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Weighted Approximation of Continuous Positive Functions

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ABSTRACT. We investigate the density of convex cones of continuous positive functions in weighted spaces and present some applications.

Keywords: convex cone, weighted space, Bernstein's Theorem.

1 INTRODUCTION AND PRELIMINARIES

Throughout this paper we shall assume, unless stated otherwise, that X is a locally compact Hausdorff space. We shall denote by $C(X; \mathcal{R})$ the space of all continuous real-valued functions on X and by $C_b(X; \mathcal{R})$ the space of continuous and bounded real-valued functions on X. The vector subspace of all functions in $C(X; \mathcal{R})$ with compact support is denoted by $C_c(X; \mathcal{R})$.

An upper semicontinuous real-valued function f on X is said to *vanish at infinity* if, for every $\varepsilon > 0$, the closed subset $\{x \in X : |f(x)| \geq \varepsilon\}$ is compact.

In what follows, we shall present the concept of *weighted spaces* as developed by Nachbin in [4]. We introduce a set V of non-negative upper semicontinuous functions on X, whose elements are called *weights*. We assume that V is directed, in the sense that, given $v_1, v_2 \in V$, there exist $\lambda > 0$ and $v \in V$ such that $v_1 \leq \lambda v$ and $v_2 \leq \lambda v$.

Let V be a directed set of weights. The vector subspace of $C(X; \mathcal{R})$ of all functions f such that vf vanishes at infinity for each $v \in V$ will be denoted by $CV_\infty(X; \mathcal{R})$.

When $CV_\infty(X; \mathcal{R})$ is equipped with the locally convex topology ω_V generated by the seminorms

$$\begin{aligned} p_v : CV_\infty(X; \mathcal{R}) &\rightarrow \mathcal{R}^+ \\ f &\mapsto sup\,\{v(x)|f(x)| : x \in X\} \end{aligned}$$

for each $v \in V$, we call $CV_\infty(X; \mathcal{R})$ a *weighted space*.

We assume that for each $x \in X$, there is $v \in V$ such that $v(x) > 0$.

In the following we present some examples of weighted spaces.

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(a) If V consists of the constant function $\mathbf{1}$, defined by $\mathbf{1}(x) = 1$ for all $x \in X$, then $CV_\infty(X;\mathcal{R})$ is $C_0(X;\mathcal{R})$, the vector subspace of all functions in $C(X;\mathcal{R})$ that vanish at infinity. In particular, if X is compact then $CV_\infty(X;\mathcal{R}) = C(X;\mathcal{R})$. The corresponding weighted topology is the topology of uniform convergence on X.

(b) Let V be the set of characteristic functions of all compact subsets of X. Then the weighted space $CV_\infty(X;\mathcal{R})$ is $C(X;\mathcal{R})$ endowed with the compact-open topology.

(c) If V consists of characteristic functions of all finite subsets of X, then $CV_\infty(X;\mathcal{R})$ is $C(X;\mathcal{R})$ endowed with the topology of pointwise convergence.

(d) If $V = \{v \in C_0(X;\mathcal{R}) : v \geq 0\}$, then $CV_\infty(X;\mathcal{R})$ is the vector space $C_b(X;\mathcal{R})$. The corresponding weighted topology is the strict topology β (see Buck [1]).

For more information on weighted spaces we refer the reader to [4, 5].

We set $CV_\infty^+(X;\mathcal{R}) = \{f \in CV_\infty(X;\mathcal{R}) : f \geq 0\}$.

A subset $W \subset CV_\infty^+(X;\mathcal{R})$ is a convex cone if $\lambda W \subset W$, for each $\lambda \geq 0$ and $W + W \subset W$.

We denote by $CV_\infty^+(X;\mathcal{R}) \bigotimes CV_\infty^+(Y;\mathcal{R})$ the subset of $CV_\infty^+(X \times Y;\mathcal{R})$ consisting of all functions of the form

$$\sum_{i=1}^{n} g_i(x)h_i(y), \qquad x \in X,\ y \in Y$$

where $g_i \in CV_\infty^+(X;\mathcal{R}),\ h_i \in CV_\infty^+(Y;\mathcal{R}),\ i = 1,\ldots,n,\ n \in \mathcal{N}$.

