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Half-Factoriality in Subrings of Trigonometric Polynomial Rings

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ABSTRACT. Trigonometric polynomials are widely used in different fields of engineering and science. Inspired by their applications, we investigate half-factorial domains in trigonometric polynomial rings. We construct the half-factorial domains T_2', T_3' and T_4' which are the subrings of the ring of complex trigonometric polynomials T', such that $T_2' \subseteq T_3' \subseteq T_4' \subseteq T'$. We also discuss among these three subrings the *Condition*: Let $A \subseteq B$ be a unitary (commutative) ring extension. For each $x \in B$ there exist $x' \in U(B)$ and $x'' \in A$ such that $x = x'x''$, where $U(B)$ denote the group of units of B.

Keywords: trigonometric polynomial, HFD, condition 1, condition 2, irreducible.

Mathematical subject classification: 13A05, 13B30, 12D05, 42A05.

1 INTRODUCTION

Often it becomes interesting and fruitful to bridge different fields of study. Some time resolving one problem becomes much easier if we translate it into some equivalent problem. As we know from computer science, all NP-hard problems are equivalent. Resolution of one NP-hard problem can benefit the other, if we are able to translate the problem correctly. For example, logical solvers like SAT Solvers are used for cryptanalysis and solving algebraic system of equations over GF(2) [3, 4]. The more recent fields of study in this direction are, approximate commutative algebra [18] and numerical algebraic geometry [22]. We are hopeful that our study in this paper could be one of such contributions.

Trigonometric polynomials are widely used in different fields of engineering and science, like trigonometric interpolation applied to the interpolation of periodic functions, approximation theory, discrete Fourier transform, and real and complex analysis, etc. We are developing this study by keeping in mind, the possibility that studding factorization properties of these polynomials could help studying the above fields and especially Fourier series, that is, the study of big waves (a trigonometric polynomial) in terms of small wavelets (irreducibles). The study of Fourier series is a vast field of study by itself and this study will help to understand a big Fourier series in terms of smaller Fourier series.

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We refer to [9, 10] and reference therein, for a short review of some of the recent interesting results on nonnegative trigonometric polynomials and their applications in Fourier series, signal processing, approximation theory, function theory and number theory. Many applications, especially in mechanical engineering and in numerical analysis lead to quantifier elimination problems with trigonometric functions involved (see [14]). Decompositions of trigonometric polynomials with applications to multivariate subdivision schemes is studied in [11], random almost periodic trigonometric polynomials and applications to ergodic theory can be found in [6], a detailed treatment of trigonometric series can be found in [26], and a new proof of a theorem of Littlewood concerning flatness of unimodular trigonometric polynomials is given in [5], this proof is shorter and simpler than Littlewood's. Inspired by the above stated applications and lot more, we investigate trigonometric polynomials using an algebraic approach. Throughout this article we follow the notation and definitions introduced in [15, 25] unless mentioned otherwise.

In polynomial rings, factorization properties of integral domains have been a frequent topic of recent mathematical literature. Recall that a Unique Factorization Domain (UFD) is an integral domain in which every non-unit element can be uniquely expressed, up to isomorphism, as a product of irreducible elements. A Principal Ideal Domain (PID) is an integral domain in which every ideal is principal, i.e., can be generated by a single element. An integral domain D is atomic if each nonzero nonunit of D is a product of irreducible elements *(atoms)* of D, [7] and it is well known that UFDs, PIDs and Noetherian domains are atomic domains. In an integral domain D, if there does not exist any infinite strictly ascending chain of principal integral ideals, it is said to satisfy the *ascending chain condition on principal ideals (ACCP).* Noetherian domains, PIDs and UFDs satisfy ACCP and domains satisfying ACCP are atomic. For examples of atomic domains which do not satisfy ACCP we refer to Grams [12] and Zaks [24].

Another interesting type of domains in commutative algebra is of half-factorial domains. An integral domain D is said to be a *half-factorial domain (HFD)* if D is *atomic* and whenever $x_1 \dots x_m = y_1 \dots y_n$, where $x_1, x_2 \dots x_m$, $y_1, y_2 \dots y_n$ are irreducibles in D, then $m = n$ [23]. Since we obtain HFDs by dropping uniqueness condition on UFDs, a UFD is obviously an HFD. But the converse is not true, since any Krull domain D with $CI(D) \cong \mathbb{Z}_2$ is an HFD [23], but not a UFD. Moreover, a polynomial extension of an HFD is not an HFD anymore, for instance $\mathbb{Z}[\sqrt{-3}][X]$ is not an HFD, as $\mathbb{Z}[\sqrt{-3}]$ is an HFD but not integrally closed [8]. In general we have,

$$\textbf{UFD} \Longrightarrow \textbf{HFD} \Longrightarrow \textbf{ACCP} \Longrightarrow \textbf{Atomic}$$

Note that none of the above implications is reversible. By $U(D)$ we denote the group of units of D.

Half-factoriality of integral domains have been a frequent topic of the recent mathematical literature, particularly for polynomial rings. In this study we would investigate half-factorial domains which are the subrings of the complex trigonometric polynomial ring T' (see [15]). The basic concepts, notions and terminology are as standard in [15], [20] and [21].

