On Complete Spacelike Hypersurfaces with Constant Scalar Curvature in the De Sitter Space

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Manuscript received on April 2, 2000; accepted for publication on May 10, 2000; presented by MANFREDIO DO CARMO

ABSTRACT

Let \( M^n \) be a complete spacelike hypersurface with constant normalized scalar curvature \( R \) in the de Sitter Space \( S^{n+1}_1 \). Let \( H \) the mean curvature and suppose that \( \overline{R} = (R - 1) > 0 \) and \( \overline{R} \leq \sup H^2 \leq C\overline{R} \), where \( C\overline{R} \) is a constant depending only on \( R \) and \( n \). It is proved that either \( \sup H^2 = \overline{R} \) and \( M^n \) is totally umbilical, or \( \sup H^2 = C\overline{R} \) and \( M^n \) is the hyperbolic cylinder \( H^1(1 - \coth 2r) \times S^{n-1}(1 - \tanh 2r) \).

Key words: hyperbolic cylinder, spacelike hypersurfaces, de Sitter space.

1. INTRODUCTION

The study of spacelike hypersurfaces with constant scalar curvature in the de Sitter space \( S^{n+1}_1 \) is related to an analogue of Goddard’s conjecture for the second elementary symmetric polynomial in the principal curvatures; more precisely: “Let \( M^n \) be a complete spacelike hypersurface with constant scalar curvature immersed in de Sitter space \( S^{n+1}_1 \). Then \( M^n \) is totally umbilical.”

S. Montiel (Montiel 1996) described the hyperbolic cylinders \( H^1(1 - \coth 2r) \times S^{n-1}(1 - \tanh 2r) \), \( n \geq 3 \), in \( S^{n+1}_1 \) with constant mean curvature \( H^2 = \frac{4(n-1)}{n^2} \) and normalized constant scalar curvature \( R = 1 + \frac{1}{n}(2 + (n - 2) \tanh 2r) \).

In (Zheng 1995), (Zheng 1996) and (Cheng & Ishikawa 1998) partial results were obtained. Recently, Haizhong Li (Li 1997) and also S. Montiel in a more general spacetime (Montiel 1999) obtained the following result: “Let \( M^n \) be a complete spacelike hypersurface immersed into the de Sitter space \( S^{n+1}_1 \) with normalized constant scalar curvature \( R \) satisfying \( R \geq 0 \). Then \( M \) is totally umbilical”.

In this note we announce the following
THEOREM 1.1. Let $M^n$ be an $n$-dimensional complete ($n \geq 3$) spacelike hypersurface immersed into the de Sitter space $S^{n+1}_1$ with constant scalar curvature $R$ such that $\bar{R} = R - 1 > 0$ and suppose that $\bar{R} \leq \sup H^2 \leq C_\bar{R}$ where

$$C_\bar{R} = \frac{1}{n} \left( (n-1)^2 \frac{n \bar{R} - 2}{n-2} + 2(n-1) + \frac{n-2}{n \bar{R} - 2} \right).$$

Then

(i) $\sup H^2 = \bar{R}$ and $M$ is totally umbilic or

(ii) $\sup H^2 = C_\bar{R}$ and $M$ is isometric to $H^1 (1 - \coth^2 r) \times S^{n-1} (1 - \tanh^2 r)$.

2. PRELIMINARIES

Let $R^{n+2}_1$ be the real vector space $R^{n+2}$ endowed with the Lorentzian metric $(\cdot, \cdot)$ given by $(v, w) = -v_0 w_0 + v_1 w_1 + \ldots + v_{n+1} w_{n+1}$ that is, $R^{n+2}_1 = L^{n+2}$ is the Lorentz-Minkowski $(n + 2)$-dimensional space. We define the de Sitter space as the following hyperquadric of $R^{n+2}_1$:

