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Cones in the Euclidean space with vanishing scalar curvature

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ABSTRACT

Given a hypersurface M on a unit sphere of the Euclidean space, we define the cone based on M as the set of half-lines issuing from the origin and passing through M. By assuming that the scalar curvature of the cone vanishes, we obtain conditions under which bounded domains of such cone are stable or unstable.

Key words: stability, r-curvature, cone, scalar curvature.

1 INTRODUCTION

A natural generalization of minimal hypersurfaces in Euclidean spaces was introduced in (Reilly 1973). Reilly considered the elementary symmetric functions S_r, $r = 0, 1, \ldots, n$, of the principal curvatures $k_1, \ldots, k_n$ of an orientable hypersurface $x : M^n \to R^{n+1}$ given by

$$S_0 = 1, \qquad S_r = \sum_{i_1 < \cdots < i_r} k_{i_1} \ldots k_{i_r} \; .$$

Here, $k_{i_1}, \ldots, k_{i_n}$ are the eigenvalues of $A = -dg$, where $g : M^n \to S^n(1)$ is the Gauss map of the hypersurface. Reilly showed that orientable hypersurfaces with $S_{r+1} = 0$ are critical points of the functional

$$\mathcal{A}_r = \int_M S_r dM$$

for variations of M with compact support. Thus, such hypersurfaces generalize the fact that minimal hypersurfaces are critical points of the area functional $A_0 = \int_M S_0 dM$ for compactly supported variations.

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A breakthrough in the study of these hypersurfaces occurred in the last five years of last century: in (Hounie and Leite 1995) and (Hounie and Leite 1999) conditions for the linearization of the partial differential equation $S_{r+1} = 0$ to be an elliptic equation were found. This linearization involves a second order differential operator L_r (see the definition of L_r in Section 2) and the Hounie-Leite conditions read as follows:

$$L_r \textit{ is elliptic} \iff \textit{rank}(A) > r+1 \iff S_{r+2} \neq 0 \textit{ everywhere.}$$

In this paper, we will be interested in the case $S_2 = 0$. For this situation, since rank(A) cannot be two, the ellipticity condition is equivalent to rank $(A) \geq 3$.

In (Alencar et al. 2003) a general notion of stability was introduced for bounded domains of hypersurfaces of Euclidean spaces with $S_{r+1} = 0$. In the case we are interested, namely $S_2 = 0$, it can be shown that if we assume that L_1 is elliptic, an orientation can be chosen so that a bounded domain $D \subset M$ is stable if

$$\frac{d^2 A_1}{dt^2}\Big|_{t=0} > 0 \quad \text{for all variations with support in (the open set) } D\,.$$

In what follows, we denote by $B_r(0)$ the ball of radius r centered at the origin 0 of R^{n+1}. Let M^{n-1} be a smooth hypersurface of the sphere $S^n(1)$. A cone $\mathcal{C}(M)$ in R^{n+1} is the union of half-lines starting at 0 and passing through the points of M. It is clear that $\mathcal{C}(M) \cap S^n(1) = M$. It is easy to show that $\mathcal{C}(M) - \{0\}$ is a smooth n-dimensional hypersurface of R^{n+1}. The manifold $\mathcal{C}(M)$ is referred to as the *cone based on* M^{n-1}. The part of the cone contained in the closure of the ring $B_1(0) \setminus B_\varepsilon(0)$, $0 < \varepsilon < 1$, is called a *truncated cone* and is denoted by $\mathcal{C}(M)_\varepsilon$.

In this note we present the following two theorems which provide a nice description of the stability of truncated cones in R^{n+1} based on compact, orientable hypersurfaces of $S^n(1)$, with $S_2 = 0$ and $S_3 \neq 0$ everywhere.

THEOREM 1. *Let M^{n-1}, $n \geq 4$, be an orientable, compact, hypersurface of $S^n(1)$ with $S_2 = 0$ and $S_3 \neq 0$ everywhere. Then, if $n \leq 7$, there exists an $\varepsilon > 0$ so that the truncated cone $\mathcal{C}(M)_\varepsilon$ is not stable.*

THEOREM 2. *For $n \geq 8$, there exist compact, orientable hypersurfaces M^{n-1} of the sphere $S^n(1)$, with $S_2 = 0$ and $S_3 \neq 0$ everywhere, so that, for all $\varepsilon > 0$, $\mathcal{C}(M)_\varepsilon$ is stable.*

Although Theorems 1 and 2 are interesting on their own right, a further motivation to prove these theorems is that, for the minimal case, they provide the geometric basis to prove the generalized Bernstein theorem, namely, that a complete minimal graph $y = f(x_1, \ldots, x_{n-1})$ in R^n, $n \leq 8$, is a linear function (See (Simons 1968), Theorems 6.1.1, 6.1.2, 6.2.1, 6.2.2).

