites, it has been shown that parasitic shear also causes qualitative errors when deformation modes couplings are present[4,5]. The two-node Timoshenko beam element possesses one parasitic shear term in its transverse shear strain expression. Such term can be precisely identified and eliminated via strain gradient notation, as it will be reviewed here. Deleterious effects of parasitic shear in the convergence characteristics of the model in computing natural frequencies and mode shapes are shown in this work. Accurate computation is achieved after elimination of parasitic shear. Natural frequencies values computed with the proposed model are compared to other results available in the literature.

2. Literature Review

This section presents a brief literature review on the free vibration analysis of beams, most specifically focused on laminated composites, but also considering works associated to free vibration of isotropic Timoshenko beams considered relevant. In the classical book Vibration Problems in Engineering, Timoshenko et al.[3] describe the inclusion of rotary inertia and transverse shear deformation in the vibration of beams. The resultant beam theory is referred to in the literature as Timoshenko beam theory (TBT). In the mechanics of laminated composites, TBT corresponds to first-order shear deformation theory (FSDT), which appears to be the simplest theory that is adequate for modeling laminated composites, considering that the classical lamination theory (CLT) does not include transverse shear deformation. Many researchers have evaluated the TBT in vibration analysis. In 1949, Kruszewski[6] considered the effects of transverse shear and rotary inertia on the natural frequencies of uniform beam problems, showing that either effect causes the decline of natural frequency values. Kapur[7] reached the same conclusion when applying the finite element method for vibration analysis of Timoshenko beam problems. Other earlier work[8-14] focused on isotropic materials and concerned themselves with formulating finite elements using different strategies to improve results, and also with investigating the effects of transverse shear and rotary inertia on the dynamic response. Han et al.[15] compared four analytical beam formulations, and broad conclusions are that transverse shear is more relevant to dynamic response than rotary inertia, and that the Timoshenko model must be employed for short beams.

Along the last four decades, there has been extensive research on free vibration analysis of laminated composite beams, and important experimental, analytical and numerical work has been produced. Much has been summarized in a paper by Sayyad and Ghugal[16] where a rather extensive review on bending, buckling and free vibration of laminated composite and sandwich beams is presented. The review performed in the present work shows that mostly analytical and numerical work on free vibration of laminated composite beams include transverse shear deformation effect and sometimes also rotary inertia effect (Teoh and Huang[17], Teh and Huang[18], Chen and Yang[19], Chandrashekhara et al.[20], Abramovich[21], Nabi and Ganesan[22]). Besides the consideration of transverse shear and rotary inertia, Chandrashekhara and Bangera[23] also included the effects of Poisson's ratio in the determination of natural frequencies of laminated composite cantilever beams via a finite element beam model with a concentrated mass at the free end.

Research has demonstrated that fiber orientation has effects on the free vibration characteristics of laminated composite beams (Abarcar and Cunniff[24], Teoh and Huang[17], Teh and Huang[18], Chandrashekhara et al.[20], Abramovich[21]). For instance, analytical studies of symmetrically laminated composite beams made of Graphite-Epoxy[20] reveal that the value of the fundamental frequency of a beam decays when the angle of fiber orientation increases.

The aforementioned works considered laminated composite beams according to first-order shear deformation theory. Naturally, the evolution of analysis led to the employment of more refined theories and/or more sophisticated numerical models for evaluating the free vibration characteristics of that type of structure. Khdeir and Reddy[25] developed analytical solutions for free vibration analysis of cross-ply laminated composite beams with arbitrary boundary conditions using refined theories along with the state space approach. Marur and Kant[26] proposed higher-order analytical models for free vibration analysis of thick sandwich and laminated composite beams. Kameswara et al.[27] developed an analytical method based on higher-order mixed theory. Chen et al.[28] employed a semi-analytical method called state-space-based differential quadrature for free vibration analysis of generally laminated composite beams. Krishnaswamy et al.[29] developed analytical solutions for free vibration of nonsymmetric laminated composite beams via a constrained variational statement where the constraints are imposed by Lagrange multipliers. Chandrashekhara and Bangera[30] developed a finite element model based on a high- order shear deformation theory, including Poisson's effect. Ramtekkar et al.[31] employed plane stress mixed finite element models where transverse stresses are included as degrees-of-freedom using displacement-stress elastic relations. Subramanian[32] employed two higher-order shear deformation theory and associated finite elements for free vibration analysis of laminated composite beams. Both theories assume quintic and quartic variation of in-plane and transverse displacements in the thickness direction, respectively, and assume zero strain/stress conditions at the top and bottom surfaces. Aydogdu[33] employed the Ritz method for the free vibration analysis of angle-ply laminated beams subjected to different sets of boundary conditions. Marur and Kant[34] studied free vibrations of angle ply laminated beams using an isoparametric finite element model which embeds a high-order theory. Carrera's Unified Formulation (CUF) allows for formulating refined beam models. A generic kinematic field, i.e, a displacement function of an arbitrary order N can give way to the formulation of high-order displacement-based beam theories from which Euler-Bernoulli and Timoshenko beam theories can be achieved as particular cases. Higher-order beam models account for transverse shear without the need for adopting shear correction factors. Further, torsion mechanics and Poisson's effect deformation coupling can be well described, and in- and out-of-plane warping of the cross-section representations do not require special functions[35]. CUF also allows for adopting arbitrary cross-section geometries and boundary conditions[36]. Giunta et al.[37] employed hierarchical models, derived using Carrera's Unified Formulation (CUF), to perform free vibration analyses of cross-ply laminated, simply-supported beams. The formulation accounts for transverse shear deformation, rotary inertia and warping. Euler-Bernoulli and Timoshenko beam theories are obtained as particular cases. Flexural, torsional, axial and shear natural frequencies are considered. Carrera et al.[38] perform free vibration analysis of laminated, sandwich and thin-walled box beams using refined one-dimensional models which are derived form CUF. In those models, displacement components are defined by using different types of expansions. It is interesting to observe the higher frequency modal shapes that are represented by the theories, mostly the complex shell-like modes of a thin-walled box structure. Tornabene et al.[39] presented a higher-order shear deformation theory for the free vibration analysis of laminated composite beams, arches and ring structures. Several orders of kinematic expansion are employed and results are compared. Qu et al.[40] derived a general high-order shear deformation theory for free vibration and transient analysis of arbitrary laminated composite beams using a modified variational principle combined with a multi-segment partition technique.

Further, it is appropriate to mention that researchers have used other numerical procedures for free vibration analysis of Timoahenko beams. For instance, Lee and Schultz[41] perform free vibration analysis of Timoshenko beams using a procedure based on the Chebyshev pseudospectral method. Xu and Wang[42] use discrete singular convolution. Ferreira and Fasshauer[43] applied radial basis functions in a pseudospectral framework to analyze Timoshenko beams and Mindlin plates. Lee and Park[44] employ the isogeometric approach, and Shang[45] employs enriched finite element methods. Lastly, more recent works are worth mentioning. Georgantzinos et al.[46] use the finite element method to evaluate the vibrational characteristics of carbon fiber-graphene-reinforced hybrid composites aiming at determining the effect of graphene inclusions in the natural frequencies. Shams et al.[47] study free vibration analysis of laminated composite beam with delamination employing Euler-Bernoulli beam model and dynamic stiffness matrix method. Pradhan at al.[48]. consider the free vibration of hybrid composite beam under thermal gradient loading and different boundary conditions.

As the work reviewed in this section, the present work discusses free vibration analysis of laminated composite beams. One of the interests here is to study how parasitic shear can affect negatively the behavior of the Timoshenko beam element in vibration analysis. Our review shows that researchers have not paid much attention to this issue, and it is very likely that it has been disregarded in general. Obviously, acceptable accuracy can only be attained after elimination of parasitic shear, and, in that sense, strain gradient notation is a convenient notation as parasitic shear terms can be precisely identified and then removed with ease. Many of the works reviewed in this section present more sophisticated formulations (for instance, Carrera's Unified Formulation), which are capable of representing different aspects of the beam mechanics such as torsion, warping, and Poisson's effects. Although the present model is a simple, in-plane model, and the aforementioned effects cannot be depicted, this is the first work that employs strain gradient notation finite elements in free vibration analysis of laminated structures, and interesting results are produced. Extension to vibration analysis of more complex beam models is the following step.