$$S^{n+1}_1 = \{ x \in R^{n+2}_1; \| x \|^2 = 1 \}.$$ 

The induced metric $(\cdot, \cdot)$ makes $S^{n+1}_1$ into a Lorentz manifold with constant sectional curvature $1$. Let $M^n$ be a $n$-dimensional orientable manifold, complete and let $f : M^n \rightarrow S^{n+1}_1 \subset L^{n+2}$ be a spacelike immersion of $M^n$ into the de Sitter $S^{n+1}_1$. Choose a unit normal $\eta$ along $f$ and denote by $A : T_p M \rightarrow T_p M$ the linear map of the tangent space $T_p M$ at the point $p \in M$, associated to the second fundamental form of $f$ along $\eta$.

$$\langle AX, Y \rangle = -\langle \nabla XY, \eta \rangle,$$

where $X$ and $Y$ are tangent vector fields on $M$ and $\nabla$ is the connection on $S^{n+1}_1$. Let $\{ e_1, \ldots, e_n \}$ be an orthonormal basis which diagonalizes $A$ with eigenvalues $k_i$ of $T_p M$, i.e., $A e_i = k_i e_i$, $i = 1, \ldots, n$. We will denote by $H = \frac{1}{n} \sum k_i$ the mean curvature of $f$ and $|A|^2 = \sum k_i^2$. In our case it is convenient to define a linear map $\phi : T_p M \rightarrow T_p M$ by

$$\langle \phi X, Y \rangle = \langle AX, Y \rangle - H \langle X, Y \rangle.$$

It is easily checked that $\text{trace}(\phi) = 0$ and that

$$|\phi|^2 = \frac{1}{2n} \sum (k_i - k_j)^2,$$

so that $|\phi|^2 = 0$ if and only if $M^n$ is totally umbilical. Let $\mu_i = k_i - H$ be the eigenvalues of $\phi$; then $\sum \mu_i = 0$, and

$$|\phi|^2 = \sum \mu_i^2 = |A|^2 - nH^2$$

$$\sum_i k_i^3 = nH^3 + 3H \sum_i \mu_i^2 - \sum_i \mu_i^3.$$  \hfill (2.1)
The standard examples of spacelike umbilical hypersurfaces with constant mean curvature in the de Sitter space are given by

\[ M^n = \{ p \in S^{n+1}_1 \mid \langle p, a \rangle = \tau \}, \]

where \( a \in R_{1}^{n+2} \), \( |a|^2 = \rho = 1, 0, -1 \) and \( \tau^2 > \rho \). The corresponding mean curvature \( H \) of such surfaces satisfies

\[ H^2 = \frac{\tau^2}{(\tau^2 - \rho)} \]

(Montiel 1988) and \( M^n \) is isometric to a hyperbolic space, an Euclidean space or a sphere according to \( \rho \) equal to 1, 0, \(-1\), respectively. On the other hand, hyperbolic cylinders are the hypersurfaces given by

\[ M^n = \{ p \in S^{n+1}_1; -p^2_o + p^2_1 + \ldots + p^2_k = -\sinh^2 r \}, \]

with \( r \in R \) and \( 1 \leq k \leq n \).

Such hyperbolic cylinders have constant mean curvature

\[ nH = [k \coth r + (n - k) \tanh r]. \]

Thus, we have

\[ H^2 \geq \frac{4(n - 1)}{n^2} \]

and the equality is attained for \( k = 1 \) and \( \coth^2 r = (n - 1) \). Their normalized scalar curvature is \( R = 1 + \frac{1}{n}(2 + (n - 2) \tanh^2 r) \).

