A note on the connectedness locus of the families of polynomials $P_c(z) = z^n - cz^{n-j}$

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ABSTRACT

Let $j$ be a positive integer. For each integer $n > j$ we consider the connectedness locus $\mathcal{M}_n$ of the family of polynomials $P_c(z) = z^n - cz^{n-j}$, where $c$ is a complex parameter. We prove that $\lim_{n \to \infty} \mathcal{M}_n = \mathbb{D}$ in the Hausdorff topology, where $\mathbb{D}$ is the unitary closed disk $\{ c; |c| \leq 1 \}$.

Key words: Julia set, connectedness locus, hyperbolic components, principal components.

1 INTRODUCTION

In (Milnor 2009), J. Milnor considers the complex 1-dimensional slice $S_1$ of the cubic polynomials that have a superattracting fixed point. He gives a detailed pictured of $S_1$ in dynamical terms. In (Roesch 2007), Roesch generalizes these results for families of polynomials of degree $n \geq 3$ having a critical fixed point of maximal multiplicity. This set of polynomials is described -modulo affine conjugacy- by the polynomials $P_c(z) = z^n - cz^{n-1}$. Roesch proved that the global pictures of the connectedness locus of this family of polynomials is a closed topological disk together with “limbs” sprouting off it at the cusps of Mandelbrot copies. In this note, we consider a positive integer $j$, and for each integer $n > j$, we consider the family of polynomials $P_c(z) = z^n - cz^{n-j}$, where $c$ is a complex parameter. By definition, the connectedness locus $\mathcal{M}_n$ of this family of polynomials consists of all parameters $c$ such that the Julia set of $P_c(z)$ is connected or equivalently if the orbit of every critical point of $P_c(z)$ is bounded (see Carleson and Gamelin 1992). Since for all parameter $c; z = 0$ is a superattracting fixed point of $P_c(z)$, we deduce that $\mathcal{M}_n$ consists of all parameter $c$ such that the orbit of every non-zero critical point of $P_c(z)$ is bounded. We also consider the space of non-empty compacts subsets of the plane equiiped with the Hausdorff distance (see Douady 1994). We obtain the following result about the size of $\mathcal{M}_n$.

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THEOREM A. \( \mathcal{M}_a \) is a non-empty compact subset of the plane and

\[
\lim_{n \to \infty} (\mathcal{M}_n) = D,
\]

in the Hausdorff topology, where \( D \) is the unitary closed disk \( \{ c; |c| \leq 1 \} \).

2 PROOF OF THEOREM A

The proof of the Theorem is based in the following results.

LEMMA 2.1. For \( n > 3j \), the closed unitary disk \( D \) is contained in \( \mathcal{M}_n \).

PROOF. Let \( c \in D \) and let \( k = \left( \frac{n-j}{n} \right)^{\frac{n-j}{j}} \). Since \( n > 3j \), we have that \( j < \frac{1}{3} \), so \( k < \frac{1}{3} \). Let \( z_c \) be a non-zero critical point of \( P_c(z) \). Then, \( z_c^n = \frac{n}{n-j} \), and this implies that

\[
P_c(z_c) = z_c^n - cz_c^{n-j} = z_c^n - \frac{n}{n-j} z_c^n = -\frac{j}{n-j} z_c^n.
\]

This and the fact that

\[
|z_c|^{n-1} = \left( \frac{n-j}{n} \right)^{\frac{n-j}{j}} |c|^{\frac{n-1}{j}}
\]

imply that

\[
|P_c(z_c)| = \frac{j}{n-j} \left( \frac{n-j}{n} \right)^{\frac{n-j}{j}} |z_c| = k|c|^{\frac{n-1}{j}} |z_c|.
\]

Hence, since \( |c| < 1 \), \( P_c(z_c) | \leq k |z_c| \).

