Revista Brasileira de Ensino de Física, vol. 40, nº 4, e4305 (2018)
www.scielo.br/rbef
DOI: http://dx.doi.org/10.1590/1806-9126-RBEF-2018-0027

The meanings of mass and $E = mc^2$: an approach based on conceptual maps

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Received on January 30, 2018; Accepted on April 14, 2018.

In this work, we discuss transformations in the meaning of the concept of mass, which are important to physics teaching, by means of three conceptual schemes, which emphasize visual patterns of knowledge organization. We begin by discussing how the meaning of a physical concept is constructed, through its relations with others. In physical theories, these connections usually involve mathematical operations. We focus on the concept of mass and its changes in three different contexts, namely classical mechanics, electromagnetism and relativity. Modifications are displayed as shifts in its relative position in conceptual schemes. These schemes enforce visual knowledge and may be used as didactical tools for teaching the subject, since they stress both the signification of mass in classical mechanics and its re-signification in relativity. This paves the way for a discussion of different approaches adopted for teaching the *mass-energy* relation in relativity, expressed by the famous formula $E = mc^2$. As the terms *rest mass*, *relativistic mass* and *invariant mass*, which have different ontological and epistemological contents, are part of this discussion, we consider the main approaches to mass present in both relativity textbooks and discourses by some influential physicists.
Keywords: mass-energy equivalence, $E = mc^2$, conceptual maps, teaching of relativity, concept of mass

1. Introduction

The *mass* is one of the most fundamental concepts in physics. In teaching, from the very beginning, mass is introduced as a basic concept, together with two other founding ones, namely *time* and *space*. In classical mechanics, mass is usually related to the amount of *matter* in a body and associated with the properties of *inertia* and *attractiveness*. This central concept was introduced in the seventeenth century, by Isaac Newton, who employed it in the systematization of the dynamical laws of moving bodies and, in particular, related it to the notions of *force*, *quantity of motion* and *acceleration*. Furthermore, Newton ascribed to matter the property of attracting other portions of matter, in his formulation of the law of gravitation. Thus, from around 1670, mass became related with both *inertia* and *gravitation*.

Along the history of physics, many concepts have been introduced, to be abandoned later, such as those of *ether* and *caloric*. Also, other quantities, which were central in the past, gradually became more peripheral and, nowadays, are meaningful only in specific contexts. The *force* is such an instance: although it has not been totally discarded, at present its importance is restricted mostly to Newtonian mechanics and its applications; it is little mentioned in both special and general relativities, as well as in quantum mechanics. Even if the word 'force' occurs in nuclear and particle physics, when the various kinds of interactions are described, this is just a nostalgic remnant from the early days of the subjects and does not have the same sense as in mechanics.

The concept of mass, on the other hand, introduced in Newtonian mechanics more than 350 years ago, is still very present in the main areas of current physics, such as both special and general relativities, quantum mechanics, nuclear and particle physics and cosmology. Very recently, at the frontier of physics, the idea of mass proved to be essential to the standard model of elementary particles, with the discovery of the Higgs boson. So, the *mass* endured four centuries of physics development and remains as a fundamental idea, with enormous vitality, in different areas of science.

However, if on the one hand, the path of the concept has been continuous, on the other, its meaning did not remain stable all the time. The idea of *energy*, in particular, which became widespread after 1850, played an important role in the development of new meanings for the mass. In spite of arising about two centuries after Newton's formulation of mechanics, the energy was eventually incorporated into it, mainly in the form of kinetic and potential energies. The former is directly related with mass and the same happens with the latter, when one deals with gravitational interactions. At the beginning of the twentieth century, with the emergence of both special and general theories of relativity, the relation between

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mass and energy underwent a great transformation. Its most widely known consequence is the formula $E = mc^2$, usually called *mass-energy equivalence.* Although this formula may suggest a simple relation between mass and energy, this is not the case, since there are important technical issues and ontological subtleties underlying this expression. Their discussion is one of the aims of this work.

In order to discuss the meaning of mass and its main changes, it is convenient to depart from the picture of scientific knowledge provided by the teaching environment where, by tradition, knowledge is organized and condensed into textbooks. These instructional books are fundamental in the teaching process and, in general, consolidate the contents of syllabuses in most of the undergraduate courses. At universities, textbooks are used repeatedly over long periods and guide the teaching practice. As stressed by Kuhn [1], textbooks are considered as a reliable documents by the physics community and are key to education. He states that

> [...] the student relies mainly on textbooks until, in his third or fourth year of graduate work, he begins his own research. Many science curricula do not ask even graduate students to read in works not written specially for students ([1], p.165).

The organization of knowledge in textbooks is usually split and organized into *didactical sectors.* Important instances are classical mechanics, electromagnetism, thermodynamics, special relativity, quantum mechanics, among others. This division into sectors is due mainly to teaching requirements. In this way, textbooks establish and organize knowledge in a rather stable way and particular approaches tend to show just small variations. Each didactical sector constitutes an organization of knowledge which provides for students relatively concise, closed and self-contained structures, a ground where they can practice before entering the world that physicists live in. This division of science into sectors allows students to get acquainted with the particular and unusual modes of thinking in physics. After the formal instruction period, when students turn into teachers, researchers or scientists, their view of physics is broadened and the initial boundaries of the didactical sectors tend to fade away. The entirety of knowledge becomes more apparent and the way which an experienced physicist organizes it acquires a maturity, owing to practice. Nevertheless, the experience of dealing with didactical sectors remains alive, especially when these physicists have to teach.

In this work, we focus on the concept of mass, which appears in almost all didactical sectors encompassed by university textbooks. In order to describe the most prominent change in its meaning, we concentrate on three sectors, namely classical mechanics, electromagnetism and special relativity. In general, modern textbooks have abandoned the old practice of defining concepts and, in particular, this applies to the mass. Nowadays, the meaning of mass tends to be given tacitly, by means of its multiple relationships with other concepts, within the theoretical structures approached by textbooks. The meaning of mass is to be apprehended after one goes through the many features of the structure, which involve concepts, formulae, relations, principles and laws. So, the perception of meanings and their changes becomes clearer when one compares the presence of mass within structures in different didactical sectors. The strategy of this work is based on this idea.

Before addressing the case of the mass, it is convenient to recall the process by which meanings are ascribed to concepts. Quite generally, the meaning of a concept is not given by itself but, rather, is developed through its relationship with both actions and other concepts. For instance, if one considers a word such as *home*, a possible definition can be found in a dictionary, as '*the house or apartment that you live in, specially with your family*' (Oxford, Advanced Learner's Dictionary). In attempting to define the concept of *home*, one usually surrounds the word with other relevant ones which, in this instance, are *house*, *apartment*, *live* and *family.* There is a net of connections between *home* and them. Part of its meaning can be transmitted by means of a verbal statement, by relying on other words, which already have meanings. This process becomes more effective when complemented by direct actions involving the concept. In many cases, a verbal definition may not be enough for conveying a meaning properly, and the surrounding context around the concept has to be taken into account. Meanings can be constructed not only by words, but also by other forms of information.

This issue was illustrated by Alves [2], with the help of a simple example. This author shows a dog, represented by several of its parts, as shown in figure 1. Each part in isolation is just a spot, as that shown on the left, which has neither meaning nor role. On the other hand, when it is inserted within the structure on the right, it acquires a meaning and becomes an *ear*. In this case, the notion of ear derives its meaning from the surrounding spots. The dog is a totality and the ear acquires its meaning from its specific position in that structure. Moreover, the notion of *dog* also derives its meaning from the element called

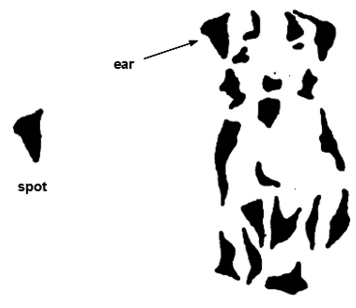

Figure 1: The part and the whole ([2], p.154)

 DOI: http://dx.doi.org/10.1590/1806-9126-RBEF-2018-0027

'ear'. Thus, this example also displays another important feature, namely the fact that the whole structure is closed and self-contained.

The case of scientific concepts is analogous, since meanings are also ascribed by relations within self-contained structures. Long ago, Bachelard already drew attention to the fact that a concept is defined by a *body of notions* and not just by a single element [3]. For instance, he argues that in Newtonian mechanics,

> the mass will be defined as the ratio of force by acceleration. Force, acceleration, mass establish themselves correlatively in a clearly rational relation, since this relation is perfectly analyzed by rational laws of arithmetic ([3], p.16, our translation).

In physics, structures tend to be more sharply defined, for the interspacing among parts is normally filled by mathematical operations. In theories, concepts are inserted into relatively stable structures, which can be displayed spatially. The meaning of concepts are, therefore, determined by a non-verbal discourse, derived from mathematical connections. Koponen and Pehkonen emphasize that structures of physics knowledge have both a *coherence* and a *conceptual hierarchy* [4]. As knowledge can be represented by a network, they stress that the *nodes* are structural elements, such as concepts, quantities, laws and fundamental principles, whereas *edges* are kinds of rules which establish connections between nodes.