For the factorization of exponential polynomials, J.F. Ritt developed: “If

$$1 + a_1 e^{\alpha_1 x} + \cdots + a_n e^{\alpha_n x} \quad \text{is divisible by} \quad 1 + b_1 e^{\beta_1 x} + \cdots + b_r e^{\beta_r x}$$

with no $b = 0$, then every β is a linear combination of $\alpha_{1,\ldots}, \alpha_n$ with rational coefficients” [17, Theorem]. Latter on getting inspired by this, G. Picavet and M. Picavet [15] investigated some factorization properties in trigonometric polynomial rings. Following [15], when we replace all α_k above by im, with $m \in \mathbb{Z}$, we obtain trigonometric polynomials. Whereas

$$T' = \left\{ \sum_{k=0}^{n} (a_k Coskx + b_k Sinkx) \ : \ n \in \mathbb{N}, \ a_k, b_k \in \mathbb{C} \right\} \quad \text{and}$$

$$T = \left\{ \sum_{k=0}^{n} (a_k Coskx + b_k Sinkx) \ : \ n \in \mathbb{N}, \ a_k, b_k \in \mathbb{R} \right\}$$

are the trigonometric polynomial rings.

Again following [15], $Sin^2 x = (1 - Cosx)(1 + Cosx)$ shows that two different non-associated irreducible factorizations of the same element may appear. Throughout we denote by $Coskx$ and $Sinkx$ the two functions $x \mapsto Coskx$ and $x \mapsto Sinkx$ (defined over $\mathbb{R}$). Also from basic trigonometric identities, it is obvious that for each $n \in \mathbb{N} \backslash \{1\}$, $Cosnx$ represents a polynomial in $Cosx$ with degree n and $Sinnx$ represents the product of $Sinx$ and a polynomial in $Cosx$ with degree $n - 1$. Conversely by linearization formulas, it follows that any product $Cos^n x Sin^p x$ can be written as:

$$\sum_{k=0}^{q} (a_k Coskx + b_k Sinkx), \text{ where } q \in \mathbb{N} \text{ and } a_k, b_k \in \mathbb{Q}.$$

Hence $T = \mathbb{R}[Cosx, Sinx] \subseteq \mathbb{C}[Cosx, Sinx] = T'$.

As proved in [15, Theorem 2.1 & Theorem 3.1], T' is a Euclidean domain and T is a Dedekind half factorial domain. We continue the investigations to find the factorization properties in trigonometric polynomial rings, begun in [15] and extended in [20] and [21], by extending this study towards finding half-factorial subrings of the complex trigonometric polynomial ring T'. In section 2, we define three subrings T_2', T_3' and T_4' of T'. We explore T_2', T_3' and T_4', and prove that each of the three new subrings, T_2', T_3', and T_4' is an HFD isomorphic to $(\mathbb{Q}+X\mathbb{R}[X])_X$, $(\mathbb{Q}+X\mathbb{C}[X])_X$ and $(\mathbb{Q}(i)+X\mathbb{C}[X])_X$ respectively. In section 3, we discussed Condition 1 (see [16, page 661]) among the three half-factorial domains. We conclude this paper by summarizing the results and discussing further research and applications.

2 THE SUBRINGS OF $\mathbb{C}[Cosx, Sinx]$

In this section we study three HFDs which are subrings of the ring of complex trigonometric polynomial ring. After knowing that an integral domain is an HFD, immediately we can deduce several useful properties.

2.1 The Construction of T_2'

Consider the following set

$$T_2' = \left\{ \sum_{k=0}^{n} (a_k Coskx + ib_k Sinkx),\ n \in \mathbb{N},\ a_k, b_k \in \mathbb{R},\ a_n = \alpha + \beta, b_n = \beta - \alpha \right\},$$

where $\alpha \in \mathbb{Q}$, $\beta \in \mathbb{R}$ and α, β are not simultaneously zero. Let $z \in T_2'$. We may write

$$z = a_0 + \sum_{k=1}^{n-1} (a_k Coskx + ib_k Sinkx) + \{(\alpha + \beta) Cosnx + i(\beta - \alpha) Sinnx\}.$$

As $Cosx = \frac{e^{ix} + e^{-ix}}{2}$ and $Sinx = \frac{e^{ix} - e^{-ix}}{2i}$, we have

$$z = e^{-inx} \left[a_0 + \sum_{k=1}^{n-1} \left\{ \left(\frac{a_k + b_k}{2} \right) e^{i(n+k)x} + \left(\frac{a_k - b_k}{2} \right) e^{i(n-k)x} \right\} + \beta e^{i2nx} + \alpha \right],$$

where $\frac{a_k + b_k}{2}, \frac{a_k - b_k}{2}, \alpha_0, \beta \in \mathbb{R}, \alpha \in \mathbb{Q}$. Since z is an arbitrary, every element of T_2' is of the form

$$e^{-inx} P(e^{ix}),\ n \in \mathbb{N},\ \text{where } P(X) \in \mathbb{Q} + X\mathbb{R}[X].$$

Conversely, for $\alpha_0 \in \mathbb{Q}$ and $\alpha_k \in \mathbb{R}$, $1 \leq k \leq 2n$, we have

$$e^{-inx} P(e^{ix}) = \alpha_0 e^{-inx} + \alpha_{2n} e^{inx} + \sum_{k=1}^{n-1} \left(\alpha_k e^{-i(n-k)x} + \alpha_{2n-k} e^{i(n-k)x} \right) + \alpha_n.$$

As $e^{ix} = Cosx + iSinx$, we have

$$\begin{aligned} e^{-inx} P(e^{ix}) &= (\alpha_0 + \alpha_{2n}) Cosnx + i(\alpha_{2n} - \alpha_0)) Sinnx \\ &\quad + \sum_{k=1}^{n-1} \{(\alpha_k + \alpha_{2n-k}) Cos(n-k)x \\ &\quad + i(\alpha_{2n-k} - \alpha_k)) Sin(n-k)x\} + \alpha_n, \end{aligned}$$

where $\alpha_n, \alpha_0 + \alpha_{2n}, \alpha_{2n} - \alpha_0, \alpha_k + \alpha_{2n-k}, \alpha_{2n-k} - \alpha_k \in \mathbb{R}$. Therefore every element which is of the form $e^{-inx} P(e^{ix})$, $n \in \mathbb{N}$, where $P(X) \in \mathbb{Q} + X\mathbb{R}[X]$, is in T_2'.