3. Timoshenko Beam Model

Laminated composites are suitably modeled by the first-order shear deformation theory because of shear stresses that develop between laminae as well as transverse shear stresses which contribute to initiate delamination. The first-order deformation theory for beams is known as Timoshenko beam theory. This section briefly describes the Timoshenko beam mathematical model and presents the formulation of the Timoshenko beam finite element employed here. (Figure 1) represents the geometry of a laminated composite beam where details of fiber directions (angle θ) and laminae

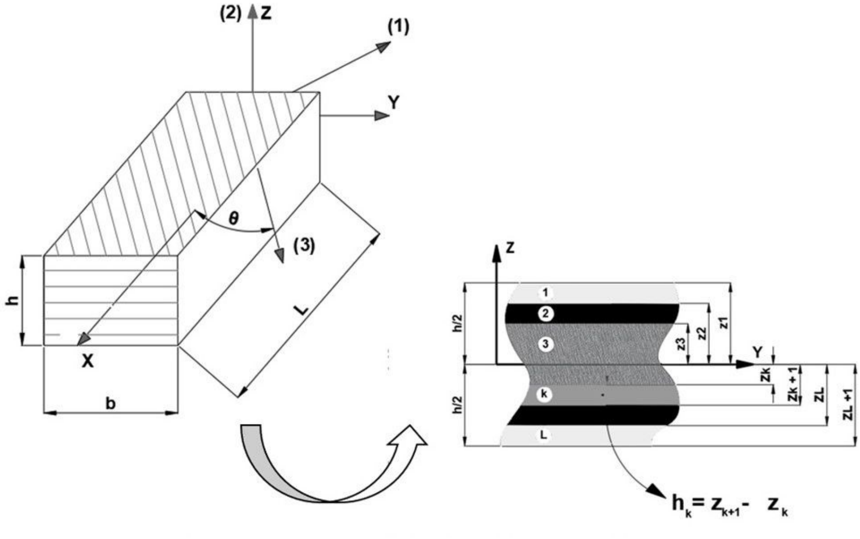

Figure 1. Geometry of a laminated composite beam.

stacking are shown. The axial and transverse displacements of a Timoshenko beam are given by:

$$u(x,z,t) = u_0(x,t) - zq(x,t) \tag{1}$$

$$w(x,t) = w_0(x,t) \tag{2}$$

where u_0, w_0, and q are the mid-surface in-plane displacement, transverse displacement, and rotation about the y-axis, respectively, x is the longitudinal axis and t is time.

The longitudinal and transverse shear strains are given by:

$$\varepsilon_{xx} = u_{0,x} - zq,x \tag{3}$$

$$\gamma_{xz} = w_{0,x} - q \tag{4}$$

where the derivatives of the mid-surface displacements and rotation with respect to x are indicated by the comma sign.

A two-node, six-degree-of-freedom beam finite element, as depicted by (Figure 2), is employed here for the free vibration analysis of laminated composite beams. Mass and stiffness matrices must be formulated.

The stiffness matrix is formulated using strain gradient notation. The formulation has been originally presented by the senior author and co-workers in[3], and it is reviewed here. The beam's displacement field in terms of arbitrary coefficients is defined according to the Timoshenko beam theory and to the element model defined in Figure 2 as follows:

$$u_0(x) = a_0 + a_1 x \tag{5}$$

$$w_0(x) = b_0 + b_1 x \tag{6}$$

$$q(x) = c_0 + c_1 x \tag{7}$$

as $u_0(x)$, $w_0(x)$ and $q(x)$ are mutually independent fields. In strain gradient notation, the arbitrary coefficients are physically determined, and the expressions above a re-written as:

$$u_0(x) = [u]_0 + [\varepsilon_x]_0 x \tag{8}$$

$$w_0(x) = [w]_0 + \left[\frac{\gamma_{xz}}{2} - q\right]_0 x \tag{9}$$

$$q(x) = \left[-\frac{\gamma_{xz}}{2} - q\right]_0 + [\varepsilon_{x,z}]_0 x \tag{10}$$

and, consequently, the longitudinal displacement $u(x,z)$ results in expression below:

$$u(x,z) = \left([u]_0 + [\varepsilon_x]_0 x\right) - z\left(\left[-\frac{\gamma_{xz}}{2} - q\right]_0 + [\varepsilon_{x,z}]_0 x\right) \tag{11}$$

Note that these expressions are written disregarding their dependence on time as stiffness is assumed to be time-dependent. These equations reveal the kinematic quantities that the model is capable of representing. They are the rigid body displacements $[u]_0$, $[w]_0$ and $[q]_0$, the constant normal and transverse shear strains $[\varepsilon_x]_0$, $[\gamma_{xz}]_0$, and the flexural strain $[\varepsilon_{x,z}]_0$. These strain states are called *strain gradients* here.

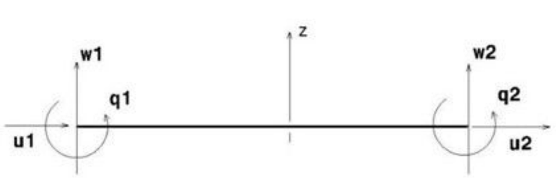

Figure 2. Two-node beam finite element.

The nodal degrees-of freedom are related to the strain gradients through the following:

$$\{d\} = [\Phi]\{\varepsilon_{sg}\} \tag{12}$$

where $[\Phi]$ is explicitly written:

$$[\Phi] = \begin{bmatrix} 1 & 0 & 0 & x_1 & 0 & 0 \\ 0 & 1 & -x_1 & 0 & x_1/2 & 0 \\ 0 & 0 & -1 & 0 & -1/2 & -x_1 \\ 1 & 0 & 0 & x_2 & 0 & 0 \\ 0 & 1 & -x_2 & 0 & x_2/2 & 0 \\ 0 & 0 & -1 & 0 & -1/2 & -x_2 \end{bmatrix} \tag{13}$$

According to Figura 2, the element's origin is defined at its mid-length. Thus, in matrix $[\Phi]$, $x_1 = -L/2$ and $x_2 = L/2$. Further, the strain field, which is defined by the derivatives of the displacements, is:

$$\varepsilon_x = [\varepsilon_x]_0 + [\varepsilon_{x,z}]_0 z \tag{14}$$

$$\gamma_{xz} = [\gamma_{xz}]_0 + [\varepsilon_{x,z}]_0 x \tag{15}$$

This is a key point in the formulation as an interesting feature is revealed. The reader should observe that both sides of Equation 14 are physically coherent. That is, normal strain is defined as a sum of terms which are associated to that normal strain. However, the same does not occur with the transverse shear strain expression (Equation 15). It is seen that the second term on the right-hand side is a normal strain gradient $[\varepsilon_{x,z}]_0$, which represents bending. That term is erroneous in the transverse shear expression in the sense that bending does not contribute to shear strain. On the contrary, that term is responsible for an undue increase in shear strain energy since there is no coupling between transverse shear and flexural strains. That is, it is responsible for the modeling error of shear locking. For this reason, such term is called parasitic shear. As a result of this precise identification of the parasitic shear term, it can be immediately eliminated *a-priori* rendering the corrected shear strain expression given below:

$$\gamma_{xz} = [\gamma_{xz}]_0 \tag{16}$$

The relation between strain and strain gradients is written via the transformation matrix $[T_{sg}]$:

$$\{\varepsilon\} = [T_{sg}]\{\varepsilon_{sg}\} \tag{17}$$

After insertion of Equation 12 and Equation 17 into the strain energy expression and proper manipulation, the *strain energy matrix* results:

$$[U_M] = \sum_{k=1}^{n} \int_{V_k} [T_{sg}]_k^T [\bar{Q}]_k [T_{sg}]_k \, dV_k \tag{18}$$

where the summation sign results from the fact that the strain energy of the laminate is the sum of strain energies of the n comprising laminae, and $[\bar{Q}]_k$ is the constitutive matrix of a typical lamina k. The strain energy matrix allows for the formation of the stiffness matrix of the finite element in the strain gradient notation as:

$$[k]=[\Phi]^{-T}[U_M][\Phi]^{-1} \tag{19}$$

Specifically for a laminated composite beam, the following stiffness definitions are present in the element's stiffness matrix:

$$A_{11}=\sum_{k=1}^{n}(\bar{Q}_{11})_k(z_k-z_{k-1}) \tag{20}$$

$$A_{55}=\frac{5}{4}\sum_{k=1}^{n}(\bar{Q}_{55})_k\left(h_k-h_{k-1}-\frac{4}{3}h_k^3-h_{k-1}^3\frac{1}{h^2}\right) \tag{21}$$

$$D_{11}=\frac{1}{3}\sum_{k=1}^{n}(\bar{Q}_{11})_k\left(z_k^3-z_{k-1}^3\right) \tag{22}$$

$$B_{11}=\frac{1}{2}\sum_{k=1}^{n}(\bar{Q}_{11})_k\left(z_k^2-z_{k-1}^2\right) \tag{23}$$

In order to study the effects of parasitic shear in the model's response, the element must be implemented in both versions with and without parasitic shear. When parasitic shear is retained, it appears in the stiffness matrix as $A_{55}\left(\frac{bL^3}{12}\right)$. The stiffness matrix can be obtained explicitly after performing closed-form integrations of its terms. The result is not shown here, and the reader is referred to[3]. The mass matrix of the element is defined as below:

$$[m]=\int_V [N]^T[\rho][N]\,dV \tag{24}$$

where $[N]$ contains the Timoshenko beam element shape functions and $[\ ]$ contains the mass densities of the laminated composite.