The idea of theoretical nets has been, for instance, used by Salém [5], to discuss the meaning of the electric charge q. Using the relatively simple framework of electrostatics, she related it with other quantities, such as electric field ($\vec{E}$), electric force ($\vec{F}$), potential energy (U) and electric potential (V), as in figure 2. This organization supports the meaning of charge and each line represents a mathematical expression, such as $\vec{E} = \frac{q}{4\pi\epsilon_0}\frac{1}{r^2}\,\hat{r}$ for the relationship between Q and $\vec{E}$ and $V = \frac{q}{4\pi\epsilon_0}\frac{1}{r}$ for that between q and V. An important difference between this concise structure and that of the dog is the presence of the straight lines in the former, associated with well defined mathematical operations. This structure illustrates a dimension of physical theories called *extension*, whereby

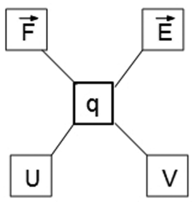

Figure 2: Part of the structure of electrostatics [5], where q is the electric charge, V is the electric potential, U is the potential energy, $\vec{E}$ is the electric field and $\vec{F}$ is the electric force.

> concepts only acquire meaning when inserted into extended frameworks, whereas these frameworks are a kind of constellation of concepts and symbols, related internally by mathematical operations ([6], p.649).

Complementarily to the extension, there is another dimension, called *depth*, more non-verbal, more intuitive, conductive to an ontology. This kind of knowledge is silent, with little verbal mediators, and is also important in physics. Its non-verbal nature is stressed by Okun [7], who says

> most people have intuitive notions of space and time. Every physicist has intuitive notions of energy, mass, and momentum. But practically everybody has difficulties in casting these notions into words without using mathematics ([7], p.3).

Therefore, the physical knowledge resembles a kind of architecture, which content cannot be expressed by words. As the seven liberal arts traditionally taught in medieval Europe[1], the physics is compatible with the *Quadrivium.* Since that time it was known that knowledge has two different natures, which according to Bernstein, the trivium explores the *word* and, the quadrivium, the *world* [9].

These features of physics knowledge are especially relevant when it comes to education, which is largely based on discourses, either oral or written, either verbal or formal, that run along time. Even in textbooks, knowledge is presented as a discourse, which runs along pages, and a linearization is necessary. In discourses, the spatiality of knowledge is partially lost, and this hampers the perception of the complementary dimensions of *extension* and *depth.* In this work, one makes an effort to recover the *space-like* structure of physics knowledge, by representing theories by means of visual conceptual schemes. This is instrumental to our aim of discussing the different meanings of mass in three of the didactical sectors where learners get their training from. These schemes resemble maps and structure physical entities in space. The importance of conceptual maps was stressed by Novak [10], who aimed at understanding the apprehension of concepts in the learning of science [11]. His research was based on the ideas of meaningful learning and cognitive structures developed by Ausubel [12]. Afterwards, other authors also used conceptual maps as didactical tools in science teaching and developed rules and hierarchies of concepts in order to organize them [13, 14]. However, the conceptual schemes employed in this work are more qualitative and akin to that for the electric charge shown in figure 2.

Our scheme for classical mechanics, presented in section 2, organizes Newtonian concepts, whereas that for

[1] In the Middle Ages, the curriculum was organized in seven subjects: the *trivium*, composed by grammar, dialectic and rhetoric, and the *quadrivium*, which encompasses arithmetic, geometry, music, and astronomy.

electromagnetism, discussed in section 3, is more general and encompasses both entities present in Maxwell's equations and the ideas of force and momentum. The scheme for special relativity, in section 4, displays the main ideas of this theory. In section 5, we discuss the changes in the meaning of mass, based on these three schemes, and recall the theoretical elements which support the emphasis of mass as a relativistic invariant. Finally, in the ensuing section, epistemological and educational implications for the teaching of $E = mc^2$ are stressed. In special, we stress the existence of two different interpretations for the mass-energy relation, namely that which takes the mass as being the velocity-dependent and the other one, which takes it to be as a frame-independent entity. The educational, structural and ontological effects of each approach are also highlighted.

2. Mass in Newtonian Mechanics

The mechanics originally proposed by Newton has been continuously revised and enlarged in the two centuries that followed, with the inclusion of several new concepts. As a consequence, what one today calls Newtonian mechanics, incorporates many modifications that did not exist originally. For example, energy and gravitational field, both included in the 19th century, were not part of Newton's thinking. Although the word *energy* was already known in ancient Greece, the modern meaning of the concept entered physics just around the 1840s, with the works of Mayer and Joule [15, 16], being endorsed by Helmhotz in the 1870s. The concept of *field* has emerged within electromagnetism and also was incorporated into Newtonian mechanics later.

The concept of *mass*, on the other hand, was introduced into Mechanics from the very beginning, already related to *inertia* and *gravitation*. The former appears into Newton's first law of dynamics, which regards the maintenance of a body's motion in the absence of forces. The latter is the property related with the idea that '*matter attracts matter*', present in Newton's law of gravitation. The concepts of *space* and *time* are also central to classical mechanics and can be considered as a kind of stage where physical phenomena take place. The classical idea of three dimensional *space* is about five centuries old and corresponds to a set of points, continuous and infinite. This space is metric, since there is the notion of distance between any two points, and its features are assumed to be always the same everywhere and not to involve privileged directions. These last two properties are known as *homogeneity* and *isotropy* of space. The classical concept of *time* corresponds to a succession of instants, unidimensional, continuous and uniform everywhere, which proceed along a single direction, going from past to future, without return. Both time and space are assumed to be absolute concepts in mechanics, independent of each other. In Newtonian mechanics, inertia is considered as a natural property of masses. This idea is closely related with the uniformity of time and homogeneity of space, since they appear entangled in Newton's first law.

Here, we adopt the usual educational notion of classical mechanics, which includes both energy and fields, and use it to infer the meaning for the concept of mass in this framework. The word *mass* or, alternatively, the symbol m, appears in textbooks surrounded by other important concepts, such as force, momentum, gravity and energy. Its meaning can be apprehended from this net of relations. The relationship between mass and other concepts in Newtonian mechanics is represented in the scheme given in figure 3. Some boxes represent concepts whereas, others, indicate either features or properties of concepts. Hence, some connective lines correspond to mathematical operations and others, just to qualitative links.

There are two main classes of ontological contents associated with mass, namely *inertia* and *attractiveness*. The relationships between mass and force and mass and momentum are particularly important for defining the features of *inertial mass* and the relevant expressions are

$$\vec{p} = m\,\vec{v} \tag{1}$$

$$\vec{F} = m\,\vec{a} \quad , \tag{2}$$

which in the scheme, are represented by blue lines. Newton called the term $m\,\vec{v}$ the *quantity of motion* of a body and, nowadays, this entity is usually referred to as *linear momentum* and represented by $\vec{p}$. Eq.(2) is Newton's second law of dynamics and corresponds to a formal relationship between mass, force and acceleration. Within

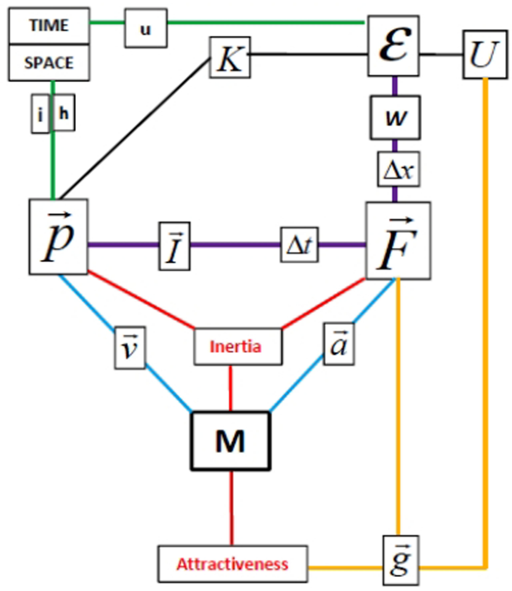

Figure 3: Conceptual Scheme of Mechanics. With u - uniformity, h - homogeneity, i - isotropy, K - kinetic energy, $\mathcal{E}$ - energy, U - potential energy, W - work, Δx - displacement, $\vec{F}$ - force, $\vec{p}$ - momentum, $\vec{I}$ - impulse, Δt - time interval, $\vec{v}$ - velocity, $\vec{a}$ - acceleration, M - mass and $\vec{g}$ - gravitational field.

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the structure of mechanics, momentum and force can be related to each other by deriving eq.(1) with respect of time. However, this does not make them ontologically equivalent, since the expression of the inertia conveyed by each of them has a different content. Some people prefer to think about the *inertia* of mass in terms of *momentum*, by means of eq.(1), whereas others may prefer to rely on the force, using the eq.(2). An instance of such a dichotomy is the competing attempts to define mass by Weyl and Mach, described by Jammer ([17], p.10). When one considers the *momentum*, the *inertia* of a body seems to be *active*. This kind of activeness manifests itself when a moving body, carrying momentum and inertia, collides with another one and imparts its movement into the target. This looks as *inertia* in *action*. On the other hand, when one thinks about eq.(2), the *force* acts over the mass and the *inertia* manifests as a reaction to this force. The force is *active*, whereas the inertia of mass seems to be more *passive* than the first one.