Conclusion 1. The consequence of above construction is: $T_2' = \{e^{-inx} P(e^{ix}), n \in \mathbb{N}$, where $P(X) \in \mathbb{Q} + X\mathbb{R}[X]$. So we have an isomorphism $f : (\mathbb{Q} + X\mathbb{R}[X])_X \longrightarrow T_2'$ through the substitution morphism $X \longrightarrow e^{ix}$. Therefore $T_2' \backsimeq (\mathbb{Q} + X\mathbb{R}[X])_X$.

Theorem 1. *The integral domain T_2' is an HFD having irreducible elements, up to units, trigonometric polynomials of the form $Cosx + iSinx - a$, where $a \in \mathbb{R} \backslash \{0\}$.*

Proof. Since X is a prime in $\mathbb{Q}+X\mathbb{R}[X]$ [1, Example 1.8 (b)], so $(\mathbb{Q}+X\mathbb{R}[X])_X$ is a localization of $\mathbb{Q}+X\mathbb{R}[X]$ by a multiplicative system generated by a prime. Also $\mathbb{Q}+X\mathbb{R}[X]$ is an HFD [2, Proposition 3.1]. Therefore $(\mathbb{Q}+X\mathbb{R}[X])_X$ is an HFD [1, Corollary 2.5]. Hence the isomorphism $T_0' \backsimeq (\mathbb{Q}+X\mathbb{R}[X])_X$ in Conclusion 1 gives the result. $\square$

2.2 The Construction of T_3'

We define the set T_0' of all polynomials of the form

$$\sum_{k=0}^{n}(a_k Coskx + b_k Sinkx),\ n \in \mathbb{N},\ a_k, b_k \in \mathbb{C} \quad \text{and}$$
$$a_n = \alpha + \gamma + i\beta, b_n = -\beta + i(\alpha - \gamma),$$

such that $\alpha, \beta \in \mathbb{R}$, $\gamma \in \mathbb{Q}$, where α, β and γ are not simultaneously zero.

Let $z \in T_3'$ be an arbitrary element, so we may write

$$z \ = \ a_0 + \sum_{k=1}^{n-1}(a_k Coskx + b_k Sinkx) + \left\{(\alpha+\gamma+i\beta)Cosnx + (-\beta + i(\alpha-\gamma))Sinnx\right\}.$$

As $Cosx = \frac{e^{ix}+e^{-ix}}{2}$ and $Sinx = \frac{e^{ix}-e^{-ix}}{2i}$, so

$$\begin{aligned} z \ &= \ a_0 + \sum_{k=1}^{n-1}\left\{\left(\frac{a_k' + b_k'' + i(a_k'' - b_k')}{2}\right)e^{ikx} + \left(\frac{a_k' - b_k'' + i(a_k'' + b_k')}{2}\right)e^{-ikx}\right\} \\ &\quad + (\alpha + i\beta)e^{inx} + \gamma e^{-inx}, \end{aligned}$$

where $a_k = a_k' + ia_k''$, $b_k = b_k' + ib_k''$ and $a_k', a_k'', b_k', b_k'' \in \mathbb{R}$, $a_0 \in \mathbb{C}$.

Setting

$$\alpha_k' = \frac{a_k' + b_k'' + i(a_k'' - b_k')}{2} \ \text{ and } \ \beta_k' = \frac{a_k' - b_k'' + i(a_k'' + b_k')}{2},$$

we have

$$z = e^{-inx}\left[a_0 e^{inx} + \sum_{k=1}^{n-1}\left\{\alpha_k' e^{i(n+k)x} + \beta_k' e^{i(n-k)x}\right\} + (\alpha+i\beta)e^{i2nx} + \gamma\right],$$

where $\alpha_k', \beta_k', a_0 \in \mathbb{C}$, and $\alpha, \beta \in \mathbb{R}$, $\gamma \in \mathbb{Q}$.