4. Natural Frequencies and Vibration Modes

The free vibration characteristics of a structure consist of its natural frequencies of vibration and associated mode shapes. Such set of data informs the analyst how the structure will vibrate when subjected to a given dynamic loading. Natural frequencies and vibration mode shapes are determined via the solution of the free vibration equation of motion cast in the eigenvalue form shown below:

$$\left([K]-\omega^2[M]\right)\{D\}=\{0\} \tag{25}$$

where $[K]$ and $[M]$ are the stiffness and mass matrices of the structural model, ω is natural frequency, and $\{D\}$ is vibration mode shape. If the model discretization has n degrees-of-freedom, Equation 25 will produce n values for ω and corresponding eigenvectors $\{D\}$.

5. Numerical Analyses

This section presents the free vibration solutions of laminated composite beams with different boundary conditions using models composed of the strain gradient notation beam finite element developed earlier. The beams considered are clamped-free (CF), clamped–clamped (CC), and clamped-supported (CS). The length-to-width ratio (L/b) adopted for the beams is 15. All laminae are made of graphite-epoxy (AS/3501) material whose mechanical properties[29] are given in Table 1:

The laminates are comprised of four laminae with a symmetric stacking sequence. In some of the analyses, fibers direction angle θ changes from 0 to 90 degrees in intervals of 15 degrees. The objective of these numerical analyses is twofold; namely, (i) assess the accuracy and convergence characteristics of the proposed Timoshenko beam element in computing natural frequencies, and (ii) assess the deleterious effects of parasitic shear in free vibration analysis of Timoshenko beam models. Further, it is observed how natural frequencies vary as the fibers direction angle increases. Each of the beam problems cited above are modeled by uniform meshes of 1, 2, 4, 8, 16, 32, 64 and 128 elements. For each mesh, two models are employed; namely, containing parasitic shear, and corrected for parasitic shear. Comparison of the two models allows for assessing how the spurious term affects the accuracy of the computed natural frequencies and delays their convergence. The model corrected for parasitic shear allows for assessing the accuracy and convergence characteristics of the proposed Timoshenko beam element in free vibration analysis. Non-dimensional natural frequencies, which are defined by $\bar{\omega}=\omega L^2\sqrt{\rho/E_1h^2}$, are plotted against the different fiber orientations. After investigating natural frequencies, a brief analysis of the quality of mode shapes is also performed.

5.1. Clamped-Free (CF) beam problem

The first problem analyzed is the clamped-free beam, which is represented in Figure 3.

The first three natural frequencies which are associated to transverse displacement mode shapes are computed for different values of fibers direction angles as explained before. Results are depicted in Figures 4, 5 and 6. In Figure 4, which depicts the first frequency results, also contains the plots of the analytic solution for reference purpose. It is first observed that parasitic shear retards convergence, which is very notorious for the first frequency results. Figure 4a shows that a great deal of refinement is necessary to attain convergence when

Table 1. Material properties of graphite-epoxy (AS/3501).

	E_1	E_2	G_{12}	G_{13}	G_{23}	ρ	υ
Material	[GPa]	[GPa]	[GPa]	[GPa]	[GPa]	[kg/m^3]	
AS4/3501-6	144.8	9.65	4.14	4.14	3.45	1389.2	0.33

the model contains parasitic shear. However, as shown by Figure 4b, convergence occurs very early in the refinement process when parasitic shear has been eliminated a-priori. Table 2 shows first natural frequency error values with respect to the analytic solution for both the models with and without parasitic shear to reinforce the behavior depicted by Figures 4a and 4b. These results are associated to the fiber direction angle of 45°. It is seen that the deleterious effect of parasitic shear is very important for the coarser models. Refinement starts to overcome such effects effectively only at the 32-element model. On the other hand, the 8-element mesh of the model corrected *a priori* is already quite accurate.

The next two figures, Figures 5 and 6, show that parasitic shear still plays an important role in the convergence characteristics of the model when computing the second and third natural frequencies associated to transverse

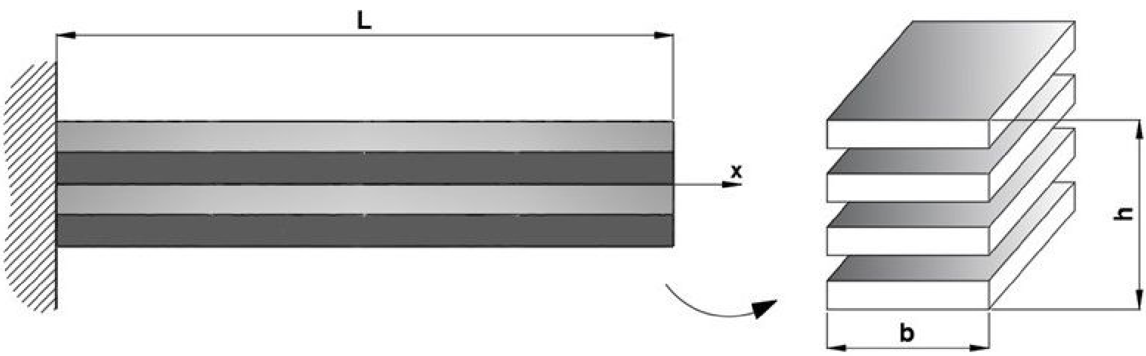

Figure 3. Clamped-Free (CF) laminated composite beam problem model.

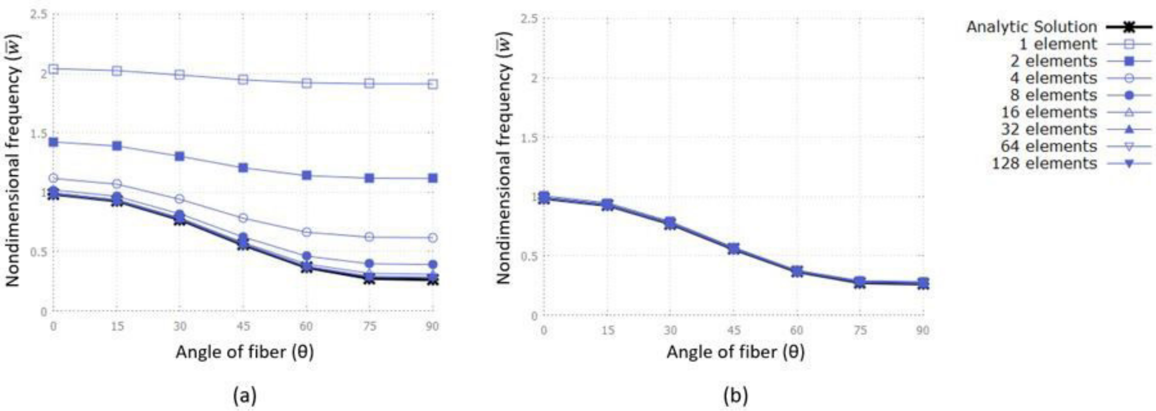

Figure 4. (**a**) First frequency for different ply fiber directions for the clamped-free (CF) beam computed using the parasitic shear finite element model; (**b**) First frequency for different ply fiber directions for the clamped-free (CF) beam computed using the corrected finite element model.