Let $W \subset CV_\infty^+(X;\mathcal{R})$ be a nonempty subset. A function $\phi \in C(X;\mathcal{R})$, $0 \leq \phi \leq 1$, is called a *multiplier* of W if $\phi f + (1-\phi)g \in W$ for every pair f and g of elements of W. The set of all multipliers of W is denoted by $M(W)$. The notion of a multiplier of W is due to Feyel and De La Pradelle [3] and Chao-Lin [2].

For any $x \in X$, $[x]_{M(W)}$ denotes the equivalence class of x, when one defines the following equivalence relation on X: $x \equiv t\,(mod M(W))$ if, and only if, $\phi(x) = \phi(t)$ for all $\phi \in M(W)$.

A subset $A \subset C(X;\mathcal{R})$ separates the points of X if, given any two distinct points s and t of X, there is a function $\phi \in A$ such that $\phi(s) \neq \phi(t)$.

Weierstrass' first theorem states that any real-valued continuous function f defined on the closed interval [0,1] is the limite of a uniformly convergent sequence of algebraic polynomials. One of the most elementary proofs of this classic result is that which uses the Bernstein polynomials of f

$$(B_n f, x) := \sum_{k=0}^{n} \binom{n}{k} f\left(\frac{k}{n}\right) x^k (1-x)^{n-k}, \qquad x \in [0,1]$$

for each natural number n. Bernstein's theorem states that $B_n(f) \to f$ uniformly on [0,1] and, since each $B_n(f)$ is a polynomial, we have as a consequence the Weierstrass approximation theorem. The operator B_n defined on the space $C([0,1])$ with values in the vector subspace of all polynomials of degree at most n has the property that $B_n(f) \geq 0$ whenever $f \geq 0$. Thus Bernstein's theorem also establishes the fact that each positive continuous real-valued function on [0, 1] is the limit of a uniformly convergent sequence of positive polynomials.

Consider a compact Hausdorff space X and the convex cone

$$C^+(X; \mathcal{R}) = \{f \in C(X; \mathcal{R}) : f \geq 0\}.$$

A generalized Bernstein's theorem would be a theorem stating when a convex cone contained in $C^+(X; \mathcal{R})$ is dense in it.

Prolla [6] proved the following result of uniform density of convex cones in $C^+(X; \mathcal{R})$.

Theorem 1.1. *Let X be a compact Hausdorff space. Let $W \subset C^+(X; \mathcal{R})$ be a convex cone satisfying the following conditions:*

(a) *given any two distinct points x and y in X, there is a multiplier ϕ of W such that $\phi(x) \neq \phi(y)$;*

(b) *given any $x \in X$, there is $g \in W$ such that $g(x) > 0$.*

Then W is uniformly dense in $C^+(X; \mathcal{R})$.

The purpose of this note is to present an extension of this result to weighted spaces and give some applications. The main tool is a Stone-Weierstrass-type theorem for subsets of weighted spaces.

2 THE RESULTS

We need the following lemma, whose proof can be found in [7].

Lemma 2.1. *Let W be a nonempty subset of $CV_\infty(X; \mathcal{R})$. Given any $f \in CV_\infty(X; \mathcal{R})$, $v \in V$ and $\varepsilon > 0$, the following statements are equivalent:*

1. *there exists $h \in W$ such that $v(x)\|f(x) - h(x)\| < \varepsilon$ for all $x \in X$;*
2. *for each $x \in X$, there exists $g_x \in W$ such that $v(t)\|f(t) - g_x(t)\| < \varepsilon$ for all $t \in [x]_{M(W)}$.*

Now we state the main result.

Theorem 2.1. *Let $W \subset CV_\infty^+(X; \mathcal{R})$ be a convex cone satisfying the following conditions:*

(a) *given any two distinct points x and y in X, there exists a multiplier ϕ of W such that $\phi(x) \neq \phi(y)$;*

(b) *given any $x \in X$, there exists $g \in W$ such that $g(x) > 0$.*

Then W is ω_V-dense in $CV_\infty^+(X; \mathcal{R})$.

