Parameter induction in continuous univariate distributions: Well-established $G$ families

MUHAMMAD H. TAHIR$^1$ and SARALEES NADARAJAH$^2$

$^1$Department of Statistics, Baghdad Campus, The Islamia University of Bahawalpur, Bahawalpur 63100, Pakistan
$^2$School of Mathematics, University of Manchester, Oxford Road, Manchester, M13 9PL, UK

Manuscript received on June 9, 2014; accepted for publication on November 25, 2014

ABSTRACT

The art of parameter(s) induction to the baseline distribution has received a great deal of attention in recent years. The induction of one or more additional shape parameter(s) to the baseline distribution makes the distribution more flexible especially for studying the tail properties. This parameter(s) induction also proved helpful in improving the goodness-of-fit of the proposed generalized family of distributions. There exist many generalized (or generated) $G$ families of continuous univariate distributions since 1985. In this paper, the well-established and widely-accepted $G$ families of distributions like the exponentiated family, Marshall-Olkin extended family, beta-generated family, McDonald-generalized family, Kumaraswamy-generalized family and exponentiated generalized family are discussed. We provide lists of contributed literature on these well-established $G$ families of distributions. Some extended forms of the Marshall-Olkin extended family and Kumaraswamy-generalized family of distributions are proposed.

Key words: Beta-distribution, exponentiated family, Kumaraswamy distribution, Marshall-Olkin family, McDonald distribution, reliability properties.

INTRODUCTION

There has been an increased interest in developing generalized (or generated) $G$ families of distributions by introducing one or more additional shape parameter(s) to the baseline distribution. There is no doubt that the popularity and the use of Euler-beta and -gamma functions in some $G$ families of distributions have attracted the attention of statisticians, mathematicians, scientists, engineers, economists, demographers and other applied researchers. One reason might be the computational and analytical facilities available in programming softwares like R (packages), ox5, Python, Matlab, Maple and Mathematica, through which researchers can easily tackle problems involved in computing incomplete-beta and -gamma functions in $G$ families. The second reason is the tail properties of $G$ distributions that can easily be explored by inducting one or more additional shape parameter(s) to the baseline distribution. Thirdly, this parameter(s) induction has also proved to be helpful in improving the goodness-of-fit of the proposed $G$ family of distributions. Fourthly, $G$ families have the ability to fit skewed data better than existing

AMS (2010): 60E05, 62E10, 62N05

Correspondence to: Saralees Nadarajah
E-mail: mbbsssn2@manchester.ac.uk
distributions (Pescim et al. 2010). Lastly, the Kumaraswamy $G$ family of distributions can generate effective models for censored data (Cordeiro and de Castro 2011).

There exists many generalized (or generated) $G$ family of distributions like Azzalini’s skewed family (Azzalini 1985), Marshall-Olkin extended (MOE) family (Marshall and Olkin 1997), exponentiated family (EF) of distributions (Gupta et al. 1998), beta-generated (beta $G$) family (Eugene et al. 2002, Jones 2004a), Ferreira and Steel’s skewed family (Ferreira and Steel 2006), transmuted family (Shaw and Buckley 2007, Aryal and Tsokos 2009, 2011), Gupta and Gupta’s skewed family (Gupta and Gupta 2008), gamma-generated (GG) families (Zografos and Balakrishnan 2009, Ristić and Balakrishnan 2012, Torabi and Montazari 2012, Nadarajah et al. 2015), transformed-transformer (T-X) family (Alza-Atreh 2011), Kumaraswamy generalized (Kw $G$) family (Cordeiro and de Castro 2011, Nadarajah et al. 2012a, Hussain 2013), generalized beta generated (GBG) or McDonald generalized (Mc $G$) family (Alexander et al. 2012), beta extended $G$ family (Cordeiro et al. 2012f), Kummer beta generalized family (Pescim et al. 2012), exponentiated transformed-transformer family (ET-X) (Alzaghal et al. 2013), exponentiated generalized (Exp $G$) family (Cordeiro et al. 2013e), geometric exponential-Poisson family (Nadarajah et al. 2013a), logistic-generated (Lo $G$) family (Torabi and Montazari 2014), Marshall-Olkin extended family (Alshangiti et al. 2014), log-gamma generated (LG $G$) families, (Amini et al. 2014), Weibull $G$ family (Bourguignon et al. 2014), Libby-Novick beta family (Cordeiro et al. 2014e), truncated negative-binomial family (Nadarajah et al. 2014a), modified beta $G$ family (Nadarajah et al. 2014b) and exponentiated exponential-Poisson family (Ristić and Nadarajah 2014). These $G$ families of distributions have received a great deal of attention in recent years. In this paper, we discuss the EF, MOE, beta $G$, Mc $G$, Exp $G$ and Kw $G$ families of distributions and provide additional literature (in chronological order) on these six families of distributions. We also propose some extended forms of the Kw $G$ families of distributions by introducing one more additional shape parameter(s).

Because of the length of this paper, we have not given details like probabilistic interpretations, analytical properties, estimation methods, simulation algorithms and applications. These details can be obtained from the cited references.