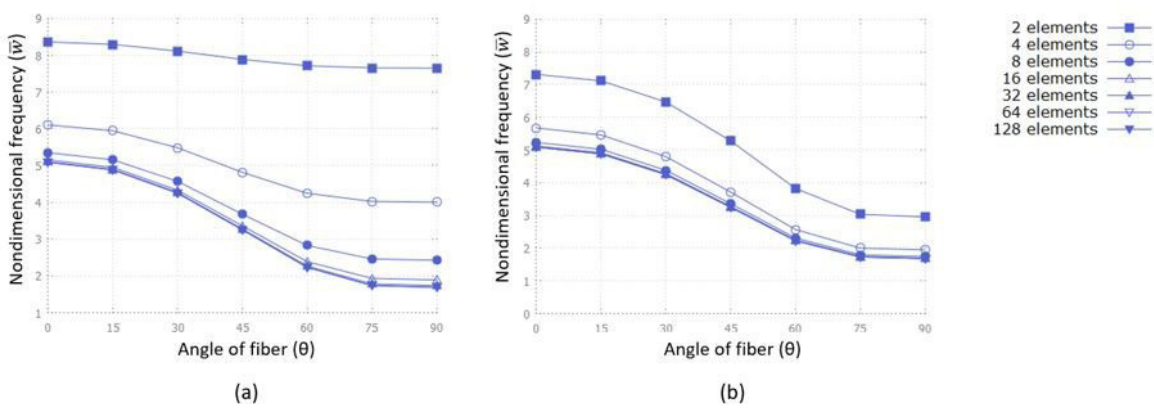

Figure 5. (**a**) Second frequency for different ply fiber directions for the clamped-free (CF) beam computed using the parasitic shear finite element model; (**b**) Second frequency for different ply fiber directions for the clamped-free (CF) beam computed using the corrected finite element model.

displacement mode shapes, although its effects are not so strong as in the first frequency values. In general, it is seen that the Timoshenko beam model converges well to natural frequency values when properly refined, but convergence is faster after removal of parasitic shear. Further, it is seen that natural frequency values decrease as fibers direction angle increases from 0 to 90 degrees. This is common behavior for all three frequencies, and, as seen for the first frequency, it matches analytic solution behavior.

Next, a clamped-free (CF) symmetric laminated composite beam with stacking sequence (0/90/90/0) is analyzed for its first eight natural frequencies. This specific beam has been chosen by these authors because results obtained by other researchers are available in the literature[20,23,26,31,34,39] and are used for comparison. Table 3 and Table 4 contain the results

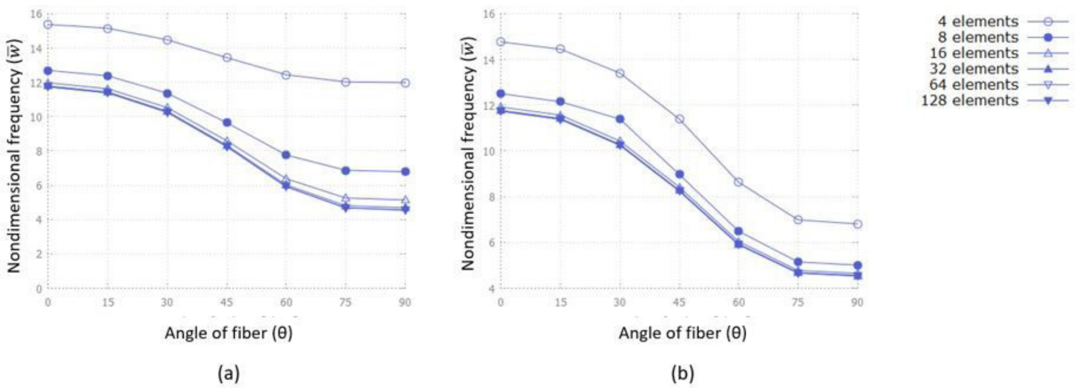

Figure 6. (**a**) Third frequency for different ply fiber directions for the clamped-free (CF) beam computed using the parasitic shear finite element model; (**b**) Third frequency for different ply fiber directions for the clamped-free (CF) beam computed using the corrected finite element model.

Table 2. Error values in the first natural frequency (CF).

	1	2	4	8	16	32	64	128
Without parasitic shear	4,7%	2,8%	1,3%	0,8%	0,65%	0,6%	0,6%	0,6%
With parasitic shear	250,5%	33,3%	18,6%	8,5%	3,2%	1,3%	0,8%	0,7%

Table 3. Natural frequencies (nondimensional values) for CF beam - Laminated composite with (0/90/90/0) lamination scheme. Model corrected for parasitic shear.

Element Nel	Present solution: mode number							
	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**
1	0,919	77,094	-	-	-	-	-	-
2	0,948	7,122	78,91	86,156	-	-	-	-
4	0,935	5,468	14,457	25,298	88,152	97,645	-	-
8	0,93	5,028	12,163	20,796	30,62	41,139	51,176	58,389
16	0,928	4,921	11,576	18,7	27,164	35,509	44,153	49,311
32	0,928	4,894	11,431	18,7	26,285	33,933	41,614	49,311
64	0,928	4,887	11,395	18,597	26,067	33,542	40,982	48,359
128	0,928	4,8865	11,391	18,585	26,042	33,498	40,909	48,250
Ref[26] ł	0,923	4,941	11,656	19,18	27,038	-	-	-
Ref[20]	0,924	4,893	11,44	18,697	26,212	-	-	-
Ref[23]	0,923	4,888	11,433	18,689	26,203	-	-	-
Ref[26]	0,924	4,985	11,832	19,573	27,72	-	-	-
Ref[31]	0,925	4,996	11,879	19,737	28,174	37,079	46,632	56,405
Ref[34]	0,921	4,888	11,433	18,689	26,203	-	-	-
Ref[39]	0,924	4,882	11,403	18,622	26,091	33,548	40,943	48,257
Note								
Ref[26] ł	Timoshenko Theory (1996)							
Ref[20]	Analytic Solution - First Order shear Deformation (1990)							
Ref[23]	Laminated Plate Theory - Mass at free End (1993)							
Ref[26]	High Order Theory (1996)							
Ref[31]	Mixed finite element modelling (2002)							
Ref[34]	High Order Theory (2007)							
Ref[39]	Generalized Differential Quadrature (2019)							

Table 4. Natural frequencies (nondimensional values) for CF beam Laminated composite with (0/90/90/0) lamination scheme. Model containing parasitic shear.

Element Nel	Present solution: mode number							
	1	2	3	4	5	6	7	8
1	2,024	88,325	-	-	-	-	-	-
2	1,389	8,296	88,429	105,490	-	-	-	-
4	1,068	5,951	15,156	25,746	88,712	102,375	-	-
8	0,965	5,161	12,385	21,046	30,855	41,324	51,298	58,436
16	0,938	4,955	11,634	19,184	27,234	35,576	44,213	53,154
32	0,931	4,902	11,446	18,717	26,303	33,951	41,630	49,327
64	0,929	4,889	11,399	18,601	26,071	33,547	40,986	48,363
128	0,929	4,887	11,390	18,579	26,026	33,469	40,861	48,177
Ref[26] ł	0,923	4,941	11,656	19,18	27,038	-	-	-
Ref[20]	0,924	4,893	11,44	18,697	26,212	-	-	-
Ref[23]	0,923	4,888	11,433	18,689	26,203	-	-	-
Ref[26]	0,924	4,985	11,832	19,573	27,72	-	-	-
Ref[31]	0,925	4,996	11,879	19,737	28,174	37,079	46,632	56,405
Ref[34]	0,921	4,888	11,433	18,689	26,203	-	-	-
Ref[39]	0,924	4,882	11,403	18,622	26,091	33,548	40,943	48,257
Note								
Ref[26] ł	Timoshenko Theory (1996)							
Ref[20]	Analytic Solution - First Order shear Deformation (1990)							
Ref[23]	Laminated Plate Theory - Mass at free End (1993)							
Ref[26]	High Order Theory (1996)							
Ref[31]	Mixed finite element modelling (2002)							
Ref[34]	High Order Theory (2007)							
Ref[39]	Generalized Differential Quadrature (2019)							

provided by strain gradient models corrected for parasitic shear and containing parasitic shear, respectively. The same uniform meshes of 1, 2, 4, 8, 16, 32, 64 and 128 elements used previously are employed here.

