Erratum to “Bernstein-type Theorems in Hypersurfaces with Constant Mean Curvature”

MANFREDO P. DO CARMO\textsuperscript{1} and DETANG ZHOU\textsuperscript{2,3}

\textsuperscript{1}IMPA, Estrada Dona Castorina, 110-Jardim Botânico 22460-320 Rio de Janeiro, Brazil,
\textsuperscript{2}Department of Mathematics, Shandong University, Jinan, Shandong 250100, China
\textsuperscript{3}Departamento de Geometria, Universidade Federal Fluminense (UFF), 24020-140 Niterói, RJ, Brazil

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ABSTRACT
An erratum to Lemma 2.1 in Do Carmo and Zhou (2000) is presented.

Key words: Riemannian manifold, eigenvalue, hypersurface, mean curvature.

ERRATUM
Replace Section 2 in Do Carmo and Zhou (2000) by the following. The resulting change in the lemma will not affect the rest of the paper.

2. A RESULT ON NODAL DOMAINS
In this section we prove a result on the nodal domains of $|\phi|$ which will be needed in our proof of main theorems. We first need to recall the definition of nodal domains.

Definition. An open domain $D$ is called the nodal domain of a function $f$ if $f(x) \neq 0$ for $x \in \text{int} D$ and vanishes on the boundary of $\partial D$. We denote by $N(f)$ the number of disjoint \textit{bounded} nodal domains of $f$.

Now we have the following lemma which follows directly from Proposition 2.2 below. We want to thank the referee who provided the clearer proof of Proposition 2.2.

Lemma 2.1. \textit{Let $M$ be a hypersurface in $\mathbb{R}^{n+1}$ with constant mean curvature $H$. Then

$$\text{ind}(M) \geq N(|\phi|).$$

(2.1)\textit{}}
Proof. Let \( N = N(\phi) \) and \( D_1, D_2, \ldots, D_N \) be the \( N \) nodal domains of \( \phi \) and let

\[
\varphi(u) = u^2 + \frac{n(n-2)}{\sqrt{n(n-1)}} Hu - nH^2.
\]

Then from (1.5) and Proposition 2.2 below we have functions \( f_1, f_2, \ldots, f_N \) with supports in \( D_1, D_2, \ldots, D_N \) respectively such that

\[
I(f_i, f_i) = \int_{D_i} (|\nabla f_i|^2 - \varphi(u)f_i^2) < 0.
\]

Denote \( W \) the linear subspace spanned by \( f_1, f_2, \ldots, f_N \). Since they have disjoint supports, they are orthogonal and thus the dimension of \( W \) is \( N \). The index form \( I(\cdot, \cdot) \) is negative definite on \( W \) so the Morse index is greater than or equal to \( N \). □

Proposition 2.2. Let \((M, g)\) be Riemannian manifold and \( u \geq 0 \) be a continuous function satisfying the following inequality of Simons’ type in the distribution sense

\[
u^2\varphi(u) \geq a|\nabla u|^2_g - u\Delta_g u,
\]

where \( a > 0 \) is a constant and \( \varphi \) is a continuous function on \( \mathbb{R} \). If \( u \) has a relatively compact nodal domain \( D \), then there exists a function \( f_D \) with support in \( D \) such that

\[
\int_D (|\nabla f|^2 - \varphi(u)f^2) < 0.
\]

Proof. Suppose that \( u \) admits a relatively compact nodal domain \( D \). Write \( q := \varphi(u) \) and \( v := \log u \) on \( D \). Thus (2.2) can be written as

\[
q \geq a|\nabla v|^2_g - \Delta_g v - |\nabla v|^2_g.
\]

Then for any Lipschitz function \( f \) with support in \( D \) and vanishing at \( \partial D \), we have

\[
\int_D (|\nabla f|^2 - qf^2) \leq -a \int_D f^2|\nabla u|^2 + \int_D |\nabla f - f\nabla v|^2.
\]

Let \( f = wu \), for some function \( w \) to be determined. We obtain

\[
\int_D (|\nabla f|^2 - qf^2) \leq -a \int_D w^2|\nabla u|^2 + \int_D u^2|\nabla w|^2.
\]

For all \( b \) such that \( U/2 \leq b \leq U \), where \( U := \sup_D u \), set

\[
w_b(x) = \begin{cases} 
b & \text{as } u(x) \leq b, \\
u(x) & \text{as } u(x) > b. \end{cases}
\]

Denote $D_+$ (resp. $D_-$) the set of points in $D$ with $u(x) \geq b$ (resp. $u(x) \leq b$). A simple calculation leads to:

$$\int_D (|\nabla f|^2 - qf^2) \leq \int_{D_+} u^2 |\nabla u|^2 - \frac{aU^2}{4} \int_D |
abla u|^2.$$  

When $b$ goes to $U$, the first term of right hand side tends to 0 (because $|\nabla u|^2$ is integrable), while the second term is fixed. It follows that $\int_D (|\nabla f|^2 - qf^2) < 0$ for all functions $f = w_b u$, when $b$ is close to $U$. The conclusion is proved.

REFERENCE