By induction, suppose that \( |P_c^{q-1}(z_c)| \leq k^q |z_c| \). Then,

\[
|P_c^{q+1}(z_c)| = |P_c^q(z_c)|^{n-j} |(P_c^q(z_c))^{n-j} - c| = |P_c^q(z_c)|^{n-j} |(P_c^q(z_c))^{n-j} - \frac{n}{n-j} z_c^n| \leq k^q |z_c| |z_c|^{n-j} \left( k^{\frac{n-j}{n-j}} + \frac{n}{n-j} \right) \]

where the last inequality is true because \( |z_c| < 1 \) and \( k < 1 \).

On the other hand, since \( n > 3j \), \( \frac{n}{n-j} < \frac{3}{2} \) and \( q(n-j-1)-1 > 1 \). Thus,

\[
k^{q(n-j-1)-1} \left( k + \frac{n}{n-j} \right) < k \left( k + \frac{3}{2} \right) < \frac{1}{2} \left( \frac{3}{2} + \frac{3}{2} \right) = 1.
\]

Comibinated with the estimate above, this gives \( |P_c^{q+1}(z_c)| \leq k^{q+1} |z_c| \). Hence, \( |P_c^q(z_c)| \leq k^q |z_c| \) for all positive integer \( q \). Since \( k < 1 \), we deduce that the orbit \( \{ P_c^q(z_c) \} \) is bounded and Lemma 2.1 is proved.

LEMMA 2.2. If \( n > j \), then \( \mathcal{M}_n \) is a subset of the disk \( \{ c; |c| \leq \left( \frac{n-j}{j} \right)^{\frac{1}{n-j}} \left( \frac{n}{n-j} \right)^{\frac{2}{j}} \} \).

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PROOF. Let $|c| > \left( \frac{n-j}{j} \right)^{\frac{q}{n-j}} \left( \frac{n}{n-j} \right)^{n-j}$. By definition of $\mathcal{M}_n$, we have that, in order to prove Lemma 2.2, it is sufficient to prove that, for each non-zero critical point $z_c$ of $P_c(z) = z^n - cz^{n-j}$, the orbit $\{P_c^q(z_c)\}$ is not bounded.

Let $k = \frac{n-j}{n-j} |z_c|^{n-1}$. We claim that $k > \left( \frac{n-j}{n-j} \right)^{\frac{n-1}{j}}$ and hence $k > 1$.

In fact, since $z_c^j = \frac{n-j}{n-j} c$,

$$k = \frac{j}{n-j} \left( \frac{n-j}{n} \right)^{\frac{n-j}{j}} |c|^{\frac{n-j}{j}} = \frac{j}{n-j} \left( \frac{n-j}{n} \right)^{\frac{n-j}{j}} \left( \frac{n-j}{n} \right) \left( \frac{n}{n-j} \right) > \left( \frac{n-j}{n-j} \right)^{\frac{n-1}{j}},$$

and the claim is proved.

Now, we have that

$$|P_c(z_c)| = |z_c^n - cz_c^{n-j}| = |z_c^n - \frac{n}{n-j} z_c^n| = \frac{j}{n-j} |z_c|^{n-1} = k |z_c|$$

By induction, suppose that $|P_c^q(z_c)| \geq k^q |z_c|$. Then,

$$|P_c^{q+1}(z_c)| = |P_c^q(z_c) z_c^n - P_c^q(z_c) | = |P_c^q(z_c)|^{n-j} |z_c|^j \left( \frac{P_c^q(z_c)}{z_c} \right)^j - n \left( \frac{P_c^q(z_c)}{z_c} \right)^j \geq k^{q(n-j)} |z_c|^n \left( k^n - \frac{n}{n-j} \right) = k^{q(n-j)} k \left( \frac{n-j}{n-j} \right) \left( \frac{n}{n-j} \right) \left( \frac{n-j}{n-j} k^{q(j-1)} \right) |z_c| \geq n \left( \frac{n-j}{n-j} k^{q(j-1)} \right) |z_c| \geq n \left( \frac{n-j}{n-j} k^{q(j-1)} - 1 \right) k^{q+1} |z_c|.$$

where the last inequality follows from the Claim above.