Comparison of the tables is done to show once again the effects of parasitic shear on the natural frequency results. The deleterious effect is greater on coarser meshes and also on the fundamental frequency value. Considering this frequency, the percent differences between the two sets of values are calculated. Results are 120%, 46.5%, 14.22%, 3.76%, 1.08%, 0.32%, 0.11%, which indicate that refinement is capable of reducing drastically the effects of parasitic shear. It is safe to state that the 64-element mesh would be necessary. However, and most importantly, if parasitic shear is eliminated *a-priori* of the computations, results within very good accuracy are already obtained with the 16-element mesh. Comparison of the strain gradient model results with other researchers' results, which are provided at the table bottoms, shows that there is good agreement in general. Only two other references[31,39] present results for all eight frequencies. It is seen that the strain gradient model results are closer to the results in reference[39], which have been obtained via the generalized differential quadrature method. For instance, the difference in the fundamental frequency is only 0.43%. It is worth observing also that results presented in reference[31], which have employed a mixed finite element formulation, appear to be poorer than the others. All frequency values are higher, and, except for the fundamental frequency, results are between the ones provided by 8- and 16-element meshes of the strain gradient model containing parasitic shear. According to the point being made in this work, those results are not sufficiently accurate.

5.2. Clamped-Clamped (CC) beam problem

The second problem analyzed is the clamped-clamped beam, which is represented in Figure 7.

Here, only the first natural frequency, which is associated to the first transverse displacement mode shape is computed. As opposed to the previous problem, results for other frequencies are not shown. Figure 8a shows values of the first natural frequency for different values of fibers direction angles computed using the model containing parasitic shear. Figure 8b shows the corresponding results provided by the corrected model. Comparison of the two figures clearly shows that parasitic shear delays convergence as results provided by coarser meshes are very far from the analytic solution results. Again, it is seen that a great deal of refinement is necessary for reaching convergence when parasitic shear is present. However, as shown by Figure 8b, convergence occurs very early in the refinement process when the model corrected for parasitic shear is employed. Table 5 shows first natural frequency error values with respect to the analytic solution for both the models with and without parasitic shear which are associated to the fiber direction angle of 45°. It is seen that errors contained in model with parasitic shear are much higher, and, again, it appears that refinement starts to attenuate the effects of parasitic shear effectively only for the 32-element mesh on.

In the following, a clamped-clamped (CC) symmetric laminated composite beam with stacking sequence (0/90/90/0)

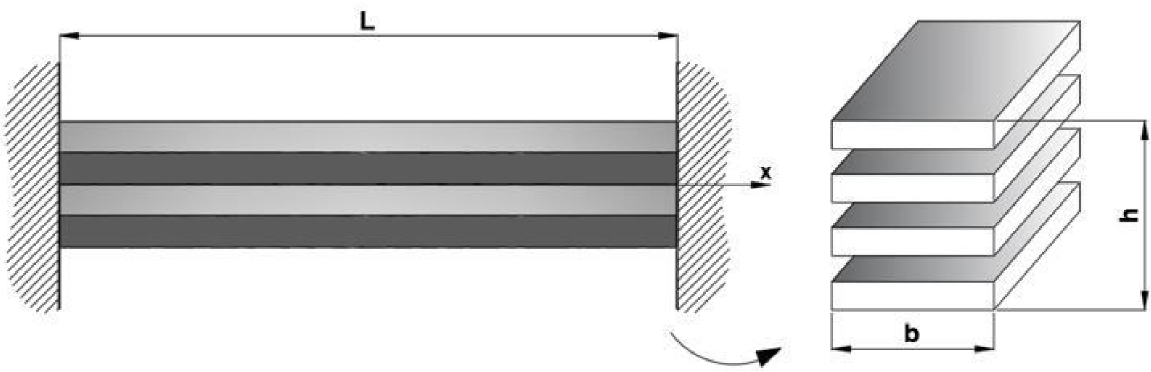

Figure 7. Clamped-Clamped (CC) laminated composite beam problem model.

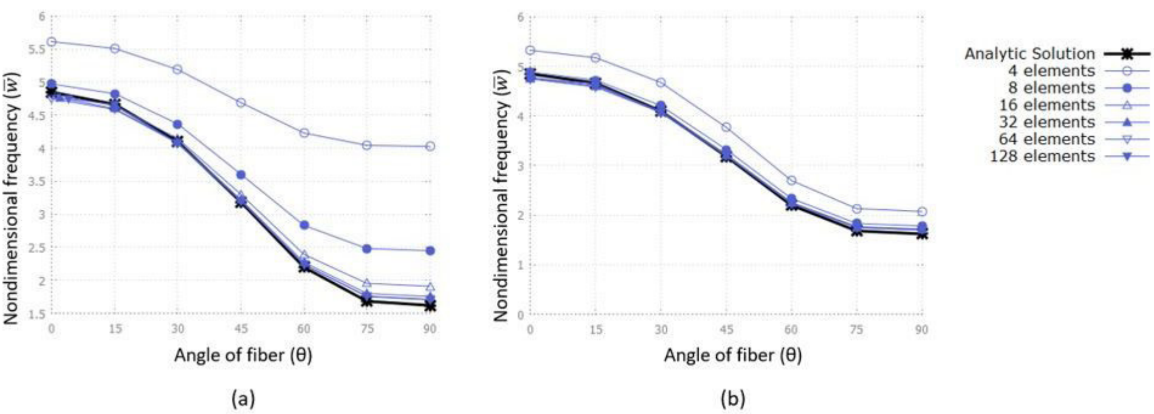

Figure 8. (**a**) First frequency for different ply fiber directions for the clamped-free (CC) beam computed using the parasitic shear finite element model; (**b**) First frequency for different ply fiber directions for the clamped-free (CC) beam computed using the corrected finite element model.

is analyzed for its first eight natural frequencies using strain gradient elements with and without parasitic shear. Results obtained by other researchers[20,31,39,49] are used for comparison. Table 6 and Table 7 contain the results provided by strain gradient models corrected for parasitic shear and containing parasitic shear, respectively. Seven uniformly refined meshes (from 2 to 128 elements) have been employed.

Again, comparison of the tables is done to show the effects of parasitic shear on the natural frequency results. A close look shows that results in Table 7 are higher than results in Table 6. However, it is seen that differences are not very significant. That is, parasitic shear does not play a very important role in this specific problem, a finding which is opposed to that in the previous problem. It is interesting to note that the fundamental frequency values for the 2-element mesh are nearly equal. The difference occurs only on the sixth decimal number, not registered on the tables for obvious reason. One of the greatest the percent differences registered is on the values of the second natural frequency of the 2-element models. Result provided by the parasitic shear model is 12,6% greater. In general, as it has been shown in this work, parasitic shear causes very significant errors. What the present problem might be showing is that the importance of parasitic shear effects might depend on the laminate being examined, and also perhaps on the boundary conditions nature.

Comparison of the strain gradient model results with reference results given at the table bottoms shows good agreement up to the fifth natural frequency. Discrepancies are found in the sixth, seventh and eighth natural frequencies. Results provided by the most refined strain gradient model are between 11% and 18% higher than values from references[39,49]. At this stage, these authors cannot assess which solutions are more accurate.

5.3. Clamped-Supported (CS) beam problem

The third problem analyzed is the clamped-supported beam, which is represented in Figure 9.

In these analyses, only the first natural frequency is computed. Figure 10a shows values of the first natural frequency for different values of fibers direction angles computed using the model containing parasitic shear, whereas Figure 10b shows the corresponding results provided by the corrected model. Once again it is observed that parasitic shear delays convergence as results provided by coarser meshes are very far from the analytic solution results. Further, it is seen that a great deal of refinement is necessary to attain convergence when parasitic shear is present. However, Figure 10b indicates that the rate of convergence of the model corrected for parasitic shear increases rapidly from the 8-element mesh on. These observations are confirmed

Table 5. Error values in the first natural frequency (CC).

	1	2	4	8	16	32	64	128
Without parasitic shear	-	62,5%	18,4%	4,3%	1,2%	0,5%	0,3%	0,3%
With parasitic shear	-	138,9%	47,3%	13,2%	3,6%	1,1%	0,5%	0,4%

Table 6. Natural frequencies (nondimensional values) for CC beam - Laminated composite with (0/90/90/0) lamination scheme. Model corrected for parasitic shear.