We point out that these examples have only two different constant principal curvatures at each point and one of them has multiplicity one. Moreover, they are isometric to the Riemannian product

\[ H^1(1 - \coth^2 r) \times S^{n-1}(1 - \tanh^2 r). \]

The Gauss equation relates the scalar curvature, the mean curvature and the square of the norm of the second fundamental formula as follows:

\[ n(n - 1)(R - 1) = n^2H^2 - |A|^2. \tag{2.2} \]

Let \( T = \sum_{i,j} T_{ij} \omega_i \otimes \omega_j \) be a symmetric tensor defined on \( M^n \), where

\[ T_{ij} = nH\delta_{ij} - h_{ij}. \]

Following (Cheng & Yau 1977), we introduce the operator \( L_1 \) associated to \( T \) acting on \( C^2 \) functions \( f \) on \( M^n \) by

\[ L_1 f = \sum_{i,j} T_{ij} f_{ij} = \sum_{i,j} (nH\delta_{ij} - h_{ij}) f_{ij}. \tag{2.3} \]
Around a given point \( p \in M \) we choose an orthonormal frame field \( \{e_1, \ldots, e_n\} \) with dual frame field \( \{w_1, \ldots, w_n\} \) so that \( h_{ij} = \delta_{ij} \) at \( p \). We have the following computation by using (2.3) and Gauss equation (2.2):

\[
L_1(nH) = nH \Delta(nH) - \sum_i k_i(nH)_{ii} = \frac{1}{2} \Delta(nH)^2 - \sum_i (nH)_i^2 - \sum_i k_i(nH) = \frac{1}{2} n(n - 1) \Delta R + \frac{1}{2} |A|^2 - n^2 |\nabla H|^2 - \sum_i k_i(nH)_{ii}.
\]

On the other hand, using Simons Formula (see Zheng 1995) we get

\[
\frac{1}{2} |A|^2 = |\nabla A|^2 + n \sum_i k_i H_{ii} + \frac{1}{2} \sum_{i,j} R_{ijij}(k_i - k_j)^2.
\]

From (2.4) and (2.5), we have:

\[
L_1(nH) = \frac{1}{2} n(n - 1) \Delta R + |\nabla A|^2 - n^2 |\nabla H|^2 + \frac{1}{2} \sum_{i,j} R_{ijij}(k_i - k_j)^2.
\]

3. SKETCH OF THE PROOF OF THE THEOREM

Since \( R \) is constant, by (2.6) we obtain

\[
L_1(nH) = |\nabla A|^2 - n^2 |\nabla H|^2 + \frac{1}{2} \sum_{i,j} (1 + k_i k_j)(k_i - k_j)^2
\]

\[
= |\nabla A|^2 - n^2 |\nabla H|^2 + \frac{1}{2} n \sum_i k_i^2 + \frac{1}{2} n \sum_j k_j^2
\]

\[
- \sum_{i,j} k_i k_j + \frac{1}{2} \sum_{i,j} k_i^3 k_j + \frac{1}{2} \sum_{i,j} k_j^3 k_i - \sum_{i,j} k_i^2 k_j^2.
\]

Making \( i = j \), we then have

\[
L_1(nH) = |\nabla A|^2 - n^2 |\nabla H|^2 + n |\phi|^2 + |\phi|^4 - n H \sum_i \mu_i^3 - 3 n H^2 |\phi|^2 - n^2 H^4 + n H \sum_i \mu_i^3.
\]

Using (2.1) in (3.1) we obtain

\[
L_1(nH) = |\nabla A|^2 - n^2 |\nabla H|^2 + n |\phi|^2 + |\phi|^4 - n H \sum_i \mu_i^3 - 3 n H^2 |\phi|^2 - n^2 H^4.
\]

Then,

\[
L_1(nH) = |\nabla A|^2 - n^2 |\nabla H|^2 + n |\phi|^2 + |\phi|^4 + n^2 H^4 + 2 n H^2 |\phi|^2
\]

\[
- 3 n H^2 |\phi|^2 - n^2 H^4 - n H \sum_i \mu_i^3.
\]
This yields
\[ L_1(nH) = |\nabla A|^2 - n^2 |\nabla H|^2 + |\phi|^2 (n - nH^2 + |\phi|^2) - nH \sum_i \mu_i^3. \] (3.4)

