A relation between the right triangle and circular tori with constant mean curvature in the unit 3-sphere

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ABSTRACT
In this note we will show that the inverse image under the stereographic projection of a circular torus of revolution in the 3-dimensional euclidean space has constant mean curvature in the unit 3-sphere if and only if their radii are the catet and the hypotenuse of an appropriate right triangle.

Key words: Flat torus, constant mean curvature, circular tori, stereographic projection.

1 INTRODUCTION
We will denote by $T(r,a)$ the standard circular torus of revolution in $\mathbb{R}^3$ obtained from the circle $\Gamma$ in the $xz -$ plane centered at $(r,0,0)$ with radius $a < r$, i.e.

$$T(r,a) = \{(x, y, z) \in \mathbb{R}^3 : (\sqrt{x^2+y^2} - r)^2 + z^2 = a^2\}.$$ 

Now let $\rho : S^3 \setminus \{n\} \to \mathbb{R}^3$ be the stereographic projection of the Euclidean sphere $S^3 = \{x \in \mathbb{R}^4 : |x|^2 = 1\}$, where $n = (0, 0, 0, 1)$ is its north pole. The inverse image of a circular torus in $\mathbb{R}^3$ under the stereographic projection will be called a circular torus in $S^3$. We would like to know when circular tori in $\mathbb{R}^3$ comes from constant mean curvature circular tori in $S^3$ under the stereographic projection. A circular torus in $S^3$ meant that it is obtained from a revolution of a circle in $S^3$ under a rigid motion. A general $T(r,a)$ will not satisfy the above requirement. For instance, it was proved by Montiel and Ros (Montiel and Ros 1981) that a compact embedded surface $S$ with constant mean curvature contained in an open hemisphere of $S^3$ must be a round sphere. Hence for $T(r,a)$ contained inside or outside of the unit ball $B(1) \subset \mathbb{R}^3$, $\rho^{-1}(T(r,a))$ will be contained in an open hemisphere of $S^3$ and can not have constant mean curvature. Then among

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all tori $T(r, a)$ which intercept the inside and the outside of the unit ball $B(1)$ we will describe those which have the desired property. We will show that to construct such a torus we take an arbitrary point $P(\alpha) = (\cos \alpha, 0, \sin \alpha)$ on the unit circle of the $xz$-plane, $0 < \alpha < \pi / 2$, draw its tangent until it meets the $x$ axis at the point $Q(\alpha) = (\sec \alpha, 0, 0)$ which will be the center of the circle $\Gamma$ whereas its radius will be $a = \tan \alpha$, i.e. the torus $T(\sec \alpha, \tan \alpha)$ will satisfy the previous requirement. We note if $O$ denotes the origin of $\mathbb{R}^3$ then the triangle $OPQ$ is a right triangle. This description will yield that the Clifford torus is associated to a right triangle with two equal sides. More precisely, our aim in this note is to present a proof of the following fact:

**Theorem 1.** Let $T^2 \subset S^3$ be a circular torus of constant mean curvature. Then

$$T^2 = \rho^{-1}(T(\sec \alpha, \tan \alpha)) = S^1(\cos \alpha) \times S^1(\sin \alpha).$$

Moreover, the mean curvature of $T^2$ is given by

$$H = \frac{\tan^2 \alpha - 1}{2 \tan \alpha}.$$  

## 2 Preliminaries

For an immersion $f : M \to \overline{M}$ between Riemannian manifolds we will denote by $ds^2_f$ the induced metric on $M$ by $f$. Now let $M^n$, $M^m_1$ and $M^m_2$ be Riemannian manifolds, where the superscript denote the dimension of the manifold. Consider $\psi : M^n \to M^m_1$ be an immersion, $\rho : M^m_1 \to M^m_2$ a conformal mapping and set $\varphi = \rho \circ \psi$. Let $\phi : M \to \mathbb{R}$ be a function verifying $ds^2_\varphi = e^{2\phi} ds^2_\psi$. If $k_i$ and $k_i$ denote the principal curvatures of $\psi$ and $\varphi = \rho \circ \psi$, respectively, then we get

$$k_i = e^{-\phi} \left( \bar{k}_i - \frac{\partial \phi}{\partial \xi} \right), \quad (1)$$

where $\xi$ is a unit normal vector field to $\psi(M)$, see for instance (Abe 1982) or (Willmore 1982). At first we will recall the following known lemma of which we sketch the proof.

**Lemma 1.** Let $\psi = (\psi_1, \psi_2, \psi_3, \psi_4) : M^2 \to S^3 \setminus \{n\}$ be an immersion of a surface $M^2$, set $\varphi = \rho \circ \psi$ and suppose $ds^2_\varphi = e^{2\phi} ds^2_\psi$. Then we get

$$k_i = e^{-\phi} (\bar{k}_i - g), \quad (2)$$

where $g = \langle v, \varphi \rangle$ denotes the support function on $M^2 \subset \mathbb{R}^3$.