Since z is an arbitrary, every element of T_3' is of the form

$$e^{-inx}P(e^{ix}),\ n \in \mathbb{N},\ \text{ where } \ P(X) \in \mathbb{Q} + X\mathbb{C}[X].$$

Conversely, for $\alpha_0 \in \mathbb{Q}$ and $\alpha_k \in \mathbb{C}$, $1 \le k \le 2n$, we have

$$e^{-inx}P(e^{ix}) = \alpha_0 e^{-inx} + \alpha_{2n}e^{inx} + \sum_{k=1}^{n-1}(\alpha_k e^{-i(n-k)x} + \alpha_{2n-k}e^{i(n-k)x}) + \alpha_n.$$

As $e^{ix} = Cosx + iSinx$, so by setting

$$\alpha_k = \alpha'_k + i\alpha''_k,\ \ \alpha_{2n-k} = \alpha'_{2n-k} + i\alpha''_{2n-k} \text{ and } \alpha_{2n} = \alpha'_{2n} + i\alpha''_{2n}.$$

We have

$$\begin{aligned} e^{-inx}P(e^{ix}) &= (\alpha_0 + \alpha'_{2n} + i\alpha''_{2n})Cosnx + (-\alpha''_{2n} + i(\alpha'_{2n} - \alpha_0))Sinnx \\ &\quad + \sum_{k=1}^{n-1}\{(\alpha'_k + \alpha'_{2n-k} + i(\alpha''_k + \alpha''_{2n-k}))Cos(n-k)x \\ &\quad + (\alpha''_k - \alpha''_{2n-k} + i(\alpha'_{2n-k} - \alpha'_k))Sin(n-k)x\} + \alpha_n. \\ &= a_nCosnx + b_nSinnx + \sum_{k=1}^{n-1}\left\{a_kCos(n-k)x + b_kSin(n-k)x\right\} + \alpha_n, \end{aligned}$$

where

$$\begin{aligned} a_n &= \alpha_0 + \alpha'_{2n} + i\alpha''_{2n}, \quad b_n = -\alpha''_{2n} + i(\alpha'_{2n} - \alpha_0), \\ a_k &= \alpha'_k + \alpha'_{2n-k} + i(\alpha''_k + \alpha''_{2n-k}) \text{ and } b_k = \alpha''_k - \alpha''_{2n-k} + i(\alpha'_{2n-k} - \alpha'_k). \end{aligned}$$

Therefore every element which is of the form $e^{-inx}P(e^{ix})$, $n \in \mathbb{N}$, where $P(X) \in \mathbb{Q} + X\mathbb{C}[X]$, is in T'_3.

Conclusion 2. The consequence of above construction is: $T'_3 = \left\{e^{-inx}P(e^{ix}),\ n \in \mathbb{N},\text{ where } P(X) \in \mathbb{Q} + X\mathbb{C}[X]\right\}$. So we have an isomorphism $f : (\mathbb{Q} + X\mathbb{C}[X])_X \longrightarrow T'_3$ through the substitution morphism $X \longrightarrow e^{ix}$. Therefore $T'_3 \backsimeq (\mathbb{Q} + X\mathbb{C}[X])_X$.

Theorem 2. *The integral domain T'_3 is an HFD having irreducible elements, up to units, trigonometric polynomials of the form $Cosx + iSinx - a$, where $a \in \mathbb{C}\backslash\{0\}$.*

Proof. Since X is a prime in $\mathbb{Q}{+}X\mathbb{C}[X]$ [1, Example 1.8 (b)], so $(\mathbb{Q}{+}X\mathbb{C}[X])_X$ is a localization of $\mathbb{Q}{+}X\mathbb{C}[X]$ by a multiplicative system generated by a prime. Also $\mathbb{Q}{+}X\mathbb{C}[X]$ is an HFD [2, Proposition 3.1]. Therefore $(\mathbb{Q}{+}X\mathbb{C}[X])_X$ is an HFD [1, Corollary 2.5]. Now use the isomorphism $T'_3 \backsimeq (\mathbb{Q}{+}X\mathbb{C}[X])_X$ in Conclusion 2. □

2.3 The Construction of T'_4

Consider the set T'_4 of polynomials of the form

$$\sum_{k=0}^{n}(a_kCoskx + b_kSinkx),\ \ n \in \mathbb{N},\ \ a_k, b_k \in \mathbb{C} \text{ and}$$

$$a_n = (\alpha + \gamma) + i(\beta + \delta), b_n = (\delta - \beta) + i(\alpha - \gamma),$$

such that $\alpha, \beta \in \mathbb{R}$, $\gamma, \delta \in \mathbb{Q}$, where α, β, γ and δ are not simultaneously zero. Let $z \in T_4'$ be an arbitrary element, so we may write it as

$$\begin{aligned} z &= a_0 + \sum_{k=1}^{n-1}(a_k Coskx + b_k Sinkx) \\ &\quad + \Big\{\big((\alpha+\gamma)+i(\beta+\delta)\big)Cosnx + \big((\delta-\beta)+i(\alpha-\gamma)\big)Sinnx\Big\}. \end{aligned}$$

As $Cosx = \frac{e^{ix}+e^{-ix}}{2}$ and $Sinx = \frac{e^{ix}-e^{-ix}}{2i}$, so

$$\begin{aligned} z &= a_0 + \sum_{k=1}^{n-1}\Big\{\Big(\frac{a_k' + b_k'' + i(a_k'' - b_k')}{2}\Big)e^{ikx} + \Big(\frac{a_k' - b_k'' + i(a_k'' + b_k')}{2}\Big)e^{-ikx}\Big\} \\ &\quad + (\alpha + i\beta)e^{inx} + (\gamma + i\delta)e^{-inx}, \end{aligned}$$

where $a_k = a_k' + ia_k''$, $b_k = b_k' + ib_k''$ and $a_k', a_k'', b_k', b_k'' \in \mathbb{R}$, $a_0 \in \mathbb{C}$.