For elliptic graphs in R^n with vanishing scalar curvature the question appears in a natural way. Of course, since we want to consider graphs with $S_2 = 0$ and S_3 never zero, we must start with $n \geq 4$, and the solution cannot be a hyperplane. Thus the question is whether there exists an elliptic graph in R^n, $n \geq 4$, with vanishing scalar curvature.

2 PRELIMINARIES

For notational reasons, it will be convenient to denote the hypersurface of the Introduction by $x : \bar{M} \to R^{n+1}$. We first need to consider the Newton Transformations P_r, that are inductively given by

$$\begin{aligned} P_0 &= I \\ P_r &= S_r I - A P_{r-1} , \end{aligned} \tag{1}$$

and then define the differential operator L_r by

$$L_r f = \mathrm{trace}\{P_r \mathrm{Hess} f\} . \tag{2}$$

It turns out that L_r is self-adjoint and that $L_r f = \mathrm{div}(P_r \mathrm{grad} f)$.

The second variation formula for the variational problem of the functional $\mathcal{A}_1$ is, up to a positive constant, given by

$$I(f) = -\int_{\bar{M}} f(L_1 f - 3S_3 f) d\bar{M} , \tag{3}$$

for test functions f of compact support in $\bar{M}$.

Consider now a compact orientable $(n-1)$-dimensional manifold M immersed as a hypersurface of the unit sphere $S^n(1)$ of the Euclidean space R^{n+1}. The cone $\mathcal{C}(M)$ based on M is described by

$$\begin{aligned} M \times (0, \infty) &\to R^{n+1} \\ (m, t) &\to tm . \end{aligned} \tag{4}$$

Of course, the geometry of $\mathcal{C}(M)$ is closely related to the one of M and it is simple to compute the second fundamental form $\bar{A}$ of $\mathcal{C}(M)$ in terms of the second fundamental form A of M. In fact, one finds

$$\bar{A} = \frac{1}{t} A .$$

From this relation on the second fundamental forms it follows that

PROPOSITION 1. *If $\bar{S}_r$ represents the elementary symmetric function of order r of $\mathcal{C}(M)$ and $\bar{P}_r$ its Newton transformations, then:*

a) $\bar{S}_r = (1/t^r) S_r$,

b) $\bar{S}_r = 0$ *if and only if* $S_r = 0$,

c) $|\bar{A}| = (1/t)|A|$,

d) $$\bar{P}_r = (1/t^r)\left[\begin{array}{c|c} S_r & 0 \\ \hline 0 & P_r \end{array}\right] .$$

PROOF. The proof is direct except for the last item. But this can be done using finite induction and the definition of $\bar{P}_r$.

Let $F : \mathcal{C}(M) \to R$ be a C^2 function. For each $t > 0$, define $\tilde{F}_t : M \to R$ by $\tilde{F}_t(m) = F(m, t)$.

Proposition 2. *With the above notation we have:*

$$\bar{L}_r F = \frac{1}{t^r} S_r \frac{\partial^2 F}{\partial t^2} + \frac{n-r-1}{t^{r+1}} S_r \frac{\partial F}{\partial t} + \frac{1}{t^{r+1}} L_r(\tilde{F}_t) .$$

PROOF. The proof of this Lemma follows the same lines used to find the expression of the Laplacian in polar coordinates and using the previous proposition.

3 SKETCH OF PROOF OF THEOREM 1

First of all let us observe that since $S_2 \equiv 0$ then $(S_1)^2 = |A|^2 \geq 0$. Hence, at a point where $S_1 = 0$ we would have that all the entries of the matrix A are zero and so $S_3 = 0$ what is forbidden by our hypothesis. Therefore, we will have $(S_1)^2 > 0$ everywhere.

According to Proposition 1, our hypotheses then imply that, for the cone $\mathcal{C}(M)$, we have $\bar{S}_2 \equiv 0$ and $\bar{S}_1$ and $\bar{S}_3$ never zero.

It was proved in (Hounie and Leite 1999) that, for a hypersurface of R^{n+1} with $\bar{S}_r \equiv 0$, $2 \leq r < n$, the operator $\bar{L}_{r-1}$ is elliptic if and only if $\bar{S}_{r+1}$ is never zero. Then we conclude that L_1 and $\bar{L}_1$ are elliptic.