Departing from Newton's second law, one gets the two important theorems involving the force, both represented by purple lines. The effect of the force over a body acting during a time interval gives rise to a variation of the velocity, and is expressed by the *Impulse Theorem*, written as $\vec{F} \ \ dt = d \ \ (m \ \ \vec{v})$. It shows that the action of force over time changes the body's velocity, in a proportion inverse to its mass. For instance, for a given force, if the mass increases, the change of velocity decreases, displaying the *inertia* of the body.

The effect of a force acting along space is expressed by the *Theorem of the Living Forces*, proposed by Leibniz. In modern terms, it can be derived by a chain of manipulations of Newton's second law, as follows

$$\vec{F} \, . \, d\vec{r} \ = \ m \, \frac{d\vec{v}}{dt} \, . \, d\vec{r} \ = \ m \, d\vec{v} \, . \, \vec{v} \ = \ d \ (\frac{m \ \ v^2}{2}). \quad (3)$$

In this approach, the result of the action of the force is proportional to the velocity squared. The term $\frac{m \ v^2}{2}$ was called *kinetic energy*, afterwards. Both the impulse and the kinetic energy theorems relate mass with force. The former emphasizes that the action of the force along *time* changes the linear momentum of mass, whereas the latter shows that the action of the force along *space* changes the kinetic energy.

Thus, in our scheme, the inertia of mass is represented by means of its relationships with both momentum and force, as blue and red lines. The relationships of force $\vec{F}$, with both linear momentum $\vec{p}$ and energy $\mathcal{E}$, given by impulse and kinetic energy theorems, are indicated by purple lines.

In order to complete the scheme for classical mechanics, it is essential to mention Newton's law of gravitation, where the mass also plays a fundamental, although distinct role. This law describes the forces owing to the interaction of two masses, which attract each other with an intensity given by

$$F = G \ \ \frac{Mm}{d^{\,2}} \quad , \qquad (4)$$

where G is the gravitation constant. Newton's law of gravitation describes a property of mass which is different from inertia, since it stresses the cause of attraction. Gravitational mass indicates *where the force comes from*, whereas inertial mass describes results from the *action of forces*.

The concept of *potential energy* is also encompassed in the scheme, applied to the gravitational case. A typical instance is the potential energy of the Sun-Earth system. Considering the Sun as fixed at the center of the reference frame, the gravitational force on the Earth is negative, as shown in figure 4.

Replacing the gravitational force of eq.(4) into eq.(3), one has

$$\vec{F} \, . \, d\vec{r} \ = - \ \frac{GMm}{r^2} \ dr \ \ = \ d \, (\frac{m \ v^2}{2}) \quad . \qquad (5)$$

In order to compare both terms, it is appropriate to express the gravitational term as a derivative, and one finds

$$d \ \ (\frac{GMm}{r}) = d \ \ (\frac{m \ \ v^2}{2}) \quad \longleftrightarrow$$
$$d \ \ (- \ \frac{GMm}{r} + \frac{m \ \ v^2}{2}) = 0$$

This result is very important, because it shows that the sum of the two terms is a constant, assumed to be the energy. The gravitational potential energy highlights the property of attractiveness, whereas the kinetic energy relies on inertia. In the scheme both energies, kinetic K and potential U, are connected with the concept of energy $\mathcal{E}$.

Supplementing the scheme of mechanics, *space* and *time* were introduced due to their importance and are placed at the top left. They represent the classical preexisting stage, where physical phenomena happen, and are related to both momentum and energy, respectively, by Noether's theorem. Derived in 1918, this theorem associates energy conservation with uniformity of time (in the figure, u, between *time* and $\mathcal{E}$) and linear and angular momentum conservations with homogeneity and isotropy of space (in the figure, h and i, between *space* and $\vec{p}$). Both connections are represented by green lines.

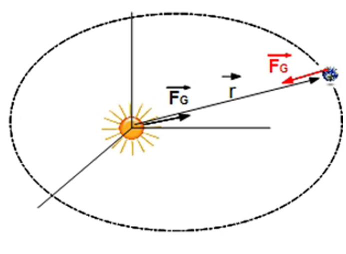

Figure 4: Gravitational force on Earth

In the scheme, the two properties of *inertia* and *attractiveness* are displayed symmetrically around the mass. The former is related directly with both the linear momentum $\vec{p}$ and the force $\vec{F}$, which are indicated by red lines. The latter is related with potential energy U, gravitation field $\vec{g}$ and the force $\vec{F}$, by yellow links.

3. Electromagnetism and Mass

We also represent the concepts of electromagnetism in a scheme, even though *mass* is not explicitly present in this theory, since the force is and this supports an association with both momentum and inertia. Besides, electromagnetic theory motivated the development of special relativity, as stated in the very title of Einstein's 1905 paper "*On the electrodynamics of moving bodies*". In this section, we outline the core of the theory and show, by means of a specific pedagogical example, a motivation for the deconstruction of Newtonian mechanics.

Despite of the vast phenomenology it accounts for, the electromagnetic theory involves just a few new concepts, such as electric charge (q), electric and magnetic fields ($\vec{E}$, $\vec{B}$), besides space and time ($\vec{r}$, t) and force ($\vec{F}$). In the twentieth century, fields acquired an important role in physics and, gradually, became seen as concrete entities.

Electromagnetic phenomena are based on interactions mediated by fields, which are described by Maxwell's four equations[2]. One of them, called Gauss' law for electricity, relates the electric field $\vec{E}$ to the charge q and to the constant ϵ_o, the electric permittivity of the vacuum. When a charge is at a point in space, there is a flux of the electric field through a closed surface which surrounds it and, in our scheme, shown in figure 5, this is represented as

$$\vec{E} \quad \longleftrightarrow \quad \frac{q}{\epsilon_o} \quad .$$

Another equation, called Faraday's electromagnetic induction law, associates the electric field $\vec{E}$ to the time variation of the magnetic field $\vec{B}$ and is indicated by

$$\vec{E} \quad \longleftrightarrow \quad \frac{\partial \vec{B}}{\partial t} \quad .$$

The Ampère-Maxwell law, in turn, ascribes the existence of a magnetic field both to an electric current and to the time variation of the electric field. Here, another constant intervenes, the magnetic permeability μ_o. The qualitative representation of this law reads

$$\vec{B} \quad \longleftrightarrow \quad \mu_o\, i \quad + \quad \mu_o \epsilon_o\, \frac{\partial \vec{E}}{\partial t} \quad .$$

The last equation is the Gauss' law for magnetism, which states that there is no magnetic charge and hence magnetic lines of force are always closed curves.

[2]We do not present the Maxwell's equations in their mathematical form. They can be found in many textbooks on Electromagnetism, such as Griffiths [18], Purcell & Morin [19], and others.

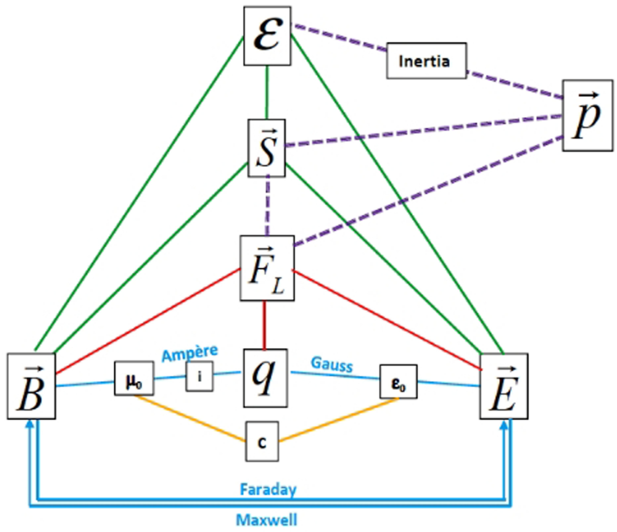

Figure 5: Conceptual scheme of electromagnetism. With q - electric charge, $\vec{E}$ - electric field, $\vec{B}$ - magnetic field, $\vec{F}_L$ - Lorentz force, $\vec{S}$ - Poynting's vector, $\mathcal{E}$ - energy, $\vec{p}$ - momentum, i - electric current, μ_o - magnetic permeability, ϵ_o - electric permittivity, c - speed of light.

When taken together, Maxwell's equations allow the derivation of wave equations, which describe light as the propagation of fields $\vec{E}$ and $\vec{B}$ in empty space. They predict the velocity c of electromagnetic waves to be

$$c \;=\; \frac{1}{\sqrt{\mu_0\ \epsilon_0}} \quad . \tag{6}$$

This result, arising from manipulations of Maxwell's equations, highlights the importance of the constants ϵ_o and μ_o. Each of them, in isolation, represents an aspect of nature but, together, determine the speed of light in vacuum, as a constant. This constancy of c can be seen as a *seed* within electromagnetism, which would germinate later on as special relativity.