On the other hand, let $s = q(n-j) - 1$. Then, $s > 1$ and

$$\frac{n}{j} \left( \frac{n-j}{n} \right)^{s-1} - 1 = \frac{n}{j} \left( \frac{n-j}{n} - 1 \right) \left( \frac{n-j}{n} \right)^{s-1} + \ldots + 1$$

$$= \frac{n-j}{n} \left( \frac{n-j}{n} \right)^{s-1} + \ldots + 1 > 1.$$

Combinated with the estimates above, this gives $|P_c^{q+1}(z_c)| \geq k^{q+1} |z_c|$. Hence, $|P_c^q(z_c)| > k^q |z_c|$ for all positive integer $q$. Since $k > 1$, we conclude that, for each critical point $z_c$ of $P_c(z)$, the orbit $\{P_c^q(z_c)\}$ is not bounded, and Lemma 2.2 is proved.

Now, we prove Theorem A. By Lemma 2.2, $\mathcal{M}_n$ is bounded.

Let $J = \left( \frac{n-j}{j} \right)^{1/j} \left( \frac{n}{n-j} \right)^{1/2}$ and let $L$ be a positive integer such that $L^j - J > 1$. Suppose by contradiction that $\mathcal{M}_n$ is not closed. Then, there exists $d$ in the boundary $\partial \mathcal{M}_n$ of $\mathcal{M}_n$ such that the orbit $\{P_d^j(z_d)\}$ is not bounded for some non-zero critical point $z_d$ of $P_d(z)$. Hence, there exists a positive integer
such that $|P_d^q(z_d)| > L$. Since $z_d = \frac{n-j}{n}d$, we can choose a local branch of $F(c) = \left(\frac{n-j}{n}c\right)^j$ in a neighborhood $V$ of $d$ such that $|P_d^q(z)| > L$, for all $c \in V$. Since $d \in \partial \mathcal{M}_n$, there exists $c \in \mathcal{M}_n \cap V$ such that $|P_c^q(z_c)| > L$. By Lemma 2.2, $|c| < j$. Let $\omega = P_c^q(z_c)$. Then,

$$|\omega| > |c| > L^j - J > 1,$$

thus,

$$|P_c(\omega)| = |\omega^{n-j}||\omega^j - c| > L.$$  

By induction, suppose that $|P_c^m(\omega)| > L^m$. Then, $|P_c^m(\omega)|^j - |c| > L^{mj} - J > L$. It follows that,

$$|P_c^{m+1}(\omega)| = |P_c^m(\omega)|^{n-j}|(P_c^m(\omega))^j - c| > L^{m(n-j)}L > L^{m+1}.$$ 

Hence, the orbit $\{P_c^j(z_c)\}$ is not bounded. This is a contradiction because $c \in \mathcal{M}_n$. Therefore, $\mathcal{M}_n$ is closed, so it is compact. Now, Lemmas 2.1 and 2.2 and the fact that $\lim_{n \to \infty} \left(\frac{1}{L^n}\right)^{n-j} \left(\frac{n}{n-j}\right)^2 = 1$ imply that $\lim_{n \to \infty} \mathcal{M}_n = D$ in the Hausdorff topology, and Theorem A is proved.

RESUMO

Seja $j$ um inteiro positivo. Para cada inteiro $n > j$, consideramos o locus conexo $\mathcal{M}_n$ da família de polinômios $P_c(z) = z^n - cz^{n-j}$, onde $c$ é um parâmetro complexo. Provamos que $\lim_{n \to \infty} \mathcal{M}_n = D$ na topologia de Hausdorff; onde $D$ é o disco unitário $\{c; |c| \leq 1\}$.

Palavras-chave: Conjunto de Julia, locus conexo, componentes hiperbólicas, componente principal.

REFERENCES