Element	Present solution: mode number							
Nel	1	2	3	4	5	6	7	8
1	-	-	-	-	-	-	-	-
2	7,609	83,491	-	-	-	-	-	-
4	5,179	13,138	23,796	90,95	-	-	-	-
8	4,728	10,923	18,726	27,825	38,01	48,52	57,287	-
16	4,624	10,41	17,316	24,841	32,824	41,181	49,922	59,062
32	4,598	10,285	16,974	24,11	31,486	38,978	46,557	54,219
64	4,592	10,254	16,89	23,929	31,156	38,434	45,722	53,006
128	4,591	10,248	16,873	23,894	31,092	38,328	45,561	52,772
Ref[20]	4,594	10,291	16,966	24,041	31,287	-	-	-
Ref[31]	4,725	10,754	17,907	25,596	33,613	-	-	-
Ref[49]	4,594	10,291	16,966	24,041	31,288	34,518	-	-
Ref[39]	4,581	10,251	16,891	23,925	31,126	34,51	38,355	45,568
Note								
Ref[20]	Analytic Solution - First Order shear Deformation (1990)							
Ref[31]	Mixed finite element modelling (2002)							
Ref[49]	First-Order shear Deformation (1995)							
Ref[39]	Generalized Differential Quadrature (2019)							

Table 7. Natural frequencies (nondimensional values) for CC beam - Laminated composite with (0/90/90/0) lamination scheme. Model containing parasitic shear.

Element Nel	Present solution: mode number							
	1	2	3	4	5	6	7	8
1	-	-	-	-	-	-	-	-
2	7,609	93,977		-	-	-	-	-
4	5,512	13,438	23,932	93,131	-	-	-	-
8	4,83	11,067	18,892	27,982	38,13	48,588	57,307	-
16	4,65	10,45	17,366	24,895	32,876	41,229	49,964	59,097
32	4,605	10,296	16,987	24,124	31,501	38,991	46,57	54,231
64	4,594	10,257	16,893	23,932	31,16	38,437	45,726	53,009
128	4,591	10,249	16,874	23,895	31,094	38,33	45,563	52,773
Ref[20]	4,594	10,291	16,966	24,041	31,287	-	-	-
Ref[31]	4,725	10,754	17,907	25,596	33,613	-	-	-
Ref[49]	4,594	10,291	16,966	24,041	31,288	34,518	-	-
Ref[39]	4,581	10,251	16,891	23,925	31,126	34,51	38,355	45,568
Note								
Ref[20]	Analytic Solution - First Order shear Deformation (1990)							
Ref[31]	Mixed finite element modelling (2002)							
Ref[49]	First-Order shear Deformation (1995)							
Ref[39]	Generalized Differential Quadrature (2019)							

by the first natural frequency error values with respect to the analytic solution contained in Table 8. These results refer to the 45° fiber direction angle. It is seen here also that refinement only attenuates the effects of parasitic shear effectively for the 32-element mesh on. Table 9 and Table 10 contain the results provided by strain gradient models corrected for parasitic shear and containing parasitic shear, respectively. Eight uniform refinement meshes of 1 to 128 elements are employed in the analyses. Comparison of the Tables 9 and 10 show the effects of parasitic shear on the natural frequency results of this specific laminate and boundary conditions.

Finally, a brief analysis of the quality of mode shapes is done. Only transverse displacement mode shapes of the cantilever beam (CF) are investigated. Further, only mode

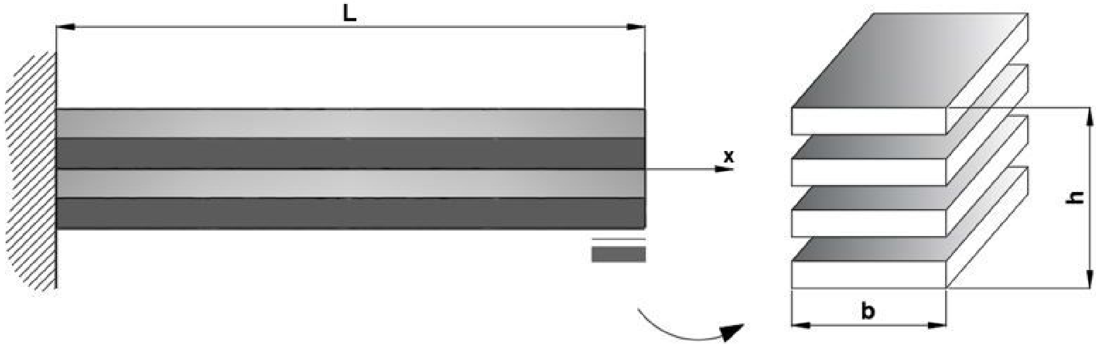

Figure 9. Clamped-Supported (CS) laminated composite beam problem model.

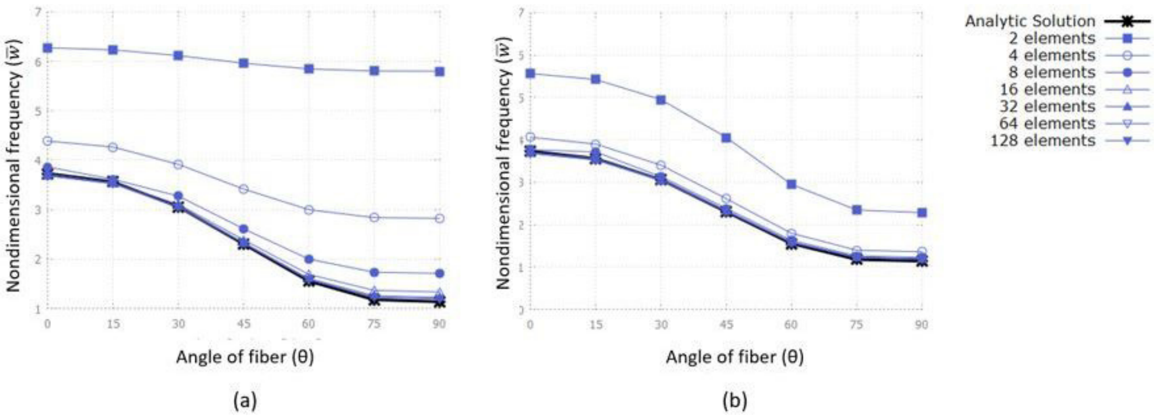

Figure 10. (**a**) First frequency for different ply fiber directions for the clamped-free (CS) beam computed using the parasitic shear finite element model; (**b**) First frequency for different ply fiber directions for the clamped-free (CS) beam computed using the corrected finite element model.

Table 8. Error values in the first natural frequency (CS).

	1	2	4	8	16	32	64	128
Without parasitic shear	-	76,2%	13,4%	3,4%	1,1%	0,6%	0,4%	0,4%
With parasitic shear	-	158,9%	48,9%	13,4%	3,7%	1,2%	0,6%	0,5%

Table 9. Natural frequencies (nondimensional values) for CS beam - Laminated composite with (0/90/90/0) lamination scheme. Model corrected for parasitic shear.

Element	Present solution: mode number							
Nel	1	2	3	4	5	6	7	8
1	77,006	-	-	-	-	-	-	-
2	5,427	78,832	-	-	-	-	-	-
4	3,904	11,997	23,319	-	-	-	-	-
8	3,617	10,008	18,108	27,479	-	-	-	-
16	3,55	9,558	16,735	24,492	32,957	-	-	-
32	3,534	9,448	16,404	23,765	31,275	38,849	46,474	54,168
64	3,53	9,421	16,322	23,586	30,945	38,304	45,639	52,952
128	3,529	9,416	16,306	23,551	30,881	38,199	45,477	52,718
Ref[20]	3,525	9,442	16,384	23,685	31,066	-	-	-
Ref[31]	3,83	9,829	16,940	23,960	26,376	-	-	-
Ref[49]	3,525	9,442	16,384	17,259	23,685	31,066	-	-
Ref[39]	3,519	9,412	16,317	23,575	30,909	38,219	45,478	51,765
Note								
Ref[20]	Analytic Solution - First Order shear Deformation (1990)							
Ref[31]	Mixed finite element modelling (2002)							
Ref[49]	First Order shear Deformation (1995)							
Ref[39]	Generalized Differential Quadrature (2019)							

shape vectors extracted from the finer mesh (128 elements) are considered. Figure 11 shows the first five transverse displacement mode shapes computed using the models corrected for parasitic shear. They correspond to the usual shapes of vibration modes for a cantilever beam found in the literature. Figure 12 shows the same mode shapes when computed using the models containing parasitic shear. The reader may observe the qualitative differences that are present when comparing the mode shapes. Apparently, there is no qualitative difference in the first mode shape. However, there are significant differences in all the other mode shapes. It is observed that there is no mode in the parasitic shear model which is similar to the correct second mode. This is as if parasitic shear has suppressed second mode. In fact, parasitic shear second mode is similar to the correct third mode. Further, in that line of thought, third to fifth parasitic shear modes are more complex than their correct counterparts. The presence of spurious terms in the model's stiffness matrix is the cause of such erroneous numerical results, which leads to the conclusion that parasitic shear might also affect free vibration analysis of laminated composites qualitatively, i.e., computing eigenvectors wrongly. This emphasizes the requirement that parasitic shear be eliminated if sound dynamic results are to be obtained. Inspection of

Table 10. Natural frequencies (nondimensional values) for CS beam - Laminated composite with (0/90/90/0) lamination scheme. Model containing parasitic shear.