The first three Maxwell's equations are represented with blue lines in the scheme and the relationship between the speed of light c and the vacuum constants ϵ_o and μ_o are given by orange lines.

Besides the fields, three other concepts are important in electromagnetic theory, namely the force $\vec{F}$, the energy $\mathcal{E}$ and the Poynting's vector $\vec{S}$. Electric and magnetic fields can create a force on a charge q, which is described by Lorentz's expression as

$$\vec{F} \;=\; q\ (\ \vec{E} \;+\; \vec{v} \times \vec{B}\ ) \quad , \tag{7}$$

where $\vec{v}$ is the velocity of charge. These two components of the force, due to the electric and magnetic fields, are shown as red links in figure 5.

The concept of energy is also fundamental in electromagnetism and is associated directly with the fields. In regions of space where fields exist, there is an electromagnetic energy $\mathcal{E}$, whose density is given by

$$\frac{d\mathcal{E}}{dV} = \frac{\epsilon_0 \vec{E}^2}{2} + \frac{\vec{B}^2}{2\,\mu_0} \quad . \tag{8}$$

This expression is very powerful, since it allows one to assess the electromagnetic energy distributed in space for any system, ranging from two point charges at rest to propagating waves, as light. The intensity of the flux of electromagnetic energy through a surface, that is the energy that crosses it per units of time and area, is described by the Poynting's vector $\vec{S}$, written as

$$\vec{S} = \frac{\vec{E} \times \vec{B}}{\mu_0} \quad . \tag{9}$$

The concepts of energy $\mathcal{E}$ and its flux $\vec{S}$, both related to the fields, are indicated by green lines in figure 5.

Our conceptual scheme for the electromagnetic theory only employs key elements, such as charge, current, magnetic and electric fields, energy, force and the speed of light. One notes that mass is absent. By themselves, Maxwell's equations are not directly relevant to the concept of mass, but some of their consequences are. In particular, there are phenomenological instances, associated with the Lorentz force, which allow inertia to be related directly with the fields. The purple dotted lines in the scheme represent important external connections of the core of the electromagnetism and, in our scheme, they work as an 'exit door' from electromagnetism towards the deconstruction of the Newtonian mass.

In order to illustrate the inertia of electromagnetic fields, we consider a situation in which a perfect absorbent body is hit by an electromagnetic wave propagating in the vacuum, normally to its flat surface. The situation described is partially similar to a fully ineslastic collision between two bodies, in which momentum is conserved. The fields $\vec{E}$ and $\vec{B}$ of the incoming wave are mutually orthogonal and the directions of $\vec{E}$, $\vec{B}$ and $\vec{c}$ in this situation are shown in figure 6. The technical details of results discussed qualitatively in the sequence are sketched in the appendix.

The absorption of the wave is due to the oscillation of electrons at the body's surface, caused by the electric force. This movement generates dissipative effects which transform the energy of the wave into heat. Here, one is interested in the fact that this wave is able to push the body, owing to the magnetic force $\vec{F} = q\,\vec{v} \times \vec{B}$, which acts along the z-axis.

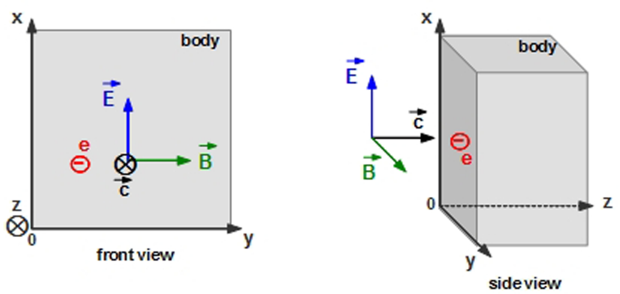

Figure 6: Electromagnetic wave hitting on a body

If one considers that magnetic force acts over all electrons moving at the surface of the body, there is a net force on the body, which pushes it. This force is related to Poynting's vector by

$$\vec{F} = \frac{1}{c}\;A\;\vec{S} \quad , \tag{10}$$

as shown in appendix. The second Newton's law relates the force with momentum through the expression $\vec{F} = d\vec{p}_{body}/dt$ and using the result (10), one establishes the relation between $\vec{p}$ and $\vec{S}$

$$d\vec{p}_{body} = \frac{1}{c}\,\vec{S}\;A\;dt \quad . \tag{11}$$

This expression is coherent with figure 6, insofar as the directions of $\vec{p}$ and $\vec{S}$ are the same. Acting along time, this force gives rise to an impulse, which corresponds to an increase of the momentum in the direction parallel to $\vec{c}$. Although the wave oscillates, the Poynting's vector has always the same direction and, if the body is initially at rest, its momentum increases continuously as time goes by. This leads one to inquire: is the momentum created continuously out of nothing or, alternatively, is it transmitted from the wave to the body? Physics chose to answer this question relying on the idea of momentum conservation. The momentum acquired by the body comes from the electromagnetic wave. Even if it has no mass! One can see in eq.(11) that momentum is related to $\vec{E}$ and $\vec{B}$. This represents an important step for the process of dissociating inertia from mass and, also, to deconstruct Newtonian mechanics. The wave has a momentum, which is transferred to the body and, in this case, it is given by the right side of eq.(11).

The instance presented establishes a tense boundary between classical mechanics and electromagnetism since, in the case of waves, the momentum $\vec{p}_{wave}$ is not related with mass. A wave has inertia, but no mass. In a empty space, waves propagate in straight lines with constant velocity, analogously to massive bodies in uniform rectilinear motion. This reinforces the idea that waves do have inertia. In order to disclose the carrier of this inertia, one recalls that, in Newtonian mechanics, $\vec{p}$ is written as

$$\vec{p}_{\,body} = m\;\vec{v} \quad , \tag{12}$$

where m is associated with the *inertia* of the body. This corresponds to the notion that

$$\vec{p}_{\,body} = \text{inertia} \times \vec{v} \quad . \tag{13}$$

In the case of electromagnetic waves, one has an analogous relation, of the form

$$\vec{p}_{\,wave} = \text{inertia} \times \vec{c} \quad . \tag{14}$$

Results discussed in the appendix indicate that the relationship between the momentum $\vec{p}$ and the energy $\mathcal{E}$

$$d\vec{p}_{\,wave} \;=\; \frac{d\mathcal{E}}{c^2}\;\vec{c} \quad . \tag{15}$$

In other words, one associates the inertia of the wave with its energy. This is a profound consequence from electromagnetism. This emphasizes the kind of conceptual differences between classical mechanics and electromagnetism. In the former, the mass is associated with *energy* $\mathcal{E}$ and *momentum* $\vec{p}$, and one uses to say that (i) *a body has kinetic energy* , given by $K = \frac{1}{2}\, m\, v^2$; (ii) *a body has linear momentum* , given by $\vec{p} = m\,\vec{v}$; (iii) *a body has inertia* . In electromagnetism, the subject underlined changes from 'body' to 'fields' and one says that (i) *fields have energy* , (ii) *fields carry linear momentum* and (iii) *fields have inertia* .

Comparing the schemes of mechanics and electromagnetism, one notes that $\vec{p}$ has a different meaning in each realm. In mechanics, inertia is related with mass, whereas in electromagnetism, it is associated with energy. In mechanics, the role of massive body is quite important to ensure the conservation of momentum and reasonings about transference of energy and momentum are made by means of massive bodies. In electromagnetism, the world-picture changes. The stuff that 'replaces' the mass in bodies is the electromagnetic field, which thereafter begins to be considered as concrete as the mass.

Although the momentum is the same entity in both contexts, it was **re-signified** by electromagnetism. In the schemes presented, this re-signification is noticeable by the change in its position. In mechanics, $\vec{p}$ is associated directly to m and, in the electromagnetism, $\vec{p}$ is related with $\vec{S}$ and $\mathcal{E}$. From a historical perspective, electromagnetism indicated some limits of Newton's laws and pointed to the need of new theoretical structures.

In a historical context, the existence of electromagnetic mass was cogitated well before relativity, as discussed by Jammer. In his words,

> The electromagnetic concept, now, proposed to deprive matter of this intrinsic nature, of its substantial mass. Although charge, to some extent at least, fulfills the function of mass, the real field of physical activity is not the bodies but, as Maxwell and Poynting have shown, the surrounding medium. The field is the seat of energy, and matter ceases to be the capricious dictator of physical events. To interpret mass as quantity of matter, or, more accurately, to regard inertial mass as the measure of quantity of matter, has now lost all meanings. For the primacy of substance has been abandoned. The electromagnetic concept of mass was not only one of the earliest field theories, in the modern sense of the word, but it also fully expressed a fundamental tenet of modern physics and of the modern philosophy of matter: matter does not do what it does because it is what it is, but it is what it is because it does what it does. ([20], p.153).

In electromagnetism, the idea of inertia moved to the fields and began to be ascribed to their energy. This motivated a reorganization of relations among concepts, which led to the development of special relativity, in which they acquire a new coherence.