Setting

$$\alpha_k' = \frac{a_k' + b_k'' + i(a_k'' - b_k')}{2} \text{ and } \beta_k' = \frac{a_k' - b_k'' + i(a_k'' + b_k')}{2},$$

we have

$$z = e^{-inx}\left[a_0 e^{inx} + \sum_{k=1}^{n-1}\{\alpha_k' e^{i(n+k)x} + \beta_k' e^{i(n-k)x}\} + (\alpha + i\beta)e^{i2nx} + (\gamma + i\delta)\right],$$

where $\alpha_k', \beta_k', a_0 \in \mathbb{C}$, and $\alpha, \beta \in \mathbb{R}$, $\gamma, \delta \in \mathbb{Q}$.

Since z is an arbitrary, every element of T_4' is of the form

$$e^{-inx}P(e^{ix}),\ n \in \mathbb{N},\ \text{where } P(X) \in \mathbb{Q}(i) + X\mathbb{C}[X].$$

Conversely, for $\alpha_0 \in \mathbb{Q}(i)$ and $\alpha_k \in \mathbb{C}$, $1 \le k \le 2n$, consider

$$e^{-inx}P(e^{ix}) = \alpha_0 e^{-inx} + \alpha_{2n}e^{inx} + \sum_{k=1}^{n-1}\big(\alpha_k e^{-i(n-k)x} + \alpha_{2n-k}e^{i(n-k)x}\big) + \alpha_n.$$

As $e^{ix} = Cosx + iSinx$, so for

$$\alpha_0 = \alpha_0' + i\alpha_0'', \alpha_k = \alpha_k' + i\alpha_k'',\ \alpha_{2n-k} = \alpha_{2n-k}' + i\alpha_{2n-k}'' \text{ and } \alpha_{2n} = \alpha_{2n}' + i\alpha_{2n}''.$$

We have

$$\begin{aligned} e^{-inx}P(e^{ix}) &= (\alpha_0 + \alpha_{2n}' + i(\alpha_0'' + \alpha_{2n}''))Cosnx + (\alpha_0'' - \alpha_{2n}'' + i(\alpha_{2n}' - \alpha_0))Sinnx \\ &\quad + \sum_{k=1}^{n-1}\{(\alpha_k' + \alpha_{2n-k}' + i(\alpha_k'' + \alpha_{2n-k}''))Cos(n-k)x \end{aligned}$$

$$
\begin{aligned}
&+(\alpha''_k-\alpha''_{2n-k}+i(\alpha'_{2n-k}-\alpha'_k))Sin(n-k)x\}+\alpha_n.\\
=\ & a_nCosnx+b_nSinnx+\sum_{k=1}^{n-1}\{a_kCos(n-k)x+b_kSin(n-k)x\}+\alpha_n,
\end{aligned}
$$

where

$$
\begin{aligned}
a_n &= \alpha_0+\alpha'_{2n}+i(\alpha''_0+\alpha''_{2n}), \quad & b_n &= \alpha''_0-\alpha''_{2n}+i(\alpha'_{2n}-\alpha_0),\\
a_k &= \alpha'_k+\alpha'_{2n-k}+i(\alpha''_k+\alpha''_{2n-k}) \ \text{ and } & b_k &= \alpha''_k-\alpha''_{2n-k}+i(\alpha'_{2n-k}-\alpha'_k).
\end{aligned}
$$

Therefore every element which is of the form $e^{-inx}P(e^{ix})$, $n\in\mathbb{N}$, where $P(X)\in\mathbb{Q}(i)+X\mathbb{C}[X]$, is in T'_4.

Conclusion 3. The consequence of above construction is: $T'_4=\{e^{-inx}P(e^{ix}),\ n\in\mathbb{N}$, where $P(X)\in\mathbb{Q}(i)+X\mathbb{C}[X]\}$. So once again we have an isomorphism $f:(\mathbb{Q}(i)+X\mathbb{C}[X])_X\longrightarrow T'_4$ through the substitution morphism $X\longrightarrow e^{ix}$. Therefore $T'_4\backsimeq(\mathbb{Q}(i)+X\mathbb{C}[X])_X$.

Theorem 3. *The integral domain T'_4 is an HFD having irreducible elements, up to units, trigonometric polynomials of the form $Cosx+iSinx-a$, where $a\in\mathbb{C}\backslash\{0\}$.*

Proof. The proof follows analogous to Theorem 1 and 2. □

The following is the analogue of [15, Corollary 2.2] and gives the factorization in T'_4 instead of T'.

Corollary 1. *Let $z=\sum\limits_{k=0}^{n}(a_kCoskx+b_kSinkx)$, $n\in\mathbb{N}\backslash\{1\}$, $a_k,b_k\in\mathbb{C}$ with $(a_n,b_n)\neq(0,0)$, such that $a_n=(\alpha+\gamma)+i(\beta+\delta)$ and $b_n=(\delta-\beta)+i(\alpha-\gamma)$, where $\alpha,\beta\in\mathbb{R}$, $\gamma,\delta\in\mathbb{Q}$. Let d be a common divisor of the integers k such that $(a_k,b_k)\neq(0,0)$. Then z has a unique factorization*

$$
\lambda(Cosnx-iSinnx)\prod_{j=1}^{\frac{2n}{d}}(Cosdx+iSindx-\alpha_j), \ \ \textit{where} \ \ \lambda,\alpha_j\in\mathbb{C}\backslash\{0\}.
$$

Proof. Since $T'_4\subseteq T'$, therefore proof follows by [15, Corollary 2.2]. □

Remark 1. The factorization in T'_2 and T'_3 is an analogue of Corollary 1.

Now onwards the symbol $\cap$ in all diagrams will represent the inclusion $\subseteq$.