The rest of the paper is organized as follows. In Section 2, the EF of distributions is defined and a list of contributed work is presented. In Section 3, we describe the MOE family and propose one generalized MOE family of distributions. The contributed literature on the MOE family is also presented. In Section 4, the beta $G$ family of distributions is discussed. The contributions to the beta $G$ family of distributions are also listed in this section. In Section 5, the McDonald distributions and Mc $G$ families of distributions are described. The contributed work on Mc $G$ families of distributions is also presented. Section 6 consists of Kumaraswamy distributions and Kw $G$ families of distributions. Some new types of the Kumaraswamy distribution and Kw $G$ families of distributions are proposed. The contributed work on the Kw $G$ family of distributions is also listed in this section. Section 7 ends the paper with some final remarks.

**EXPONENTIATED FAMILY (EF) OF DISTRIBUTIONS**

The genesis of this family can be traced back to the first half of the nineteenth century when Gompertz (1825) and Verhulst (1838, 1845, 1847) used the cumulative distribution function (cdf) $G(t) = (1 - \rho e^{-\lambda t})^\alpha$ for $t > \lambda^{-1} \log \rho$, where $\rho, \alpha$ and $\lambda$ are positive real numbers. Ahuja and Nash (1967) introduced the generalized Gompertz-Verhulst family of distributions to study growth curve mortality. Gompertz-Verhulst’s cdf was the first member of the EF of distributions. The exponentiated exponential (EE) distribution is its particular case.
for $\rho = 1$. The properties and estimation methods for parameters of the EF of distributions have been studied by many authors, see Mudholkar and Srivastava (1993), Mudholkar and Hutson (1996), Mudholkar et al. (1995), Gupta and Kundu (1999, 2001a, b, 2007), Pal et al. (2006), Nadarajah and Kotz (2006a), Nadarajah (2011) and Nadarajah et al. (2013b). The EF of distributions is also known as Lehmann alternatives (LAs) (Lehmann 1953) or proportional reversed hazard rate model (PHRM) (see Gupta et al. 1998, Gupta and Gupta 2007, Martinez-Florez et al. 2013), while other authors referred to the EF of distributions as max-stable family (Sarabia and Castillo 2005) and $F^\alpha$- distributions (Gupta et al. 1998, Al-Hussaini, 2010a, b, 2012, Shakil and Ahsanullah 2012, Hamedani 2013 and Ghitany et al. 2013).

In literature there exist four different ways for obtaining the EF of distributions.

LEHMANN ALTERNATIVE 1 (LA1)

The method of Lehmann alternative 1 (LA1) (due to Lehmann (1953)) has received a great deal of attention in developing the EF of distributions.

If $G(z)$ is the cdf of the baseline distribution, then an EF of distributions is defined by taking the $\alpha$th-power of $G(z)$ as

$$ F(z) = G(z)^\alpha, $$(2.1)

where $\alpha > 0$ is a positive real parameter. The variable $z$ can take any of the form $z = x$ or $z = x - \mu$ or $z = \frac{x - \mu}{\sigma}$ or $z = k \left( \frac{x - \mu}{\sigma} \right)^2$. The probability density function (pdf) corresponding to (2.1) is

$$ f(z) = \alpha g(z) G(z)^{\alpha - 1}, $$ (2.2)

where $g(z) = dG(z)/dz$ denotes the pdf of $G$. For any lifetime random variable $t$, the survival (reliability) function ($sf$), $F(t)$, the hazard (failure) rate function ($hrf$), $h(t)$, the reversed hazard rate function ($rhrf$), $r(t)$, and the cumulative hazard rate function (chrf), $H(t)$, associated with (2.1) and (2.2) are

$$ F(t) = 1 - G(t)^\alpha, $$
$$ h(t) = \alpha g(t) G(t)^{\alpha - 1} \left[ 1 - G(t)^\alpha \right]^{-1}, $$
$$ r(t) = \alpha g(t) G(t)^{-1}, $$

and

$$ H(t) = -\log \left[ 1 - G(t)^\alpha \right]. $$

LEHMANN ALTERNATIVE 2 (LA2)

The method of Lehmann alternative 2 (LA2) (due to Lehmann (1953)) has received less attention.

If $G(z)$ is the cdf and $\overline{G}(z) = 1 - G(z)$ is the sf of the baseline distribution, then an EF of distributions is defined by taking one minus the $\alpha$th-power of $\overline{G}(z)$ as

$$ F(z) = 1 - [\overline{G}(z)]^\alpha, $$

where $\alpha$ is a positive real parameter. The LA2 cdf may also be written as

$$ F(z) = 1 - [1 - G(z)]^\alpha. $$ (2.3)

The pdf corresponding to (2.3) is

$$ f(z) = \alpha g(z) \left[ 1 - G(z) \right]^{\alpha-1}. $$ (2.4)
For any lifetime random variable \( t \), the sf, hrf, rhrf and chrf associated with (2.3) and (2.4) are

\[
\bar{F}(t) = [1 - G(t)]^α,
\]

\[
h(t) = αg(t) [1 - G(t)]^{α-1},
\]

\[
r(t) = αg(t) [1 - G(t)]^{α-1} \{1 - [1 - G(t)]^{α}\}^{-1},
\]

and

\[
H(t) = -α\log [1 - G(t)].
\]