4. Mass in Special Relativity

In special relativity, the meaning of mass is different from that in classical mechanics. Our conceptual scheme for relativity does not include the whole theoretical framework, which represents an important *didactical sector* and can be found in many textbooks. We focus just on the relevant concepts associated with *mass*, displaying important relationships and highlighting mathematical operations which support them. Our scheme is represented in figure 7.

Special relativity deals with the description of physical phenomena in different reference frames, with relative uniform motion. The theory relates these descriptions and relies on absolute laws, which underlie different descriptions. An important feature is the absence of privileged reference frames. Physics is assumed to be unique, even though its form can vary from frame to frame. In this way, the theory distinguishes and correlates two different realms. One of them is associated with relative, frame-dependent manifestations of physical entities, whereas the other one regards absolute quantities, which cannot be reached directly, by either measures or experiments. This kind of dichotomy was emphasized by Minkowski, in the case of space and time, when he stated

> The views of space and time which I wish to lay before you have sprung from the soil of experimental physics, and therein lies their strength. They are radical. Henceforth space by itself, and time by itself, are doomed to fade away into mere shadows, and only a kind of union of the two will preserve an independent reality ([21], p.75).

This passage, at the very opening of this famous paper *Space and Time*, is remarkable for the idea of an *independent reality.* Later on, in his text, the notion of an *absolute world* appears:

> Since the *relativity-postulate* comes to mean that only the four-dimensional world in space and time is given by phenomena, but that the projection in space and time may still be undertaken with a certain degree of freedom, I prefer to call it the *postulate of the absolute world* (or briefly, the world-postulate)([21], p.83).

This splitting of the conceptual world into two parts, associated with either invariant aspects of nature or with

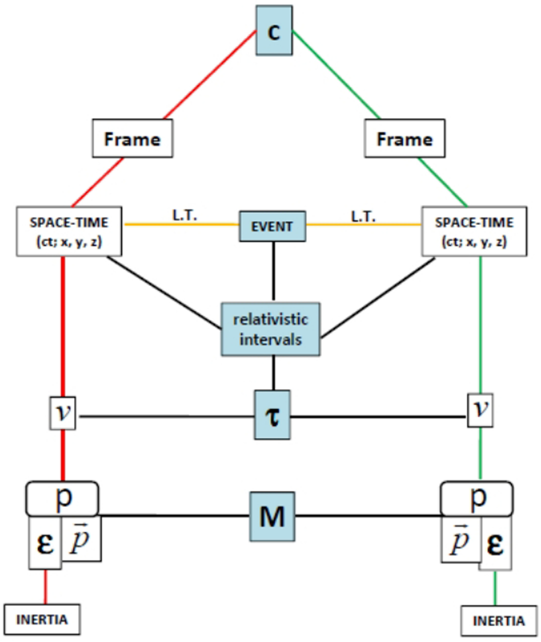

Figure 7: Conceptual scheme of Special Relativity. With c - speed of light, v - four-velocity, τ - proper time, M - mass, p - four-momentum, $\vec{p}$ - three-momentum, $\mathcal{E}$ - energy.

what can be observed, is essential to the discussion of mass and fully supported by the mathematical formalism.

The special theory is based on two principles. The first one is known as "Principle of Relativity" (1905) or "Special Principle of Relativity" (1916) and states that '*[...] the same laws of electrodynamics and optics will be valid for all frames of reference for which the equations of mechanics hold good*' ([21], p.37) or '*if a system of co-ordinates K is chosen so that, in relation to it, physical laws hold good in their simplest form, the same laws also hold good in relation to any other system of co-ordinates K' moving in uniform translation relatively to K*' (p.111). It means that all frames are equivalent and, mathematically, corresponds to the idea that physical laws must be covariant.

The second one is more specific and sets the speed of light as an absolute entity. In his 1905 paper, Einstein states that '*light is always propagated in empty space with a definite velocity c which is independent of the state of motion of the emitting body*'([21], p.38).

In order to produce a visual representation for both postulates, our scheme departs from c and is divided into two branches, which represent two different reference frames. Each branch has definite color, either red or green, so as to stress the idea that many observable concepts are frame-dependent.

Events are particularly importante in relativity. They are occurrences which can be described by four coordinates in a given frame: an instant in time and a point in tridimensional space. Minkowski introduced this concept as a *world-point*, stating that

> The objects of our perception invariably include places and times in combination. Nobody has ever noticed a place except at a time, or a time except at a place. But I still respect the dogma that both space and time have independent significance. A point of space at a point of time, that is, a system of values x, y, z, t, I will call a *world-point*. The multiplicity of all thinkable x, y, z, t systems of values we will christen the *world* ([21], p.76).

The *occurrence* of an event is something absolute, i.e., if it occurs in a given frame, it will also occur in all the other ones. However, the instant and the position which they occur at depend on the frame. The theory is based on the idea that different observers describe the **same** event in different ways, by means of different coordinates. As the occurrence of the event does not depend on the reference frame, we represent it along the central axis of our conceptual scheme. This central axis splits the scheme into two symmetric sides, and is the locus of relativistic invariants, represented as blue boxes.

If two observers, 'Mary' and 'John' for definiteness, are in different reference frames S_M and S_J, an event E is described by them as points with four coordinates:

$$\begin{aligned} &S_M:\ P_M = (\ c\,t_M;\ x_M, y_M, z_M\ ) \text{ and} \\ &S_J: P_J = (\ c\,t_J;\ x_J, y_J, z_J\ )\ , \end{aligned} \tag{16}$$

where the time coordinates were multiplied by c, so that all components have the same dimension.

If John moves relatively to Mary with a constant velocity v, along the y-axis, and they adopt a single space and time origin, the mathematical operations which relate their sets of four coordinates are the Lorentz transformations, written as $x_J = x_M$, $y_J = \gamma\,(\,y_M - v\,t_M\,)$, $z_J = z_M$, $t_J = \gamma\,(\,t_M - \frac{v}{c^2}\,y_M)$, where $\gamma = 1/\sqrt{1-\frac{v^2}{c^2}}$ is called *Lorentz factor*. These equations are represented by orange lines (L.T.) in the scheme and join the two sets of space-time coordinates, passing by *event*, which is absolute.

Lorentz transformations merge the concepts of space and time into a more general entity, called *space-time*, and the stage of physical occurrences becomes four-dimensional. The four coordinates of an event, as in eq.(16), can be considered as components of a *four-vector*, which defines its position in space-time.

Four-vectors are frame-dependent, but they can be used to construct invariant entities, which do not depend on particular frames. This construction is made by means of an operation called *scalar product*, which is analogous to that employed in spatial geometry. In Euclidean space, the scalar product of a vector $\vec{r} = (x, y, z)$ by itself, expressed as $\vec{r}.\vec{r} = x^2 + y^2 + z^2 = r^2$, determines its modulus, which is a scalar, frame-independent quantity. The definition of scalar product in relativity is similar, but a little different, since the space-time is four-dimensional and the temporal component does not have the same quality as the spatial ones [22]. This difference gives rise to a minus sign in the operation, which for $P_M = (\ c\, t_M; x_M, y_M, z_M\ )$ and $P_J = (\ c\, t_J; x_J, y_J, z_J\ )$ yields

$$\begin{aligned} e^2 \quad &= \quad P_M^2 \ = \ c^2\, t_M^2 \ - \ x_M^2 \ - \ y_M^2 \ - \ z_M^2 \quad , \\ &= \quad P_J^2 \ = \ c^2\, t_J^2 \ - \ x_J^2 \ - \ y_J^2 \ - \ z_J^2 \quad , \end{aligned} \tag{17}$$

where the equality $P_M^2 \ = \ P_J^2$ can be verified with the help of the Lorentz transformations.

All scalar products of four-vectors are relativistic invariants. The inner product of the four-vectors, given in eq.(17), generalizes the Euclidean notion of distance. The quantity e^2 is called *relativistic four-distance* between the event E and the origin of the coordinate system. This notion is quite important in relativity and, as it involves different signs for temporal and spatial coordinates, e^2 can be positive, negative or nil. These three possibilities generate three different kinds of interval, called *time-like*, *space-like* and *light-like* ([23], p.31).

The case of $e^2 > 0$ is especially relevant for us, since it gives rise to a relativistic invariant called *proper time*, represented by τ. It was introduced by Minkowski and is instrumental to understanding the role of *mass* in special relativity. According to Minkowski,

> if we imagine at a world-point $P\ (x, y, z, t)$ the world-line of a substancial point running through that point, the magnitude corresponding to the time-like vector dx, dy, dz, dt laid off along the line is therefore $d\tau = 1/c\sqrt{c^2dt^2 - dx^2 - dy^2 - dz^2}$. The integral $\int d\tau = \tau$ of this amount, taken along the world-line from any fixed starting-point P_0 to the variable end-point P, we call the proper time of the substancial point at P ([21], p.85).