Remark 2. Consider the following ascending chain of substructures of $\mathbb{C}[X]$

$$
\mathbb{Q}+X\mathbb{R}[X]\subseteq\mathbb{Q}+X\mathbb{C}[X]\subseteq\mathbb{Q}(i)+X\mathbb{C}[X]\subseteq\mathbb{C}[X],
$$

where $\mathbb{Q}+X\mathbb{R}[X]$, $\mathbb{Q}+X\mathbb{C}[X]$ and $\mathbb{Q}(i)+X\mathbb{C}[X]$ are HFDs, whereas $\mathbb{C}[X]$ is a Euclidean domain. The localization of all these four by a multiplicative system generated by X preserves their factorization properties.

$$\begin{array}{ccccccc} \mathbb{Q}+X\mathbb{R}[X] & \subseteq & \mathbb{Q}+X\mathbb{C}[X] & \subseteq & \mathbb{Q}(i)+X\mathbb{C}[X] & \subseteq & \mathbb{C}[X] \\ \cap & & \cap & & \cap & & \cap \\ (\mathbb{Q}+X\mathbb{R}[X])_X & \subseteq & (\mathbb{Q}+X\mathbb{C}[X])_X & \subseteq & (\mathbb{Q}(i)+X\mathbb{C}[X])_X & \subseteq & \mathbb{C}[X]_X. \end{array}$$

Using Conclusion 1, Conclusion 2, Conclusion 3 and [15, Theorem 2.1], we have

$$\begin{array}{ccccccc} \mathbb{Q}+X\mathbb{R}[X] & \subseteq & \mathbb{Q}+X\mathbb{C}[X] & \subseteq & \mathbb{Q}(i)+X\mathbb{C}[X] & \subseteq & \mathbb{C}[X] \\ \cap & & \cap & & \cap & & \cap \\ T_2' & \subseteq & T_3' & \subseteq & T_4' & \subseteq & T', \end{array}$$

where T_2', T_3', T_4' are HFDs contained in the Euclidean domain T'. So we can see that there is a ascend of factorization properties for each of these four structures.

Remark 3. Due to the domain extensions

$$\begin{array}{ccccccc} \mathbb{Q}+X\mathbb{R}[X] & \subseteq & (\mathbb{Q}+X\mathbb{R}[X])_X, & & \mathbb{Q}+X\mathbb{C}[X] & \subseteq & (\mathbb{Q}+X\mathbb{C}[X])_X \\ \mathbb{Q}(i)+X\mathbb{C}[X] & \subseteq & (\mathbb{Q}(i)+X\mathbb{C}[X])_X & \text{and} & \mathbb{C}[X] & \subseteq & (\mathbb{C}[X])_X, \end{array}$$

we obtain the following four extended ideals.

1. Consider the domain extension $\mathbb{C}[X] \subseteq (\mathbb{C}[X])_X$. Since $X\mathbb{C}[X]$ is a maximal ideal of $\mathbb{C}[X]$ and $X\mathbb{C}[X] \cap (X) \neq \phi$, the extended ideal $(X\mathbb{C}[X])^e = (\mathbb{C}[X])_X$ [25, Corollary 2]. Hence $(X\mathbb{C}[X])^e \backsimeq T'$ by [15, Theorem 2.1].

2. If we consider the domain extension $\mathbb{Q}+X\mathbb{R}[X] \subseteq (\mathbb{Q}+X\mathbb{R}[X])_X$, we observe that $X\mathbb{R}[X]$ is a maximal ideal of $\mathbb{R}[X]$ and $X\mathbb{R}[X] \cap (X) \neq \phi$. Therefore the extended ideal $(X\mathbb{R}[X])^e = (\mathbb{Q}+X\mathbb{R}[X])_X$ [25, Corollary 2]. Hence $(X\mathbb{R}[X])^e \backsimeq T_2'$ by Conclusion 1.

3. Analogous to (2) we have $(X\mathbb{C}[X])^e \backsimeq T_3'$ using Conclusion 2, since $X\mathbb{C}[X]$ is a maximal ideal of $\mathbb{C}[X]$ for the domain extension $\mathbb{Q}+X\mathbb{C}[X] \subseteq (\mathbb{Q}+X\mathbb{C}[X])_X$.

4. Once again analogous to (2) and (3) we have $(X\mathbb{C}[X])^e \backsimeq T_4'$ using Conclusion 3, since $X\mathbb{C}[X]$ is a maximal ideal of $\mathbb{C}[X]$ for the domain extension $\mathbb{Q}(i)+X\mathbb{C}[X] \subseteq (\mathbb{Q}(i)+X\mathbb{C}[X])_X$.

Definition 1. *We denote the set T_2' for $\alpha = 0$ by I_2 and is defined as*

$$I_2 = \left\{ \sum_{k=0}^{n} (a_k Coskx + ib_k Sinkx) \ : \ n \in \mathbb{N},\ a_k, b_k \in \mathbb{R} \ \text{and} \ a_n = b_n = \beta \right\}.$$

Definition 2. *We denote the set T_3' for $\gamma = 0$ by I_3 and is defined as*

$$I_3 = \left\{ \sum_{k=0}^{n} (a_k Coskx + b_k Sinkx) \ : \ n \in \mathbb{N},\ a_k, b_k \in \mathbb{C} \text{ and } a_n = \alpha + i\beta, b_n = -\beta + i\alpha \right\}.$$

Definition 3. *We denote the set T_4' for $\gamma = \delta = 0$ by I_4 and is defined as*

$$I_4 = \left\{ \sum_{k=0}^{n} (a_k Coskx + b_k Sinkx) \ : \ n \in \mathbb{N},\ a_k, b_k \in \mathbb{C} \text{ and } a_n = \alpha + i\beta, b_n = -\beta + i\alpha \right\}.$$

Remark 4. By Definition 2 and Definition 3, we have $I_3 = I_4$, so onwards we use I to denote I_3 and I_4.

