A duality result between the minimal surface equation and the maximal surface equation

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ABSTRACT
In this note we show how classical Bernstein’s theorem on minimal surfaces in the Euclidean space can be seen as a consequence of Calabi-Bernstein’s theorem on maximal surfaces in the Lorentz-Minkowski space (and viceversa). This follows from a simple but nice duality between solutions to their corresponding differential equations.

Key words: Minimal surface equation, Maximal surface equation, Bernstein’s theorem, Calabi-Bernstein’s theorem.

1. INTRODUCTION
A minimal surface in Euclidean space $\mathbb{R}^3$ is a surface with zero mean curvature. Bernstein (1915-1917) proved that the planes are the only minimal entire graphs in $\mathbb{R}^3$.

Theorem 1. (Bernstein’s theorem). The only entire solutions to the minimal surface equation

$$\text{Minimal}[u] = \text{Div} \left( \frac{Du}{\sqrt{1 + |Du|^2}} \right) = 0$$

are affine functions.

On the other hand, a maximal surface in the Lorentz-Minkowski space $\mathbb{L}^3$ is a spacelike surface with zero mean curvature. Here by spacelike we mean that the induced metric from the Lorentzian...
metric in \( L^3 \) is a Riemannian metric on the surface. Calabi (1970) obtained the corresponding version of Bernstein’s theorem for the case of maximal surfaces.

**Theorem 2. (Calabi-Bernstein’s theorem).** The only entire solutions to the maximal surface equation

\[
\text{Maximal}[u] = \text{Div} \left( \frac{Du}{\sqrt{1 - |Du|^2}} \right) = 0, \quad |Du|^2 < 1,
\]

are affine functions.

Here the condition \(|Du|^2 < 1\) means precisely that the graph defined by \( u \) is spacelike.

In this note we show how classical Bernstein’s theorem on minimal surfaces in the Euclidean space \( \mathbb{R}^3 \) can be seen as a consequence of Calabi-Bernstein’s theorem on maximal surfaces in the Lorentz-Minkowski space \( L^3 \) (and vice versa). This follows from the following duality between solutions to their corresponding differential equations.

**Theorem 3.** Let \( \Omega \subseteq \mathbb{R}^2 \) be a simply connected domain. There exists a non-affine \( C^2 \) solution to the minimal surface equation on \( \Omega \)

\[
\text{Minimal}[u] = \text{Div} \left( \frac{Du}{\sqrt{1 + |Du|^2}} \right) = 0
\]

if and only if there exists a non-affine \( C^2 \) solution to the maximal surface equation on \( \Omega \)

\[
\text{Maximal}[w] = \text{Div} \left( \frac{Dw}{\sqrt{1 - |Dw|^2}} \right) = 0, \quad |Dw|^2 < 1.
\]

**2. Proof of Theorem 3**

**Proof.** Assume that \( u \) is a non-affine solution of \( \text{Minimal}[u] = 0 \) on the domain \( \Omega \). Recall that for a vector field \( X \) on \( \mathbb{R}^2 \) it holds that

\[
(\text{Div} X) dx \wedge dy = d\omega_{JX},
\]

where \( J \) denotes the positive \( \pi/2 \)-rotation in the plane and \( \omega_{JX} \) denotes the 1-form on \( \mathbb{R}^2 \) which is metrically equivalent to the field \( JX \), that is, \( \omega_{JX} \) satisfies

\[
\omega_{JX}(Y) = \langle JX, Y \rangle
\]

for every vector field \( Y \) on \( \mathbb{R}^2 \). Then \( \text{Minimal}[u] = 0 \) is equivalent to the fact that \( \omega_{JU} \) is closed on \( \Omega \), where \( U \) is the field on \( \Omega \) given by

\[
U = \frac{Du}{\sqrt{1 + |Du|^2}}.
\]
Then since the domain $\Omega$ is simply connected, we can write
\[ J \left( \frac{Du}{\sqrt{1 + |Du|^2}} \right) = Dw \] (1)
for a certain $C^2$ function $w$ on $\Omega$. Since $J$ is an isometry, there follows
\[ |Dw|^2 = \frac{|Du|^2}{1 + |Du|^2} < 1, \] (2)
and also
\[ 1 + |Du|^2 = \frac{1}{1 - |Dw|^2}. \] (3)

From (2), we see that $w$ satisfies the spacelike condition. Besides, using that $J^2 = -\text{id}$, we obtain from (1) and (3) that
\[ J \left( \frac{Dw}{\sqrt{1 - |Dw|^2}} \right) = \sqrt{1 + |Du|^2} J(Dw) = D(-u), \]
and so Maximal[$w$] = 0 follows on $\Omega$.

If $w$ were affine, then $Dw$ is a constant vector, $|Dw|^2 \equiv \text{constant}$, and then it follows from (3) that $|Du|^2$ is a constant also. It then follows from (1) that $Du$ is a constant vector, contradicting the assumption that $u$ is non-affine.

A very similar argument, starting with a non-affine solution of Maximal[$w$] = 0 on $\Omega$ with $|Dw|^2 < 1$, produces a non-affine solution of Minimal[$u$] = 0 on $\Omega$.

In particular, when $\Omega$ is the whole plane $\mathbb{R}^2$ we obtain the following.

**Corollary 4.** There exists an entire, non-affine $C^2$ solution to the minimal surface equation
\[ \text{Minimal}[u] = \text{Div} \left( \frac{Du}{\sqrt{1 + |Du|^2}} \right) = 0 \]
on $\mathbb{R}^2$ if and only if there exists an entire, non-affine $C^2$ solution to the maximal surface equation
\[ \text{Maximal}[w] = \text{Div} \left( \frac{Dw}{\sqrt{1 - |Dw|^2}} \right) = 0, \quad |Dw|^2 < 1 \]
on $\mathbb{R}^2$.

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RESUMO

Nesta nota, mostramos como o clássico teorema de Bernstein sobre as superfícies mínimas no espaço Euclideano pode ser visto como uma consequência do teorema de Calabi-Bernstein sobre as superfícies máximas no espaço de Lorentz-Minkowski (e vice-versa). Isto decorre de uma simples, mas elegante, dualidade entre soluções a suas correspondentes equações diferenciais.

Palavras-chave: Equações de superfícies mínimas, Equações de superfícies máximas, teorema de Bernstein, teorema de Calabi-Bernstein.

REFERENCES