Hence, in our scheme, both relativistic intervals and proper time are represented by blue boxes, placed along the central axis. Although τ bears the name *time*, it is not the same concept as the coordinate t. It is, in fact, a relativistic invariant and, therefore, does not depend on the reference frame. It lives in the abstract world of invariant entities and is the same for all observers. For a clock at rest in a frame, the measure of τ coincides with the t-coordinate in this frame. However, it is important to emphasize that when one expresses this coincidence as '$t = \tau$', one is equating a scalar quantity with the fourth component of a four-vector. This equation is therefore ontologically misleading, unless accompanied by the provisos that it holds only in the specific frame where $v = 0$ and $\gamma = 1$.

In order to discuss the mass in special relativity, one needs to introduce two other quantities, namely the *four-velocity* and the *four-momentum*. The four-velocity of a particle, in a frame S_M, was defined by Minkowski as the derivative of the coordinates of the particle with respect to τ

$$\begin{aligned} v_M = \frac{dP_M}{d\tau} = (v_{Mt} = \frac{c\ dt_M}{d\tau}\ ;\ v_{Mx} = \frac{dx_M}{d\tau}\ , \\ v_{My} = \frac{dy_M}{d\tau}\ ,\ v_{Mz} = \frac{dz_M}{d\tau})\ . \end{aligned}$$

This definition ensures that v_M is a four-vector, since it is given by the ratio of the four-vector dP_M by the Lorentz scalar $d\tau$. This feature of τ is essential and, in case it were frame-dependent, the ratio $\frac{dP_M}{d\tau}$ would not be a four-vector.

The four-momentum of a particle is constructed in a similar way. Again, one wants it to be a four-vector and hence the four-velocity has to be multiplied by a Lorentz scalar. And, for historical reasons, this scalar is called mass. In S_M, one has

$$\begin{aligned} p_M \ &= \ m\, \frac{dP_M}{d\tau} \ = \ (\ m\, \frac{c\ dt_M}{d\tau}\ ; \\ & m\, \frac{dx_M}{d\tau}\ ,\ m\, \frac{dy_M}{d\tau}\ ,\ m\, \frac{dz_M}{d\tau}\ )\ . \end{aligned} \tag{18}$$

Once more, the definition of the four-momentum by eq.(18) requires that the entity which multiplies the four-velocity, which is the mass, to be a Lorentz scalar. Expression (18) amounts to a conceptual construction of the four-momentum relying on mathematics. Once this is accomplished, one can derive back the mass from the four-momentum, by means of the scalar product. This relativistic invariant p^2 is directly related to the mass, as we show below. If the particle is at rest in the frame S_J, which moves with velocity $\vec{v} = (v_x, v_y, v_z)$ with respect to S_M, one has

$$\begin{aligned} p^2 \ = \ p_M^{\ 2} \quad &= \quad p_{tM}^{\ 2} \ - \ p_{xM}^{\ 2} \ - \ p_{yM}^{\ 2} \ - \ p_{zM}^{\ 2} \\ &= \quad \gamma^2\, m^2\ \ (\ c^2 - \vec{v}^2)\ \ . \\ = \ p_J^{\ 2} \quad &= \quad m^2\ \ c^2 \end{aligned} \tag{19}$$

This result is frame-independent, since the relative velocity $\vec{v}$ is not present in the final form. The important thing is that the inner product is proportional to the **mass**. Thus, in our scheme, the **mass** is a relativistic invariant, placed in the central axis and connected with frame-dependent four-momenta.

The construction presented so far relies on mathematics. In order to complete our scheme, we now address the concept of energy $\mathcal{E}$ and its relationship with other quantities. The *energy* is associated with the time-component p_t of the four-momentum of a particle which, in frame S_M, is given by

$$E_M \ = \ c\ \ p_{tM} \quad . \tag{20}$$

This is the reason why many textbooks write directly $p = (\ E/c\ ; \vec{p}\ )$, with $\vec{p} = (p_x, p_y, p_z)$. In our scheme for relativity, we employ three separate boxes for the four-momentum p, the energy $\mathcal{E}$ and the three-momentum $\vec{p}$.

 DOI: http://dx.doi.org/10.1590/1806-9126-RBEF-2018-0027

Using eq.(20) into eq.(19), we get the well-known result

$$E^2 = \vec{p}^2 c^2 + m^2 c^4 \quad . \tag{21}$$

In any reference frame as, for instance, S_M, the four-momentum of a particle can be written in terms of its velocity $\vec{u}_M = (u_{Mx}, u_{My}, u_{Mz})$, in that frame. Using eq.(18), we obtain

$$\frac{E_M}{c} = \gamma \; m \; c \tag{22}$$

$$\vec{p}_M = \gamma \; m \; \vec{u}_M \quad , \tag{23}$$

where m is the relativistic invariant mass, as in eq.(19).

The relativistic energy of the particle in a frame S_M is given by eq.(22), which is

$$E_M = \gamma \; m \; c^2 \quad . \tag{24}$$

This energy depends on the factor γ, which in turn, depends on the velocity of the particle. Thus, the energy of this particle is frame-dependent. The mass m, on the other hand, is an invariant, as we have already shown.

The last major issue to be considered in our scheme is the idea of inertia. The reorganization of concepts promoted by the theory of relativity sheds new light into this notion. In Newtonian mechanics, it is associated to momentum through the equation $\vec{p} = m\vec{v}$, which can be understood as the product of inertia by velocity. In relativity, the three-momentum is given by equation (23), which suggests that the inertia of a particle is represented by γm. The physical meaning of the factor γm becomes more transparent by replacing eq.(22) into (23), which yields

$$\vec{p}_M = \frac{E_M}{c^2} \; \vec{u}_M \quad . \tag{25}$$

This relativistic version of the Newtonian momentum makes it clear that, in relativity, the inertia of a body is associated with its energy and, no longer, with mass. It is no coincidence that, in the sequel of his paper on special relativity, Einstein wrote another one, called '*Does the inertia of a body depend on its energy content?*' ([21], p.69). From then on, energy replaces mass as the entity representing the inertia of a body.

This new world-picture is coherent with electromagnetic phenomena. In the case of photons, the quanta of energy in electromagnetic waves, which have $m = 0$, eq.(21) becomes

$$E^2 = \vec{p}^2 c^2$$

and eq.(25), with $v \to c$, remains unchanged: $\vec{p} = \frac{E}{c^2}\,\vec{c}$. This indicates that the inertia of photon is given by its energy, as discussed in section 3, for the case of classical waves.

These very important consequences from special relativity are represented in our scheme by placing inertia as connected to energy, in each reference frame.

The three schemes presented in this work allow one to visualize the positions occupied by the concept of mass in classical mechanics, electromagnetism and relativity. They indicate that it has different **meanings** in these didactical sectors. Albeit our focus is on the mass, other concepts related to it, such as space and time, also have different meanings when one moves from a framework to another. The same happens with the underlying mathematical structures. In classical mechanics, covariance is implemented within three-dimensional Euclidian geometry, by means of vectors with three coordinates. A novelty of relativity is the four-dimensional geometry, associated with the four-vectors needed for describing events.

Relativity also brought a new attitude towards concepts, strongly emphasized by Minkowski, regarding the existence of entities that can be directly observed and others, that cannot. The first class concerns frame-dependent concepts, such as space and time, which can be measured by instruments, such as rules and clocks, set in that frame. The other kind of entities are the frame-independent quantities, which cannot be measured directly, but nevertheless, can be accessed through mathematical operations. These absolute entities become part of an abstract world, not measurable, known just with the help of theoretical constructions.

After relativity, all concepts had to be rethought and reclassified according to this new world view. The basic concepts of *space* and *time*, which in classical mechanics are both absolute and mutually independent, in relativity, are no longer absolute and become part of a broader entity, the *space-time*. They do depend on the reference frame in which one is using. Correspondingly, the concepts of scalar and vectors had also to be modified. In Newtonian mechanics, *time* and *energy*, for instance, are scalar entities, because they do not depend on particular spatial frame adopted. In relativity, however, *time* and *energy* are no longer scalar entities and become components of four-vectors.

In relativity, a scalar entity also known as Lorentz scalar, remains invariant by changes of frames. As we have discussed, powerful mathematics determines the mass m to be a Lorentz scalar. The energy, on the other hand, is expressed as $E = \gamma\, mc^2$, where γ is Lorentz factor, which embodies the notion of a reference frame. The quantity of energy of a particle is an observable, which depends on the frame used.

5. Conceptual maps and meanings

Our three schemes display the changes in the position of mass and the reorganization of other concepts in the realms of classical mechanics, electromagnetism and relativity. In principle, the discussion made in sections 2, 3 and 4 could be developed without the help of visual schemes. However, these schemes are powerful tools for representing how physics knowledge is organized in conceptual patterns. In particular, they emphasize that, in theories, sets of physical concepts are organized in stable, *space-like* structures [6], which are just tacit in both oral and written forms of discourse. The effort of constructing space-like structures is not part of the daily practice of teachers, who deal mostly with information disposed into linear patterns along *time*, as in the case of courses, which require their ordering. Thus, in physics teaching, disclosing the meanings of concepts becomes a challenge, owing to the *time-like* features of the discourse employed. In this context, visual schemes can prove to be quite useful.