Element Nel	Present solution: mode number							
	1	2	3	4	5	6	7	8
1	88,260	-	-	-	-	-	-	-
2	6,233	88,045	-	-	-	-	-	-
4	4,270	12,519	23,6	-	-	-	-	-
8	3,718	10,188	18,317	27,668	-	-	-	-
16	3,576	9,605	16,794	24,553	32,957	-	-	-
32	3,540	9,460	16,419	23,781	31,291	38,863	46,488	54,180
64	3,531	9,424	16,325	23,59	30,949	38,308	45,642	52,955
128	3,529	9,417	16,307	23,553	30,883	38,200	45,479	52,719
Ref[20]	3,525	9,442	16,384	23,685	31,066	-	-	-
Ref[31]	3,83	9,829	16,94	23,96	26,376	-	-	-
Ref[49]	3,525	9,442	16,384	17,259	23,685	31,066	-	-
Ref[39]	3,519	9,412	16,317	23,575	30,909	38,219	45,478	51,765
Note								
Ref[20]	Analytic Solution - First Order shear Deformation (1990)							
Ref[31]	Mixed finite element modelling (2002)							
Ref[49]	First Order shear Deformation (1995)							
Ref[39]	Generalized Differential Quadrature (2019)							

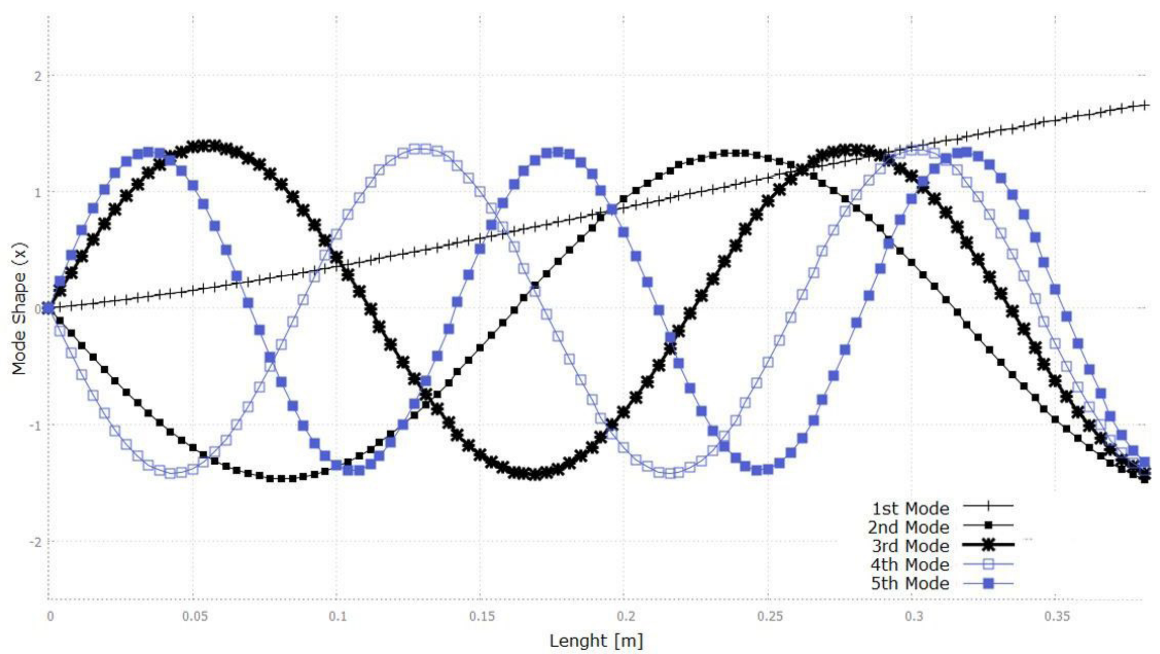

Figure 11. First five transversal mode shapes of the clamped-free (CF) beam computed using the corrected finite element model.

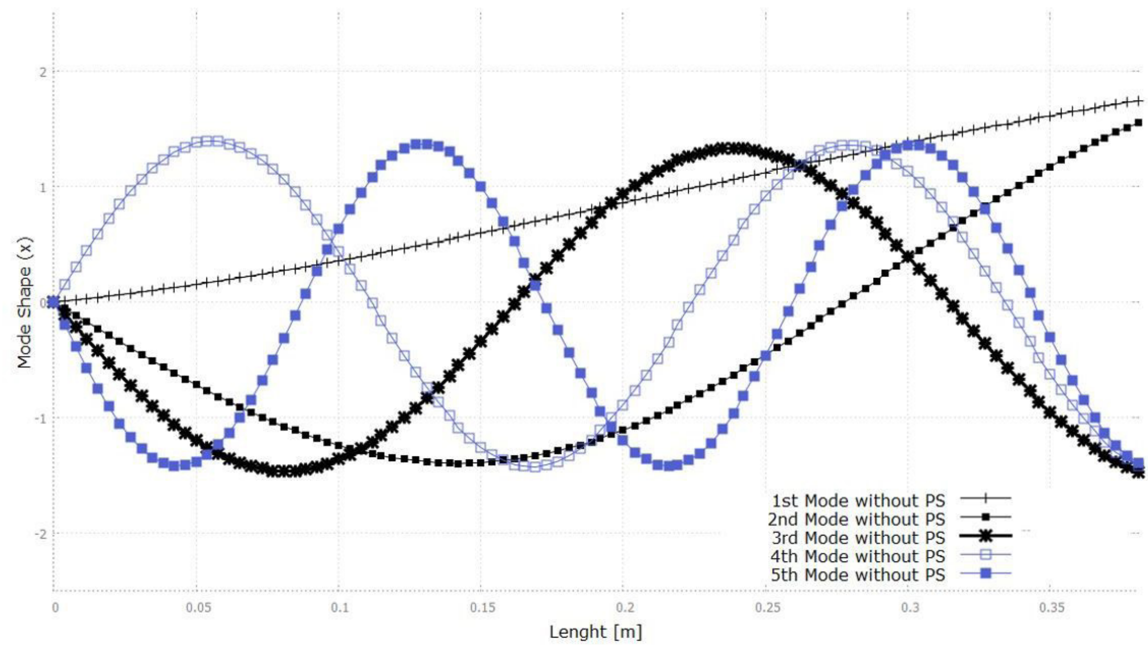

Figure 12. First five transversal mode shapes of the clamped-free (CF) beam computed using the parasitic shear finite element model.

transverse displacement mode shapes of the beam with the other boundary conditions have also been inspected, but no qualitative error has been detected.

6. Conclusions

This paper has presented an investigation on the free vibration analysis of laminated composite beams using the finite element method. Timoshenko beam models formulated via strain gradient notation have been employed for that purpose. The formulation of the model has been reviewed as well as the procedure for identification and elimination of the parasitic shear term present for completeness. This work aimed at showing that the strain gradient model is capable of computing natural frequencies and mode shapes accurately. Also, it aimed at showing how deleterious parasitic shear can be in laminated composites free vibration modeling. Different angle-ply laminated composite beams subjected to different boundary conditions, namely, clamped-free (CF), clamped-clamped (CC), and clamped-supported (CS) have been analyzed. Natural frequency results have been compared to results provided by different authors for validation purposes. It has been found that results provided by strain gradient models corrected for parasitic shear agrees with many of other researchers' results. In particular, they are very similar to results provided in reference[39] where the generalized differential quadrature method has been employed. Further, it has been demonstrated that parasitic shear slows convergence of computed natural frequencies, which requires higher degree of mesh refinement and, consequently, more computational effort. However, it was observed that parasitic shear affected more significantly the CF beam than the CC and CS beams, which indicates that boundary conditions have an influence on the parasitic shear role. Finally, it has been demonstrated here that parasitic shear also causes errors in vibration mode shapes. The nature of the error is qualitative in the sense that mode shapes might be misrepresented by the model containing parasitic shear. In general, it can be concluded that parasitic shear is a menace in free vibration analysis of laminated composites and that measures to eliminate it or at least reduce its effects must be taken. In that sense, strain gradient notation is efficient as it allows for the clear identification and accurate elimination of parasitic shear terms. Furthermore, it can be stated that strain gradient notation finite element models are competitive alternatives for free vibration analysis of laminated composite beams.