**Proof.** If we put $\psi = \psi(u_1, u_2)$ then a direct computation gives

$$\langle \frac{\partial \varphi}{\partial u_i}, \frac{\partial \varphi}{\partial u_j} \rangle = \lambda^2 \langle \frac{\partial \psi}{\partial u_i}, \frac{\partial \psi}{\partial u_j} \rangle, \quad (3)$$

where $\lambda = (1 - \psi_4)^{-1} = \frac{1 + |w|^2}{2}$. So we can write $ds^2_\varphi = e^{2\phi} ds^2_\psi$ with $e^\phi = \frac{1 + |w|^2}{2}$. Thus if $v$ denotes a unit normal vector field to $\varphi(M^2)$ then $v = e^{-\phi} \xi$, where $\xi$ stands for a unit normal vector field to $\psi(M^2)$. Hence we have from (1)

$$k_i = e^{-\phi} (\bar{k}_i - \frac{\partial \phi}{\partial v}) = e^{-\phi} (\bar{k}_i - \langle v, \varphi \rangle) = e^{-\phi} (\bar{k}_i - g),$$

as we wished to prove. 

\[ \square \]
3 PROOF OF THE THEOREM

Proof. First we note that the circle \( \Gamma = \{(x, 0, z) \in \mathbb{R}^3 : (x-r)^2 + z^2 = a^2\} \) can be parametrized by the map \( \gamma : [0, 2\pi] \rightarrow \mathbb{R}^3 \) defined by

\[
\gamma(t) = \left( \frac{r^2 - a^2}{r - a \sin t}, \frac{a\sqrt{r^2 - a^2} \cos t}{r - a \sin t}, 0 \right).
\]

In fact, it is enough to check that

\[
\left( \frac{r^2 - a^2}{r - a \sin t} - r \right)^2 + \left( \frac{a\sqrt{r^2 - a^2} \cos t}{r - a \sin t} \right)^2 = a^2.
\]

Representing by \( R_\theta \) a rotation on \( \mathbb{R}^3 \) around the \( z \)-axis, we see that \( R_\theta(\gamma(t)) \) is a circular torus \( T (r, a) \) if \( \gamma \) is a parametrization of the circle \( \Gamma \) given above. We put now \( \sigma = \sqrt{r^2 - a^2} \), \( \theta = ru_1/\sigma^2 \) and \( t = ru_2/\sigma \). We note that such a choice implies \( 0 \leq u_1 \leq (2\pi \sigma^2)/r \) and \( 0 \leq u_2 \leq (2\pi a \sigma)/r \). Let us call \( R_\theta(\gamma(t)) \) of \( \varphi(u_1, u_2) \), i.e.

\[
\varphi(u_1, u_2) = \sigma (r - a \sin t)^{-1} (\sigma \cos \theta, \sigma \sin \theta, a \cos t).
\]

Hence we have

\[
e^\varphi = \frac{1 + |\varphi|^2}{2} = \frac{q(t)}{2(r - a \sin t)},
\]

where \( q(t) = a(\sigma^2 - 1) \sin t + r(\sigma^2 + 1) \). Now a straightforward computation yields

\[
\begin{align*}
\frac{\partial \varphi}{\partial u_1} &= \frac{r}{(r - a \sin t)} (-\sin \theta, \cos \theta, 0), \\
\frac{\partial \varphi}{\partial u_2} &= \frac{r}{(r - a \sin t)} (\sigma \cos t \cos \theta, \sigma \cos t \sin \theta, a - r \sin t).
\end{align*}
\]

From that we derive that \( \varphi \) is a conformal parametrization of \( T (r, a) \) satisfying

\[
\left( \frac{\partial \varphi}{\partial u_i}, \frac{\partial \varphi}{\partial u_j} \right) = \frac{r^2 \delta_{ij}}{(r - a \sin t)^2}.
\]

Moreover, a unit vector field normal to \( \varphi \) is given as follows:

\[
v(u_1, u_2) = -\frac{1}{(r - a \sin t)} ((a - r \sin t) \cos \theta, (a - r \sin t) \sin \theta, -\sigma \cos t).
\]

Therefore we conclude that

\[
g = \frac{\sigma^2 \sin t}{(r - a \sin t)}.
\]
On the other hand a new computation gives us

\[
\begin{align*}
\frac{\partial \nu}{\partial u_1} &= -\frac{(a - r \sin t)}{\sigma^2} \frac{\partial \varphi}{\partial u_1}, \\
\frac{\partial \nu}{\partial u_2} &= \frac{1}{a} \frac{\partial \varphi}{\partial u_2}.
\end{align*}
\]

(8)

From this we have \( k_1 = \frac{(a - r \sin t)}{(r^2 - a^2)} \) and \( k_2 = -\frac{1}{a} \). Taking into account (5), (7) and (8) we conclude from Lemma 1 that

\[ H = \frac{1}{4a} \left( ra \left( \sigma^2 - 1 \right) \sin t + (\sigma^2 + 1) \left( 2a^2 - r^2 \right) \right). \]

Now we have that \( H = 0 \) if and only if \( \sigma^2 = 1 \). Moreover, \( \sigma^2 = 1 \) yields \( H = \frac{1}{2a} \left( a^2 - 1 \right) \).