To prove the theorem, we are going to show the existence of a truncated cone $\mathcal{C}(M)_\epsilon$ for which the second variation formula attains negative values. Hence, from now on we are going to work on a truncated cone, with test functions f that have a support contained in the interior of the truncated cone. As we did before, for each test function $f : \mathcal{C}(M)_\epsilon \to R$ and each fixed t we define $\tilde{f}_t : M \to R$ by $\tilde{f}_t(m) = f(m, t)$. From Proposition 2 we have that

$$\bar{L}_1 f = \frac{1}{t} S_1 \frac{\partial^2 f}{\partial t^2} + \frac{n-2}{t^2} S_1 \frac{\partial f}{\partial t} + \frac{1}{t^3} L_1(\tilde{f}_t) \quad . \tag{5}$$

The volume element of $\mathcal{C}(M)$ is easily seen to be

$$d\bar{M} = t^{n-1} dt \wedge dM \quad . \tag{6}$$

Hence, using (3), (5) and the expression of the volume, the second variation formula on f becomes

$$\begin{aligned} I(f) &= -\int_{M\times[\epsilon,1]} \big(\tilde{f}_t L_1(\tilde{f}_t) - 3S_3(\tilde{f}_t)^2\big)t^{n-4}dt\wedge dM - \\ &\quad -\int_{M\times[\epsilon,1]} \left(t^2 f\frac{\partial^2 f}{\partial t^2} + (n-2)tf\frac{\partial f}{\partial t}\right)t^{n-4}S_1 dt\wedge dM \ . \end{aligned} \tag{7}$$

Since $S_1 > 0$, then $t^{n-4}S_1 dt\wedge dM$ is a volume element in $\mathcal{C}(M)$, in particular in $\mathcal{C}(M)_\epsilon$. We will represent it by dS. In fact, dS is a product of two measures. The first one on the real line: $d\xi = t^{n-4}dt$; the second, on M, given by $d\mu = S_1 dM$. So, $dS = d\xi\wedge d\mu$. We can then rewrite the second variation formula on f as:

$$\begin{aligned} I(f) = &-\int_{M\times[\epsilon,1]} \frac{1}{S_1}\big(\tilde{f}_t L_1(\tilde{f}_t) - 3S_3(\tilde{f}_t)^2\big)d\xi\wedge d\mu \\ &-\int_{M\times[\epsilon,1]} \left(t^2 f\frac{\partial^2 f}{\partial t^2} + (n-2)tf\frac{\partial f}{\partial t}\right)d\xi\wedge d\mu \ . \end{aligned} \tag{8}$$

Define, now, the following two operators:

$$\begin{aligned} &\mathcal{L}_1 : C^\infty(M)\to C^\infty(M) && \text{by:} \quad \mathcal{L}_1 f = -(1/S_1)L_1 f + 3(S_3/S_1)f \ . \\ &\mathcal{L}_2 : C^\infty[\epsilon,1]\to C^\infty[\epsilon,1] && \text{by:} \quad \mathcal{L}_2 g = -t^2 g'' - (n-2)tg' \ . \end{aligned} \tag{9}$$

Observe that we are considering the space $C^\infty(M)$ with the inner product:

$$\ll f_1, f_2 \gg = \int_M f_1 f_2 d\mu \tag{10}$$

and $C^\infty[\epsilon, 1]$ with the inner product:

$$\langle g_1, g_2\rangle = \int_\epsilon^1 g_1 g_2 d\xi \ . \tag{11}$$

Since L_1 is elliptic and M is compact then L_1, and so $\mathcal{L}_1$, is strongly elliptic. The same is true for the operator $\mathcal{L}_2$. Let $\lambda_1 \le \lambda_2 \le \cdots \nearrow \infty$ be the eigenvalues of $\mathcal{L}_1$ and $\delta_1 < \delta_2 < \cdots \nearrow \infty$ be the eigenvalues of $\mathcal{L}_2$. Using orthonormal bases of eigenfunctions for theses operators one deduces the following Lemma:

LEMMA 1. *For any test function f we have*

$$I(f) \ge (\lambda_1 + \delta_1)\int_{M\times[\epsilon,1]} f^2 d\xi\wedge d\mu \ .$$

There exists a test function f such that $I(f) < 0$ if and only if $\lambda_1 + \delta_1 < 0$.

The operator $\mathcal{L}_2$ is well known. In fact it has been used in (Simmons 1968) to prove his celebrated theorem. The following lemma contains all the information we need about this operator:

LEMMA 2. *The operator $\mathcal{L}_2$ has eigenvalues*

$$\delta_k = \left(\frac{n-3}{2}\right)^2 + \left(\frac{k\pi}{\log\epsilon}\right)^2 \quad , \tag{12}$$

where $1 \leq k < \infty$.

We will also need the following lemma whose proof uses Lemmas (3.7) and (4.1) in (Alencar et al. 1993).

LEMMA 3. *Let M^{n-1} be a compact, orientable, immersed hypersurface of $S^n(1)$ with $S_2 \equiv 0$ e S_3 never zero. Suppose $n \geq 4$. The first eigenvalue of the operator $\mathcal{L}_1$ in M satisfy: $\lambda_1 \leq -(n-2)$.*

Finally, we observe that the lemma below completes the proof of Theorem 1.