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8. References

1. Singiresu SR. The finite element method in engineering. Reino Unido: Butterworth-Heinemann; 2011.
2. Dow JAT. A unified approach to the finite element method and error analysis procedures. Cambridge: Academic Press; 1999.
3. Timoshenko S. Vibration problems in engineering. 4th ed. New York: Wolfenden Press; 1974.
4. Abdalla Fo JE, Fagundes FA, Machado RD. Identification and elimination of parasitic shear in a laminated composite beam finite element. Adv Eng Softw. 2006;37(8):522-32.
5. Dow JO, Abdalla JE. Qualitative errors in laminated composite plate models. Int J Numer Methods Eng. 1994;37:1215-30.
6. Kruszewski ET. National Advisory Committee for Aeronautics. Technical Note 1909. Effect of Transverse Shear and Rotary

Inertia on the natural frequency of a uniform beam. Washington: NACA; 1949.
7. Kapur KK. Vibrations of a timoshenko beam, using finite-element approach. J Acoust Soc Am. 1966;40(5):1058-63.
8. Thomas DL, Wilson JM, Wilson RR. Timoshenko beam finite elements. J Sound Vibrat. 1973;31(3):315-30.
9. Thomas J, Abbas BAH. Finite element model for dynamic analysis of Timoshenko beam. J Sound Vibrat. 1975;41(3):291-9.
10. Dawe DJ. A finite element for the vibration analysis of Timoshenko beams. J Sound Vibrat. 1978;60(1):11-20.
11. Tessler A, Dong SB. On a hierarchy of conforming timoshenko beam elements. Comput Struc. 1981;14(3-4):335-44.
12. Bakr EM, Shabana AA. Timoshenko beams and flexible multibody system dynamics. Top Catal. 1987;116(1):89-107.
13. Friedman Z, Kosmatka JB. An improved two-node timoshenko beam finite element. Comput Struc. 1993;47(3):473-81.
14. Corn S, Bouhaddi N, Piranda J. Transverse vibrations of short beams: finite element models obtained by a condensation method. J Sound Vibration. 1997;201(3):353-63.
15. Han SM, Benaroya H, Wei T. Dynamics of transversely vibrating beams. J Sound Vibrat. 1999;225(5):935-88.
16. Sayyad AS, Ghugal YM. On the free vibration analysis of laminated composite and sandwich plates: a review of recent literature with some numerical results. Compos Struct. 2015;129:177-201.
17. Teoh LS, Huang CC. The vibration of beams of fiber reinforced material. J Sound Vibrat. 1977;51(4):467-73.
18. Teh KK, Huang CC. The vibrations of generally orthotropic beams, a finite element approach. Sound Vibrat. 1979;62:195-206.
19. Chen AT, Yang TY. Static and dynamic formulation of a symmetrically laminated beam finite element for a microcomputer. J Compos Mater. 1985;19(5):459-75.
20. Chandrashekhara K, Krishnamurthy K, Roy S. Free vibration of composite beams including rotary inertia and shear deformation. Compos Struct. 1990;14(4):269-79.
21. Abramovich H. Shear deformation and rotary inertia effects of vibrating composite beams. Compos Struct. 1992;20(3):165-73.
22. Nabi SM, Ganesan N. A generalized element for the free vibration analysis of composite beams. Comput Struc. 1994;51(5):607-10.
23. Chandrashekhara K, Bangera KM. Vibration of symmetrically laminated clamped-free beam with a mass at the free end. J Sound Vibrat. 1993;160(1):93-101.
24. Abarcar RB, Cunniff PF. The vibration of cantilever beams of fiber reinforced material. J Compos Mater. 1972;6(3):504-17.
25. Khdeir AA, Reddy JN. Free vibration of cross-ply laminated beams with arbitrary boundary conditions. Int J Eng Sci. 1994;32(12):1971-80.
26. Marur SR, Kant T. Free vibration analysis of fiber reinforced composite beams using higher order theories and finite element modelling. J Sound Vibrat. 1996;194(3):337-51.
27. Kameswara Rao M, Desai YM, Chitnis MR. Free vibrations of laminated beams using mixed theory. Compos Struct. 2001;52(2):149-60.
28. Chen WQ, Lv CF, Bian ZG. Free vibration analysis of generally laminated beams via state-space-based differential quadrature. Compos Struct. 2004;63(3–4):417-25.
29. Krishnaswamy S, Chandrashekhara K, Wu WZB. Analytical solutions to vibration of generally layered composite beams. J Sound Vibrat. 1992;159(1):85-99.
30. Chandrashekhara K, Bangera KM. Free vibration of composite beams using a refined shear flexible beam element. Comput Struc. 1992;43(4):719-27.
31. Ramtekkar GS, Desai YM, Shah AH. Natural vibrations of laminated composite beams by using mixed finite element modelling. J Sound Vibrat. 2002;257(4):635-51.
32. Subramanian P. Dynamic analysis of laminated composite beams using higher order theories and finite elements. Compos Struct. 2006;73(3):342-53.
33. Aydogdu M. Free vibration analysis of angle-ply laminated beams with general boundary conditions. J Reinf Plast Compos. 2006;25(15):1571-83.
34. Marur SR, Kant T. On the angle ply higher order beam vibrations. Comput Mech. 2007;40(1):25-33.
35. Carrera E, Giunta G. Refined beam theories based on a unified formulation. Int J Appl Mech. 2010;2(1):117-43.
36. Carrera E, Giunta G, Nalli P, Petrolo M. Refined beam elements with arbitrary cross-sections. Computer Structure. 2010;88:283-93.
37. Giunta G, Biscani F, Belouettar S, Ferreira AJM, Carrera E. Free vibration analysis of composite beams via refined theories. Compos, Part B Eng. 2013;44(1):540-52.
38. Carrera E, Filippi M, Zapino E. Free vibration analysis of laminated beam by polinomial, trigonometric, exponential and zig-zag theories. J. Compos. Matls. 2014;48(19):2299-316.
39. Tornabene F, Fantuzzi N, Bacciocchi M. Refined shear deformation theories for laminated composite arches and beams with variable thickness: natural frequency analysis. Eng Anal Bound Elem. 2019;100:24-47.
40. Qu Y, Long X, Li H, Meng G. A variational formulation for dynamic analysis of composite laminated beams based on a general higher-order shear deformation theory. Compos Struct. 2013;102:175-92.
41. Lee J, Schultz WW. Eigenvalue analysis of Timoshenko beams and axisymmetric Mindlin plates by the pseudospectral method. J Sound Vibrat. 2004;269(3-5):609-21.
42. Xu S, Wang X. Free vibration analyses of Timoshenko beams with free edges by using the discrete singular convolution. Adv Eng Softw. 2011
43. Ferreira AJM, Fasshauer GE. Computation of natural frequencies of shear deformable beams and plates by an RBF-pseudospectral method. Comput Methods Appl Mech Eng. 2006;196:134-46.
44. Lee SJ, Park KS. Vibrations of Timoshenko beams with isogeometric approach. Appl Math Model. 2013
45. Shang HY. Enriched finite element methods for Timoshenko beam free vibration analysis. Appl Math Model. 2016;40(15-16):7012-33.
46. Georgantzinos SK, Giannopoulos GI, Markolefas SK. Vibration analysis of carbon fiber-graphene-reinforced hybrid polymer composites using finite element techniques. Materials (Basel). 2020;13(19):4225.
47. Shams S, Torabi AR, Narab MF, Amiri Atashgah MA. Free vibration analysis of a laminated beam using dynamic stiffness matrix method considering delamination. This-walled Struct. 2021;166:107952.
48. Pradhan P, Sutar MK, Pattnaik S. Free vibration analysis of hybrid composite beam under different boundary conditions and thermal gradient loading. In: Akinlabi ET, Ramkumar P, Selvaraj M , editors. Trends in mechanical and biomedical engineering, lecture notes in mechanical engineering. USA: Springer Nature Singapore Pte Ltd.; 2021, pp. 915-24.
49. Eisenberger M, Abramovich H, Shulepov O. Direct stiffness analysis of laminated beams using first order shear deformation theory. Compos Struct. 1995;31:265-71.