Proof. Let x be an arbitrary element of X. Condition (a) implies that $[x]_{M(W)} = \{x\}$. By condition (b), there exists $g \in W$ such that $g(x) > 0$. Then, for any $f \in CV_\infty^+(X; \mathcal{R})$, $v \in V$ and $\varepsilon > 0$, we have

$$v(x) \left\| f(x) - \frac{f(x)}{g(x)} g(x) \right\| = 0 < \varepsilon.$$

Since W is a convex cone, $\frac{f(x)}{g(x)} g \in W$. Then, it follows from Lemma 2.1 that there exists $h \in W$ such that $v(t)\| f(t) - h(t)\| < \varepsilon$ for all $t \in X$. □

Corollary 2.1. *Let X and Y be locally compact Hausdorff spaces. Then*

$$CV_\infty^+(X; \mathcal{R}) \bigotimes CV_\infty^+(Y; \mathcal{R})$$

is dense in $CV_\infty^+(X \times Y; \mathcal{R})$.

Proof. It follows from Urysohn's Lemma [8] that for any two distinct elements (s, t) and (u, v) of $X \times Y$, there exist functions $h_1 \in C_c(X; \mathcal{R})$ and $h_2 \in C_c(Y; \mathcal{R})$, $0 \le h_1, h_2 \le 1$, such that $\varphi(x, y) := h_1(x)h_2(y)$ is a multiplier of $CV_\infty^+(X; \mathcal{R}) \bigotimes CV_\infty^+(Y; \mathcal{R})$ and $\varphi(s, t) = 1$ and $\varphi(u, v) = 0$. Hence, condition (a) of Theorem 2.1 is satisfied.

By using Urysohn's Lemma again, given $(x, y) \in X \times Y$, there exist $\phi \in C_c(X; \mathcal{R})$ and $\psi \in C_c(Y; \mathcal{R})$ such that $\phi(x) = 1$ and $\psi(y) = 1$ so that $\phi(x)\psi(y) > 0$,

$$\phi\psi \in CV_\infty^+(X; \mathcal{R}) \bigotimes CV_\infty^+(Y; \mathcal{R}).$$

Then, condition (b) of Theorem 2.1 is satisfied. Hence, the assertion follows by Theorem 2.1. □

Example 2.1. Consider $CV_\infty^+(\mathcal{R}; \mathcal{R})$, where V is the set of characteristic functions of all compact subsets of $\mathcal{R}$. Let $\psi \in C(\mathcal{R}; \mathcal{R})$, $0 \le \psi \le 1$, be a one-to-one function. Let W be the set of all functions g of the form

$$g(x) = \sum_{i+j \le n} b_{ij} \psi(x)^i (1 - \psi(x))^j, \qquad x \in \mathcal{R}$$

where each b_{ij} is a non-negative real number and i, j, n are non-negative integers numbers. Note that $W \subset CV_\infty^+(\mathcal{R}; \mathcal{R})$ is a convex cone.

Since $\psi \in M(W)$ and W contains positive constant functions, it follows from Theorem 2.1 that W is dense in $CV_\infty^+(\mathcal{R}; \mathcal{R})$.

Example 2.2. Let a be a fixed positive real number. Let W be the set of all functions of the form

$$f(x)e^{-ax}, x \in [0, \infty), \;\; f \in C_b^+([0, \infty); \mathcal{R}).$$

Clearly, W is a convex cone contained in $C_0^+([0, \infty); \mathcal{R})$. The function e^{-ax}, $x \in [0, \infty)$, belongs to W and is a multiplier of W that separates the points of X. Hence, by Theorem 2.1 W is dense in $C_0^+([0, \infty); \mathcal{R})$.

RESUMO. Investigamos a densidade de cones convexos de funções contínuas positivas em espaços ponderados e apresentamos algumas aplicações.

Palavras-chave: cone convexo, espaço ponderado, Teorema de Bernstein.

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