LEMMA 4. *Let M^{n-1} be a compact, orientable, immersed hypersurface of $S^n(1)$ with $S_2 \equiv 0$, S_3 never zero and $n \geq 4$. If $n \leq 7$ then there exists $\epsilon > 0$ such that the truncated cone $\mathcal{C}M_\epsilon$ is not stable.*

PROOF OF THE LEMMA: From Lemmas 2 and 3 we have

$$\lambda_1 + \delta_1 \leq -(n-2) + \left(\frac{n-3}{2}\right)^2 + \left(\frac{\pi}{\log\epsilon}\right)^2 .$$

It is trivial to verify that the sum of the first two terms of the right hand side of this inequality is a quadratic polynomial, with positive second order term, whose roots are approximately 2.2 and 7.8 . Hence it is strictly negative for values of $n \in \{4, 5, 6, 7\}$, in fact, it is less than or equal to -1. Hence,

$$\lambda_1 + \delta_1 \leq -1 + \left(\frac{\pi}{\log\epsilon}\right)^2 \quad .$$

Choosing ϵ sufficiently small we can guarantee that the right hand side is negative. Now, by Lemma 1, we see that $\mathcal{C}M_\epsilon$ is not stable. This proves Lemma 4 and completes the proof of the Theorem 1.

4 EXISTENCE OF STABLE CONES

In this section we sketch the proof o Theorem 2.

The following example has been considered by various people in different contexts (see e.g. (Chern 1968) and (Alencar et al. 2002). Consider $R^{p+2} = R^{r+1} \oplus R^{s+1}$, $r + s = p$. Write down the vectors of R^{p+2} as $\xi_1 + \xi_2$, $\xi_1 \in R^{r+1}$, $\xi_2 \in R^{s+1}$. When ξ_1 describes $S^r(1) \subset R^{r+1}$ and ξ_2 describes $S^s(1) \subset R^{s+1}$, by taking positive numbers a_1 and a_2 with $a_1^2 + a_2^2 = 1$, we have that

$$x = a_1\xi_1 + a_2\xi_2$$

describes a submanifold M of dimension $p = r + s$ of the sphere $S^{p+1}(1) \subset R^{p+2}$. The manifold M is diffeomorphic to $S^r(1) \times S^s(1)$ and so is compact and orientable. It can be shown that a_1 and a_2 can be chosen so that $S_2 = 0$ and $S_3 \neq 0$. We will show that, in this case, the truncated cone $\mathcal{C}(M)_\varepsilon$ is stable as a hypersurface of R^{r+s+1} when $r + s + 1 \geq 8$.

It can be shown that the L_1 operator on M is given by

$$L_1 f = \left[(r-1)\frac{a_2}{a_1} - s\frac{a_1}{a_2}\right]\Delta^r f + \left[r\frac{a_2}{a_1} - (s-1)\frac{a_1}{a_2}\right]\Delta^s f \ ,$$

where Δ^r and Δ^s denote the Laplacian operators in the Euclidean spheres $S^r(a_1)$ and $S^s(a_2)$, respectively. Since the first nonzero eigenvalue of the Laplace operator on a sphere $S^k(b)$ is known to be k/b^2, the first nonzero eigenvalue of L_1 will be

$$\tilde{\lambda}_1 = \min\left\{\left[(r-1)\frac{a_2}{a_1} - s\frac{a_1}{a_2}\right]\frac{r}{a_1^2}\ ,\ \left[r\frac{a_2}{a_1} - (s-1)\frac{a_1}{a_2}\right]\frac{s}{a_2^2}\right\} \ .$$

It will then follow that the first eigenvalue of the operator

$$\mathcal{L}_1 = \frac{1}{S_1}L_1 + 3\frac{S_3}{S_1}$$

will be given by

$$\lambda_1 = 3\frac{S_3}{S_1} = -(p-1) \ ,$$

where the last equality comes from a long but straightforward computation.

Therefore, using Lemma 2, the above value for λ_1, and the fact that, in our case, $n = p+1$, we obtain

$$\lambda_1 + \delta_1 = -(n-2) + \left(\frac{n-3}{2}\right)^2 + \left(\frac{\pi}{\log\varepsilon}\right)^2 \ .$$

For $n \geq 8$, the sum of the first two terms becomes $> 1/4$. Thus, for any choice of ε, $\lambda_1 + \delta_1 > 0$. Together with Lemma 1, this completes the proof of Theorem 2.

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RESUMO

Dada uma hipersuperfície M de uma esfera unitária do espaço euclidiano, definimos o cone sobre M como o conjunto das semi-retas que saem da origem e passam por M. Admitindo que a curvatura escalar de um dado cone é nula, estabelecemos condições para que os seus domínios limitados sejam estáveis ou instáveis.

Palavras-chave: estabilidade, r-curvatura, cone, curvatura escalar.

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