Since \( a < r \) we put \( a = r \sin \alpha, r = \sec \alpha \) and this completes the proof of the theorem.

We point out that \( H = 0 \) if and only if \( \sigma = 1 \) and \( r = \sqrt{2} \) which corresponds to the right triangle with two equal sides.

### 4 THE WILLMORE MEASURE ON \( T(r, a) \)

In this section we will present a simple way to compute \( \int_{T(r, a)} H^2 dA \) by using the parametrization of \( T(a, r) \) given by (4). We observe that if \( dA \) denotes the element of area of \( T(r, a) \) then its Willmore measure is given by

\[ (H^2 - K) dA = \frac{r^4}{4a^2\sigma^4} du_1 du_2. \]

Hence, using Gauss-Bonnet theorem, we easily conclude that

\[ \int_{T(r, a)} H^2 dA = \frac{r^4}{4a^2\sigma^4} \int_0^{2\pi} \int_0^{2\pi} u_1 du_2 = \frac{r^2}{a\sqrt{r^2 - a^2}} \pi^2. \]

(9)

Therefore the family of tori \( T \left( \sqrt{2}a, a \right) \), which corresponds to the family of right triangles with two equal sides, yields the minimum for \( \int_{T(r, a)} H^2 dA \) among all circular tori. Moreover, from (9) its value is (see also Willmore 1982)

\[ \int_{T(\sqrt{2}a, a)} H^2 dA = 2\pi^2. \]

Since \( a < r \), if we choose \( \alpha \) such that \( \sin \alpha = \frac{a}{r} \), we conclude from (9) the following corollary.

**Corollary 1.** Given a circular torus \( T(r, a) \subset \mathbb{R}^3 \) we have a circular torus \( T(\sec \alpha, \tan \alpha) \subset \mathbb{R}^3 \) such that \( \int_{T(r, a)} H^2 dA = \int_{T(\sec \alpha, \tan \alpha)} H_a^2 dA_a \). In other words, the family of circular tori with constant mean curvature in \( \mathbb{S}^3 \) cover all values of \( \int_{T(r, a)} H^2 dA. \)

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5 CONCLUDING REMARKS

We point out that Theorem 2 of K. Nomizu and B. Smyth (Nomizu and Smyth 1969) guarantees that a flat torus of constant mean curvature in $S^3$ is isometric to a product of circles. Then $\rho^{-1} T(a, r)$ is flat if and only if it has constant mean curvature. We notice if we set $\psi = \rho^{-1} \varphi$ where $\varphi$ was given by (4) then we have

$$
\psi(u_1, u_2) = \frac{1}{q(t)} (2\sigma^2 \cos \theta, 2\sigma^2 \sin \theta, 2a\sigma \cos t, r(\sigma^2 - 1) + a(\sigma^2 + 1) \sin t),
$$

where $q(t) = a(\sigma^2 - 1) \sin t + r(\sigma^2 + 1)$, (see(5)). Hence by using (3), (5), (6) and putting $z = u_1 + iu_2$ we conclude that

$$
d_{S^2}^2 \psi = e^{-2\varphi} d_{S^2}^2 \varphi = \frac{4r^2}{q^2(t)} |dz|^2.
$$

According to our theorem the metric $d_{S^2}^2 \psi$ is flat if and only if $\rho^{-1} T(r, a)$ has constant mean curvature in $S^3$. In this case we have

$$
\psi(u_1, u_2) = \frac{1}{\sqrt{a^2 + 1}} (\cos \theta, \sin \theta, a \cos t, a \sin t),
$$

i.e. $\rho^{-1} T(r, a)$ is isometric to the product of circles $S^1(\frac{1}{\sqrt{a^2 + 1}}) \times S^1(\frac{a}{\sqrt{a^2 + 1}})$. We note that this yields $\cos \alpha = \frac{1}{\sqrt{a^2 + 1}}$ and $\sin \alpha = \frac{a}{\sqrt{a^2 + 1}}$, i.e. $\rho(S^1(\cos \alpha) \times S^1(\sin \alpha)) = T(\sec \alpha, \tan \alpha)$.

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