We need to estimate \( tr(\phi^3) \) in (3.4). First we recall an algebraic lemma (Okumura 1974) which asserts
\[ -\frac{(n-2)}{\sqrt{n(n-1)}} (|\phi|^2)^{3/2} \leq \sum_i \mu_i^3 = tr(\phi^3) \leq \frac{(n-2)}{\sqrt{n(n-1)}} (|\phi|^2)^{3/2} \] (3.5)
and the equality holds on the right hand side if and only if
\[ \mu_1 = ... = \mu_{n-1} = -\sqrt{\frac{1}{n(n-1)}} |\phi| \quad \text{and} \quad \mu_n = \sqrt{\frac{n-1}{n}} |\phi|. \]

Using (3.5) in (3.4), we obtain
\[ L_1(nH) \geq |\nabla A|^2 - n^2 |\nabla H|^2 + |\phi|^2 (n - nH^2 + |\phi|^2) - \frac{n(n-2)}{\sqrt{n(n-1)}} |H||\phi|. \] (3.6)

Since \( R \) is constant, by (Alencar et al. 1993)
\[ |\phi|^2 = |A|^2 - nH^2 = \frac{n-1}{n} (|A|^2 - n\bar{R}). \] (3.8)

By Gauss equation (2.1) we know that
\[ |\phi|^2 = |A|^2 - nH^2 = \frac{n-1}{n} (|A|^2 - n\bar{R}). \] (3.8)

Using (8.3) and (7.3) in (6.3) we obtain
\[ L_1(nH) \geq \frac{n-1}{n} (|A|^2 - n\bar{R}) P_H(|\phi|), \] (3.9)
where \( P_H \) is a polynomial given by
\[ P_H(|\phi|) = (n - nH^2 + |\phi|^2 - \frac{(n-2)}{\sqrt{n(n-1)}} |H||\phi|). \] (3.10)

By (3.8) we may write the above polynomial as
\[ P_R(|A|) = n - 2(n-1)\bar{R} + \frac{n-2}{n} |A|^2 - \frac{(n-2)}{n} \sqrt{n(n-1)\bar{R} + |A|^2(|A|^2 - n\bar{R})}. \] (3.11)

Therefore (3.9) becomes
\[ L_1(nH) \geq \frac{n-1}{n} (|A|^2 - n\bar{R}) P_R(|A|). \] (3.12)
Using the hypothesis that $\overline{R} \leq \sup H^2 \leq C\overline{R}$, one proves that

$$P_{\overline{R}}(\sqrt{\sup |A|^2}) \geq 0.$$ \hspace{1cm} (3.13)

On the other hand,

$$L_1(nH) = \sum_{i,j} (nH \delta_{ij} - nh_{ij})(nH)_{ij} = \sum_{i} (nH - nh_{ii})(nH)_{ii}$$

$$= n \sum_{i} H(nH)_{ii} - n \sum_{i} k_i(nH)_{ii} \leq n(|H|_{\text{max}} - C)\Delta(nH),$$ \hspace{1cm} (3.14)

where $|H|_{\text{max}}$ is the maximum of the mean curvature $H$ in $M$ and $C = \min k_i$ is the minimum of the principal curvatures in $M$.

Now, we need the maximum principle at infinity for complete manifolds by Omori and Yau (Omori 1967):

“Let $M^n$ be an $n$-dimensional complete Riemannian manifold whose Ricci curvature is bounded from below. Let $f$ be a $C^2$-function bounded from below on $M^n$. Then for each $\varepsilon > 0$ there exists a point $p_\varepsilon \in M$ such that

$$\|\nabla f\|_{p_\varepsilon} < \varepsilon, \quad \Delta f(p_\varepsilon) > -\varepsilon \quad \text{and} \quad \inf f \leq f(p_\varepsilon) < \inf f + \varepsilon.$$ \hspace{1cm} (3.15)