Lemma 1. *For the maximal ideal $X\mathbb{R}[X]$ (respectively $X\mathbb{C}[X]$) of $\mathbb{Q}+X\mathbb{R}[X]$ (respectively of $\mathbb{Q}+X\mathbb{C}[X]$ and $\mathbb{Q}(i)+X\mathbb{C}[X]$), we have $(X\mathbb{R}[X])_X \backsimeq I_2$ (respectively $(X\mathbb{C}[X])_X \backsimeq I$).*

Proof. Follows by Conclusion 1 (respectively Conclusion 2 and Conclusion 3). □

3 CONDITIONS SATISFIED BY TRIGONOMETRIC POLYNOMIAL RING EXTENSIONS

In this section we study two special conditions among trigonometric polynomial ring extensions. First one known as Condition 1, is borrowed from [16, page 661] and the second one is borrowed from [20]. We denote the group of units of a ring B by $U(B)$.

Condition 1. *Let $A \subseteq B$ be a unitary (commutative) ring extension. For each $x \in B$ there exist $x' \in U(B)$ and $x'' \in A$ such that $x = x'x''$.*

Example 1. Following [16, Example 1.1].

a) If the ring extension $A \subseteq B$ satisfies Condition 1, the ring extension $A + XB[X] \subseteq B[X]$ (or $A + XB[[X]] \subseteq B[[X]]$) also satisfies Condition 1.
b) If the ring extensions $A \subseteq B$ and $B \subseteq C$ satisfy Condition 1, so does the ring extension $A \subseteq C$.
c) If B is a fraction ring of A, the ring extension $A \subseteq B$ satisfies Condition 1. Hence the ring extension $A \subseteq B$ satisfying Condition 1 is a generalization of localization.
d) If B is a field, the ring extension $A \subseteq B$ satisfies Condition 1.

Condition 2. *Let $A \subseteq B$ be a unitary (commutative) ring extension and let $A_1 \subseteq B_1$ be a unitary (commutative) ring extension where $A \subseteq A_1$ and $B \subseteq B_1$. Then for each $x \in B_1$ there exist $x' \in U(B)$ and $x'' \in A_1$ such that $x = x'x''$.*

Lemma 2. *Let $A \subseteq B$ be a unitary (commutative) ring extension which satisfies Condition* 1. *If N is a multiplicative system in A then the ring extension $N^{-1}A \subseteq N^{-1}B$ satisfies Condition* 2.

Proof. Since the ring extension $A \subseteq B$ satisfies Condition 1, for each $a \in B$ there exist $b \in U(B)$ and $c \in A$ such that $a = bc$. Obviously $N^{-1}A \subseteq N^{-1}B$. Let $x = \frac{a}{s} \in N^{-1}B$. Then $x = \frac{a}{s}$, $a \in B$, $s \in N$. This implies $x = \frac{bc}{s} = b\frac{c}{s}$, where $b \in U(B)$ and $\frac{c}{s} \in N^{-1}A$. $\square$

The converse of Lemma 2 does not hold in general. The following proposition addresses this fact.

Proposition 1. *Let $A \subseteq B$ be a unitary (commutative) ring extension. Let the domain extension $A_1 \subseteq B_1$satisfies Condition* 2 *and the domain extension $A \subseteq A_1$ satisfies Condition* 1*, where $A \subseteq A_1$ and $B \subseteq B_1$. If $U(B_1) = U(B)$ and $U(B_1) \cap A_1 = U(A_1)$. Then the domain extension $A \subseteq B$ satisfies Condition* 1.

Proof. Since the domain extension $A_1 \subseteq B_1$ satisfies Condition 2, for each $x \in B \subseteq B_1$ there exist $x_1' \in U(B)$ and $x_1'' \in A_1$ such that $x = x_1'x_1''$. Now if $x_1'' \in A$, there is nothing to prove. If $x_1'' \in A_1 \backslash A$, we proceed as follows. Since the domain extension $A \subseteq A_1$ satisfies Condition 1, for $x_1'' \in A_1$ there exist $x_2' \in U(A_1)$ and $x_2'' \in A$ such that $x_1'' = x_2'x_2''$. It follows that $x = x_1'x_2'x_2''$. As $U(B_1) = U(B)$ and $U(B_1) \cap A_1 = U(A_1)$, so $x_1'x_2' \in U(B)$ and $x_2'' \in A$. Therefore $A \subseteq B$ satisfies Condition 1. $\square$

Proposition 2. *Let $A \subseteq B \subseteq B_1$ be a unitary (commutative) ring extension. Assume that the domain extension $A \subseteq B_1$ satisfies Condition* 2*. Then the domain extension $A \subseteq B$ satisfies Condition* 1.