In this work, each scheme is a representation of a didactical sector determined by textbooks and displays knowledge organized into stable and time-independent forms. Conceptual maps are didactical tools which mobilize a dimension of knowledge complementary to that associated with either written or spoken speeches. Resource to vision allows the simultaneous perception of positions occupied by concepts and their mutual relationships. This is definitely instrumental for attribution of meanings. Visual schemes organize knowledge spatially and make explicit the relations between concepts which, many times, cannot be easily perceived in written texts. It transforms *time-like* discourses into *space-like* structures.

Taken together, our three schemes convey yet another message. While each individual scheme promotes the signification of mass in a given context, used together they allow the realization of a conceptual **re-signification**. Comparison among the three schemes promote epistemology. This kind of movement, from a framework to another one, accelerates the development of a skill that is normally only achieved by practitioners after a long time of immersion in physics. The existence of three schemes allows students to overfly theories and understand both epistemological and conceptual changes occurred along the construction of science. They act conveying a broader form of knowledge, which encompasses specific contents. In the present work, our three schemes are an instrument which allows a broader panorama to be seen and their use promotes a teaching with elements of the nature of science [24]. The insertion of epistemology in physics teaching develops in students an attitude towards mature reflections on scientific knowledge. The possibility of comparison embodied in our schemes acts similarly to looking at a forest from the top of a mountain, since it allows one to focus on more general aspects, which could not be seen when one is inside it.

A single scheme allows one to see the extension of space-like knowledge, whereas three schemes allow one to see the 'extension of extended structures' and, especially, to move quickly from a didactical sector to another one, realizing that discontinuities do exist among them. This kind of process turns the extended structures representing each sector into substructures of a more comprehensive organization pattern, which conducts to the *re-signification* of concepts.

In the case of the concept of mass, our schemes illustrate important changes. In Newtonian mechanics, the mass is connected directly with force and momentum, carrying the properties of inertia and attractiveness. In electromagnetism, the mass does not appear explicitly, but inertia does, associated with momentum and energy. Finally, our scheme for relativity shows that inertia is a property of energy, whereas mass is an entity which does not depend on the reference frame. We emphasized that mass is a Lorentz scalar and, being an absolute entity, the access to it is not made through measurements, but rather, by means of a mathematical construction, the inner product of the four-momentum. This feature of mass is indicated by its presence in the central axis of the scheme. The association of inertia directly to energy is compatible with the idea, which originated in electromagnetism, that massless entities, such as the light, also carry this property.

Conceptual changes, when one moves from classical mechanics to relativity, can be clearly identified using the features pointed out by Koponen and Pehkonen [4], which are present in the schemes, namely new links between the boxes, changes in their hierarchies and changes associated with the restructuring of the links.

In particular, there is a conceptual tension between mass and energy. By the end of the nineteenth century, energy was introduced into Newtonian mechanics and linked to mass by means of the expressions for kinetic and gravitational potential energies. In this context, as bodies with mass carry both energy and inertia, it was plausible that inertia is related with mass. In the same period of the nineteenth century, electromagnetic theory indicated that electric and magnetic fields also carry both energy and inertia. Phenomena involving light pushed the restructuring of physics. Relativity emerged in the midst of discussions of the so-called *electromagnetic mass*, well described by Jammer [20], and introduced a new relationship between mass and energy.

In figure 8, we sketch the paths followed by energy and mass from classical mechanics to relativity. The former corresponds to the top of the figure where, the properties of *inertia* and *attractiveness* are connected with *mass*. In that period, mass and energy were not identical but had, nevertheless, the same mathematical status as Euclidian scalar quantities. The *first conceptual rupture*, motivated by electromagnetism, developed into the idea that, in Special Relativity (SR), mathematics forces the splitting between mass and energy. The former is a Lorentz scalar, whereas the latter, the component of a four-vector. This led inertia to be ascribed to energy. The *second rupture* happened with General Relativity (GR), where the gravitational "attraction" is also transferred from mass to energy. As a consequence of general relativity, energy, besides having inertia, also has "weight"[3]. This could be observed by the deflection of light crossing the gravita-

[3] Here, we have used traditional language to express effects associated with the curvature of space-time in GR.

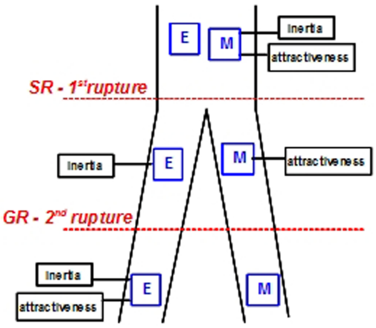

Figure 8: First and second ruptures with SR and GR

tional field of stars. Thus, the bottom of the figure shows the transference of *attractiveness* from mass to energy.

6. $E = mc^2$ and implications for teaching

The ways of approaching the theory of relativity in physics teaching have been discussed by many groups of physicists and educators [8, 25–34]. In particular, the so-called mass-energy equivalence, represented by the famous formula $E = mc^2$, opens a loophole for different modes of thought and, as a consequence, one finds divergent approaches in both textbooks and discourses by physicists. The two main trends employ different interpretations of the relationship between mass and energy, with respect to: (i) the conceptual equality between both entities and (ii) the possibility of a velocity-dependent mass. While a group advocates in favor of the idea that mass depends on the velocity of the body, the other disagrees and takes the mass as a relativistic invariant. In the sequence, these two forms of thinking are called, respectively, *mode A* and *mode B*. Each of them belongs to a specific *culture*, associated with conventions and ways of approaching knowledge. Nevertheless, from an operational point of view, both approaches are equivalent, since they coexist in the practice of paradigmatic physics. However, their ontological contents are very different and this has implications for education.

It is especially important to compare the meanings of the famous formula $E = mc^2$, which belongs to *mode A*, with the expression $E = \gamma\, mc^2$, associated with *mode B*, and presented as *relativistic energy* at the end of section 4. The differences between them can only be overcome by allowing the factor γ, in *mode B*, to be incorporated into the symbol m, in *mode A*. The symbol m then becomes polysemic. As a consequence, for the sake of precision, *mode A* requires two names for the mass, the *inertial* or *relativistic mass*, represented by m, and the *rest mass*, usually indicated by m_0. Formally, these two masses are related by

$$m = \gamma\, m_0 \quad . \tag{26}$$

Here, the ontological status of m_0 is not that of Lorentz scalar, since it seems to be just the value of m in the particular frame where $v = 0$.

In the table 1, we summarize the main differences between the two forms of the $E-m$ relation, together with kinetic energy and inertia. As mentioned before, both approaches are operationally equivalent and yield the same predictions for problems involving high velocities. In this sense, they convey the same physics. Nevertheless, they have different epistemological and ontological implications, which are quite relevant for education.

In physics teaching, the mathematical way of representing an idea affects directly the construction of meanings by students. This issue was emphasized by Feynman [35], in a lecture, when he stressed that the law of Gravitation could be expressed in three different ways, namely by means of the use of the concept of force, the concept of potential and, lastly, by using the principle of least action. Although these three approaches are mathematically equivalent, Feynman argued that each one describes nature in a different way. In his own words

> in the particular case I am talking about the theories are exactly equivalent. Mathematically each of the three different formulations, Newton's law, the local field method and the minimum principle, gives exactly the same consequences. What do we do then? You will read in all the books that we cannot decide scientifically on one or the other. That is true. They are equivalent scientifically. It is impossible to make a decision, because there is no experimental way to distinguish between them if all the consequences are the same. But **psychologically** they are very different in two ways. First, **philosophically** you like them or do not like them; and training is the only way to beat that disease. Second, **psychologically** they are different because they are completely unequivalent when you are trying to guess new laws ([35], p.53, our emphases).

As a more concrete example, in electrostatics, the electric field $\vec{E}$ can be related to the gradient of the potential V by $\vec{E} = -\vec{\nabla}V$. Therefore, from the formal point of view, we could write the Coulomb field as either

$$\vec{E} = \frac{q}{4\,\pi\,\epsilon_o}\,\frac{1}{r^2}\,\hat{r}, \quad \text{or} \quad -\vec{\nabla}V = \frac{q}{4\,\pi\,\epsilon_o}\,\frac{1}{r^2}\,\hat{r},$$
$$\text{or} \quad \vec{E} = \vec{\nabla}\frac{q}{4\,\pi\,\epsilon_o}\,\frac{1}{r} \quad .$$

Although the mathematical content of these three expressions is exactly the same, their ontological and psychological implications are not.