The hypothesis $\overline{R} \leq \sup H^2 \leq C\overline{R}$ together with Gauss equation implies that $Ric_M \geq (n - 1) - \frac{2H^2}{4}$, so the Ricci curvature is bounded below. Thus we may apply Omori and Yau’s result to the function

$$f = \frac{1}{\sqrt{1 + (nH)^2}}.$$  

We have

$$|\nabla f|^2 = \frac{1}{4} \frac{|\nabla(nH)|^2}{(1 + (nH))^2}$$ \hspace{1cm} (3.16)

and

$$\Delta f = -\frac{1}{2} \frac{\Delta(nH)}{(1 + (nH))^2} + \frac{3|\nabla(nH)|^2}{4(1 + nH)^2}.$$ \hspace{1cm} (3.17)

Let $\{p_k\}, k \in \mathbb{N},$ be a sequence of points in $M$ given by (3.15) such that

$$\lim_{k \to \infty} f(p_k) = \inf f, \quad \Delta f(p_k) > -\frac{1}{k} \quad \text{and} \quad |\nabla f|^2(p_k) < \frac{1}{k^2}.$$ \hspace{1cm} (3.18)

Using (3.18) in the two equations (3.16) and (3.17) and the fact that

$$\lim_{k \to \infty} (nH)(p_k) = \sup_{p \in M} (nH)(p),$$

we obtain
\[-\frac{1}{k} < -\frac{1}{2} \frac{\Delta(nH)}{(1 + (nH))^2} (p_k) + \frac{3}{k^2} (1 + nH(p_k))^3.\]  
(3.19)

Hence
\[\frac{\Delta(nH)}{(1 + nH)^2} (p_k) < \frac{2}{k} \left( \frac{1}{\sqrt{1 + (nH)(p_k)}} + \frac{3}{k} \right).\]  
(3.20)

On the other hand, by (3.12) and (3.14), we have
\[\frac{n - 1}{n} (|A|^2 - n\overline{R}) P_{\overline{R}}(|A|) \leq L_1(nH) \leq n(|H|_{\text{max}} - C) \Delta(nH).\]  
(3.21)

At points $p_k$ of the sequence given in (3.18), this becomes
\[\frac{n - 1}{n} (|A|^2(p_k) - n\overline{R}) P_{\overline{R}}(|A|(p_k)) \leq L_1(nH(p_k))\]  
(3.22)

\[\leq n(|H|_{\text{max}} - C) \Delta(nH)(p_k).\]

Making $k \to \infty$ and using (3.20) we have that the right hand side of (3.22) goes to zero, so by (3.15) either $\frac{n - 1}{n} (\sup |A|^2 - n\overline{R}) = 0$ or $P_{\overline{R}}(\sqrt{\sup |A|^2}) = 0$. But by (3.8) $|\phi|^2 = \frac{n - 1}{n} (|A|^2 - n\overline{R})$ and so $\sup |\phi|^2 = \frac{n - 1}{n} (\sup |A|^2 - n\overline{R}) = 0$, then $|\phi|^2 = 0$, proving that $M^n$ is totally umbilical.

If $P_{\overline{R}}(\sqrt{\sup |A|^2}) = 0$, it can be proved that $\sup H^2 = C_{\overline{R}}$ and so the equality holds on the right hand side of (3.9) and we obtain
\[L_1(n \sup H) = \frac{n - 1}{n} (\sup |A|^2 - n\overline{R}) P_{\overline{R}}(\sqrt{\sup |A|^2}).\]

One proves that the equality also holds in Okumura’s lemma (3.5). After reenumeration, we finally have
\[k_1 = k_2 = \ldots = k_{n-1}, k_1 \neq k_n, \text{ where } k_1 = \tanh r \text{ and } k_n = \coth r.\]

Therefore, $M^n$ is isometric to $H^1(1 - \coth^2 r) \times S^{n-1}(1 - \tanh^2 r)$, finishing the proof.

REFERENCES