Proof. The domain extension $A \subseteq B_1$ satisfies Condition 2, so for each $x \in B \subseteq B_1$ there exist $x' \in U(B)$ and $x'' \in A$ such that $x = x'x''$. Therefore $A \subseteq B$ satisfies Condition 1. $\square$

Example 2. Let $A \subseteq B$ be a unitary (commutative) ring extension. Let $N = U(A)$ is a multiplicative system in A. Assume that the domain extension $N^{-1}A \subseteq N^{-1}B$ satisfies Condition 2. Then the domain extension $A \subseteq B$ satisfies Condition 1.

Example 3. Some more interesting examples satisfying Condition 1 are as follows.

(a) If the ring extensions $A \subseteq B$ and $B \subseteq C$ satisfy Condition 2, so does the ring extension $A \subseteq C$.

(b) For $A = A_1$ and $B = B_1$ the Condition 1 and Condition 2 coincides.

(c) If the ring extension $A_1 \subseteq B_1$ satisfies Condition 2, it does satisfies Condition 1.

(d) By Lemma 2, the ring extensions $T_2' \subseteq T_3'$, $T_3' \subseteq T_4'$, and $T_4' \subseteq T'$ satisfy Condition 2 so do the ring extensions $T_2' \subseteq T_4'$, $T_2' \subseteq T'$, and $T_3' \subseteq T'$.

3.1 Conclusion

Consider the commutative inclusion diagram due to Remark 2.

$$\begin{array}{ccccccc} \mathbb{Q}+X\mathbb{R}[X] & \subseteq & \mathbb{Q}+X\mathbb{C}[X] & \subseteq & \mathbb{Q}(i)+X\mathbb{C}[X] & \subseteq & \mathbb{C}[X] \\ \cap & & \cap & & \cap & & \cap \\ T_2' & \subseteq & T_3' & \subseteq & T_4' & \subseteq & T' \end{array}$$

The following table concludes our discussion on Condition 1 and Condition 2 among trigonometric polynomial ring extensions.

Ring Extension	Condition 1	Condition 2
$\mathbb{Q}+X\mathbb{C}[X] \subseteq \mathbb{C}[X]$	Yes	Yes
$\mathbb{Q}(i)+X\mathbb{C}[X] \subseteq \mathbb{C}[X]$	Yes	Yes
$\mathbb{Q}+X\mathbb{R}[X] \subseteq T_2'$	Yes	Yes
$\mathbb{Q}+X\mathbb{R}[X] \subseteq T_3'$	Yes	Yes
$\mathbb{Q}+X\mathbb{R}[X] \subseteq T_4'$	Yes	Yes
$\mathbb{C}[X] \subseteq T'$	Yes	Yes
$T_2' \subseteq T_3'$	No	Yes
$T_3' \subseteq T_4'$	No	Yes
$T_4' \subseteq T'$	No	Yes
$\mathbb{Q}+X\mathbb{R}[X] \subseteq T_3'$	No	Yes
$\mathbb{Q}+X\mathbb{C}[X] \subseteq T_4'$	No	Yes
$\mathbb{Q}(i)+X\mathbb{C}[X] \subseteq T'$	No	Yes
$T_2' \subseteq T_4'$	No	Yes
$T_2' \subseteq T'$	No	Yes
$T_3' \subseteq T'$	No	Yes
$\mathbb{Q}+X\mathbb{R}[X] \subseteq \mathbb{Q}+X\mathbb{C}[X]$	No	No
$\mathbb{Q}+X\mathbb{C}[X] \subseteq \mathbb{Q}(i)+X\mathbb{R}[X]$	No	No

Note that the extensions $\mathbb{Q}+X\mathbb{R}[X] \subseteq T_3'$, $\mathbb{Q}+X\mathbb{C}[X] \subseteq T_4'$, $\mathbb{Q}(i)+X\mathbb{C}[X] \subseteq T'$, $T_2' \subseteq T_4'$, $T_2' \subseteq T'$ and $T_3' \subseteq T'$ satisfy Condition 2 due to transitivity.

3.2 Future Work and Applications

It is exciting that at the close of this paper, there are still some directions for future research work. In polynomial rings, factorization properties of integral domains have been a frequent topic of recent mathematical literature but the study of factorization properties in trigonometric polynomials is not addressed that much. So it seems to be really interesting to investigate factorization properties of trigonometric polynomial rings and this can open a new challenge for the researchers.

In addition to the applications mentioned in the introduction, we would like to highlight an application of trigonometric polynomials in symbolic computation. In [13], J. Mulholland and M. Monagan presented algorithms for simplifying ratios of trigonometric polynomials and algorithms for dividing, factoring and computing greatest common divisors of trigonometric polynomials. The provided algorithms do not always lead to the simplest form. A possible direction of study could be to provide enough general algorithms for finding a simplest form.

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RESUMO. Polinômios trigonométricos são amplamente utilizados em diferentes áreas de engenharia e ciências. Inspirado por suas aplicações, investigamos domínios semi-fatorial em anéis de polinômios trigonométricos. Nós construímos o domínios semi-fatoriais T_2', T_3' e T_4' que são os subanéis do anel de complexos, polinômios trigonométricas T', de tal forma que $T_2' \subseteq T_3' \subseteq T_4' \subseteq T'$. Discutimos, também, entre estes três subanéis a *Condição*: Seja $A \subset B$ ser uma extensão unitária (comutativa) do anel. Para cada $x \in B$ existem $x' \in U(B)$ e $x'' \in A$ tal que $x = x'x''$, onde $U(B)$ é o grupo de unidades de B.

Palavras-chave: polinômios trigonométricos, HFD, condição 1, condição 2, irredutível.

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