This same pattern is also present in expressions used for describing the *mass* in relativity. The approach based on *mode A* uses two symbols to describe mass, namely m_0 e m, each one with its own physical and mathematical meaning. The mass in *mode A* is frame-dependent whereas, in the *mode B*, it is a relativistic scalar. The use of the same word, *mass*, with two different meanings, is an

Table 1: Approaches for the mass

	mode A	mode B	frame
relativistic mass	m		depends
rest mass	m_0		ambiguous
mass		m	does not depend
energy	$E = mc^2$	$E = \gamma\ mc^2$	depends
kinetic energy	$K = (m - m_0)\ c^2$	$K = (\gamma - 1)\ mc^2$	depends
inertia	$\rightarrow\ \ m$	$\rightarrow\ \ E/c^2$	depends

undesirable source of ambiguity in teaching, specially for students not acquainted with the theory. In A-approach, this ambiguity is fostered by the formula $E = mc^2$, where m is called *inertial mass*. This immediately suggests that, in relativity, inertia is still carried by mass. In the educational context, this makes students to live together with two masses, wherein one is constant and the other increases with velocity. The coexistence of two names also propagates into the expression of the kinetic energy which, in *mode A*, is written as $K = (m - m_0)c^2$ and, as in the formula $E = mc^2$, promotes an ontological identification between mass and energy. The meaning of the factor '$m - m_0$', which involves two different frames at once, is difficult to grasp and may lead students into a conceptual mess. In *mode B*, $K = (\gamma - 1)mc^2$ and, again, the two forms are operationally equivalent.

In *mode A*, the problem is enhanced by the fact that m_0 has an ambiguous meaning, since one is left without knowing whether it does refer to a relativistic invariant or not. It is normally used just as a ground value, a numerical point of departure of measuring increases in mass. This feeling is reinforced by the name *rest mass*, which refers to a state of lack of motion and not to frame independence. This kind of ambiguous information may generate *didactical obstacles* to a proper understanding of Einstein's theory. These obstacles, according to Brousseau [36,37], do not belong to theoretical knowledge but, rather, are generated by the learning process. They can be created by unsuitable choices during teaching, which bring unwanted misleading ideas about scientific knowledge.

In many cases, the choices made by individuals between modes A and B remain private. However, when it comes to the relationship between mass and energy, there are instances in which the ontologies associated with each camp are made explicit. Representatives of group A tend to say that mass is *equal* to energy, as the philosopher of physics, Max Jammer,

> In view of the fact that in relativity there is only one conservation law of mass or energy ("massergy"), the rigorous answer to these questions [equality between the two entities, equivalence in reality an identity, "mass" and "energy" merely synonyms for the same physical reality (p.184)] undoubtedly is: mass and energy are identical, they are synonyms for the same physical substratum ([20], p.188).

This view is corroborated by the editor of *Physics World*, Peter Rodgers,

> The special theory of relativity completely changed our notions of space and time, while $E = mc^2$ led to the remarkable conclusion that mass and energy are one and the same [38].

This way of interpreting relativity concepts resound with the idea that mass increases with velocity, which is also stated by Jammer, in the following excerpt:

> "Mass", in relativity, is merely the result of certain operations, the definitions or specifications of which are intimately connected with spatio-temporal considerations. Only in virtue of these connections does the outcomes of the measuring operations depend on the velocity. With increase of velocity, mass increases ([20], p.170).

This view of a velocity-dependent mass is widespread in significant books, being adopted, among others, by Rindler [39,40], Sandin [34], Ungar [41] and Jammer [17]. On the opposite camp B, others side with the idea that mass is a relativistic invariant. Among them, we quote Wheeler and Taylor [23], which emphasize the invariance of the mass as a result coming from the inner product of four-momenta, in their influential textbook on relativity, and other physicists such as Hecht [42,43], Adler [25], Roche [44], Oas [45] and Silagadze [46,47].

The Russian physicist Lev Okun [8,30,32], also committed to *mode B*, criticizes in strong terms the ideas of *mode A*, both with respect to $E = mc^2$ and the uses of terms *rest mass* and *relativistic mass*. For instance, he states

> The notion of "relativistic mass" presents a kind of pedagogical virus which very effectively infects new generations of students and professors and shows no signs of decline. Moreover in the Year of Physics (2005) it threatens to produce a real pandemia ([31], p.470).

> To reach a consensus in the community of experts in Relativity Theory on the concept of unique relativistically invariant mass, m. Experts should discard from their writings the terms "rest mass" and "relativistic mass" and the famous but wrong formula $E = mc^2$ ([8], p.1331).

In his second book on mass, Jammer acknowledges Okun as a prominent particle physicist and emphasizes his persistence and care in stressing that mass is a relativistic invariant ([17], p.51).

The way *mode B* deals with the concepts of *mass* and the *energy*, by means of the expression $E = \gamma mc^2$, incorporates explicitly the mathematical differences between $\mathcal{E}$ and m and amounts to a didactical option which promotes a corresponding ontological distinction between these concepts. In relativity, there is a tacit hierarchy among physical concepts that cannot be neglected and it precludes one to say that mass is equal to energy. This is especially important in educational context, where one should not write formulae, such as $m = \gamma m_0$, which relates m to m_0, a concept to itself. This odd use of concepts qualified by adjectives is a didactical creation and the utilization of two terms for the *mass*, *rest* and *inertial*, is not part of the theoretical structure. If one considers such a structure to be coherent, closed and self-contained, these two masses look paradoxical and may prevent students to understand the theory. The existence of two names, which correspond to two ideas, does not promote the feeling of an internal consistency, since their strong verbal content makes it difficult for one to move safely within the theoretical structure, relying mostly on mathematics. The understanding of a concept involves the apprehension of the whole structure and the ability to move from a place to another, without wandering. In this sense, Okun emphasizes that

> the use of the proper terminology is extremely important in explaining our science to other scientists, to the taxpayers and especially to students in high school an colleges. Nonrational, confusing language prevents many students from grasping the essence of special relativity and from enjoying its beauty ([30], p.32).

In the B-approach, which is employed in our scheme of relativity, there is a unique mass and inertia is carried by the energy. In the teaching context, the use of a single word to denote *mass*, without further qualifications, contributes to a proper understanding of its meaning.

7. Concluding remarks

The concept of mass has remained alive for more than four centuries, but its meaning was not that stable and changed drastically with relativity. In this work we employ conceptual schemes to illustrate this process. Our main purpose is to provide a visual framework which could be instrumental to an introductory discussion of this important subject. Our schemes display the relationships between key concepts in three different didactical sectors, namely classical mechanics, electromagnetism and special relativity. The main motivation for highlighting the changes in the meaning of mass is to allow a more mature approach to the famous mass-energy equivalence formula $E = mc^2$. In spite of being more than a century old, the interpretation of this expression still divides the physics community. The two main trends can be accommodated when it comes to calculations. On the other hand, in physics education the mutually exclusive possibilities of mass being either dependent on the velocity or a Lorentz scalar convey different epistemological and ontological implications.

In the relativity teaching context, textbooks normally respect the main ideas of the theory and, among them, the re-signification of mass. However, most of them do not provide elements that allow one to realize clearly the connections between mass and other concepts and tend to skip comparisons with the structure of mechanics, although this would allow students to understand better the nature of modifications which take place when one moves from a theory to the other. This kind of situation seems to suggest that ontological elements are not relevant for a proper understanding of theories, that their absence does not compromise learning. Students submitted to this kind of teaching may acquire the information that, in relativity, *mass* is a relativistic invariant. However, the possession of this information does not mean that *mass* was re-signified completely. The possibility remains open that, as time goes on, the understanding of this important aspect of relativity could fade away.

The explicit contrast of ideas is very important for both determining and fixing their places in conceptual structures. The less ambiguous the perception of this place is, the more stable knowledge is. This motivates our emphasis on signifying and re-signifying physical concepts. The use of conceptual schemes in physics lessons can contribute to achieve this goal, since they are stable objects which can be put side by side. In this way, they are epistemological tools that can contribute both to learning and to the structuring of physics into stable frameworks. The comprehensive goal of re-signifying with epistemology may provide the understanding of *how*, *why*, *where* and *when*, in the process of changes of meaning. In the case of *mass*, our three schemes are instruments for promoting the visual awareness of its re-signification, which is associated with concrete changes of position of the symbol m, from mechanics to the relativity. As meanings are determined by neighboring concepts, they may clarify to students which are the main similarities and differences between Newtonian and relativistic masses.

The mass in relativity acquires an abstract meaning, quite different from its Newtonian counterpart. As discussed by Bachelard ([3], p.13), the primitive notion of mass is related to a quantity and 'materializes the very desire of eating'. Although our senses feel the mass as something concrete, the concepts of physics transcended continuously the sensorial world and, as time went by, it became more and more abstract. The theory of relativity places the mass into a rather abstract world, possibly real, but which cannot be reached directly by the senses. It can only be accessed by means of a mathematical operations: it is a Lorentz scalar and this makes it to possess an almost intangible ontology. Introducing students to this way of thinking represents a huge challenge and we hope that our schemes could contribute to making this task both more efficient and more attractive.

Supplementary material

The following online material is available for this article:
Appendix

Acknowledgements

It is my pleasure to thank Ricardo Karam for providing ideal conditions for working at the Department of Science Education of Copenhagen University, Andrea diSessa for reading an early version of this work and giving numerous suggestions for improvement and Manoel R. Robilotta for continuous discussions and a final revision of the paper. I also thank the financial support provided by CAPES (Coordenação de Apoio à Pesquisa - Brazilian Agency) (BEX 0816/15-9).

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