**Using Transformation** \( z = \log(x) \), \( x > 0 \)

Nadarajah (2005a) developed exponentiated distributions by applying the transformation \( z = \log(x) \) to (2.3). The cdf, pdf and the hrf of the exponentiated distribution are

\[
F(x) = 1 - [1 - G(e^x)]^α,
\]

\[
f(x) = ae^{x}g(e^x) [1 - G(e^x)]^{α-1}
\]

and

\[
h(x) = ae^{x}g(e^x) [1 - G(e^x)]^{-1}.
\]

**Using Transformation** \( z = -\log(x) \), \( x > 0 \)

Nadarajah (2005b) developed exponentiated distributions by applying the transformation \( z = -\log(x) \) to (2.3). The cdf, pdf and the hrf of the exponentiated distribution are

\[
F(x) = [1 - G(e^{-x})]^{α},
\]

\[
f(x) = ae^{-x}g(e^{-x}) [1 - G(e^{-x})]^{α-1}
\]

and

\[
h(x) = ae^{-x}g(e^{-x}) [1 - G(e^{-x})]^{α-1} \{1 - [1 - G(e^{-x})]^{α}\}^{-1}.
\]

A list of papers on the EF of distributions is presented in Table I.

<table>
<thead>
<tr>
<th>S.No.</th>
<th>Pioneer year</th>
<th>Distribution</th>
<th>Author(s)</th>
</tr>
</thead>
</table>
| 1     | 1967         | Exponentiated exponential distribution | Ahuja and Nash (1967)  
Gupta et al. (1998)  
Gupta and Kundu (1999, 2001a, b, 2007)  
Nadarajah (2011)  
Venkatesan and Sundaram (2011) |
| 2     | 1993         | Exponential Weibull distribution | Mudholkar and Srivastava (1993)  
Mudholkar et al. (1995)  
Mudholkar and Hutson (1996)  
Gupta et al. (1998)  
Jiang and Murthy (1999) |

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<th>S.No.</th>
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<td>Nadarajah and Kotz (2006a)</td>
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<td>Nadarajah and Gupta (2007)</td>
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<td>Shawky and Abu-Zinadah (2009)</td>
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<td>Abd-Elfattah (2011)</td>
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<td>Nadarajah and Kotz (2006a)</td>
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<td>Jamjoom and Al-Saiary (2012)</td>
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<td>Al-Nasser and Al-Omari (2013)</td>
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<td>Marwa et al. (2013)</td>
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<td>Adeyemi and Adebani (2006)</td>
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<td>2005</td>
<td>Exponentiated beta distribution</td>
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<td>Exponentiated log-logistic distribution</td>
<td>Rosaiyah et al. (2006)</td>
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<td>Aslam and Jun (2010)</td>
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<td>Rao et al. (2012, 2013)</td>
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<td>Persson and Rydén (2010)</td>
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<td>2008</td>
<td>Exponential modified Weibull</td>
<td>Carrasco et al. (2008)</td>
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<td>Elbatal (2011)</td>
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<td>2009</td>
<td>Exponentiated extreme value</td>
<td>Cho et al. (2009)</td>
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<td>Exponentiated Lindley distribution</td>
<td>Nadarajah et al. (2011)</td>
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<td>16</td>
<td>2011</td>
<td>Extended generalized exponential</td>
<td>Kundu and Gupta (2011)</td>
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TABLE I (continuation)

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<th>S.No.</th>
<th>Pioneer year</th>
<th>Distribution</th>
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<td>17</td>
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<td>Exponentiated Burr XII distribution</td>
<td>Al-Hussaini and Hussein (2011a, b)</td>
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<td>Maswadah (2013)</td>
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<td>18</td>
<td>2011</td>
<td>Exponentiated generalized gamma distribution</td>
<td>Cordeiro et al. (2011a)</td>
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<td>2011</td>
<td>Exponentiated generalized inverse Gaussian distribution</td>
<td>Lemonte and Cordeiro (2011)</td>
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<td>2012</td>
<td>Exponentiated inverted Weibull distribution</td>
<td>Flaih et al. (2012)</td>
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<td>Kim et al. (2012)</td>
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<td>Maswadah (2013)</td>
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<td>21</td>
<td>2012</td>
<td>Exponentiated Kumaraswamy distribution</td>
<td>Kumar (2012)</td>
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<td>24</td>
<td>2013</td>
<td>Exponentiated modified Weibull extension distribution</td>
<td>Sarhan and Apaloo (2013)</td>
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<tr>
<td>25</td>
<td>2013</td>
<td>Exponentiated generalized linear exponential distribution</td>
<td>Sarhan et al. (2013)</td>
</tr>
<tr>
<td>26</td>
<td>2013</td>
<td>Exponentiated Dagum distribution</td>
<td>Khan (2013)</td>
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<tr>
<td>27</td>
<td>2013</td>
<td>Exponentiated sinh Cauchy distribution</td>
<td>Cooray (2013)</td>
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</table>

MARBELL-OLKIN EXTENDED (MOE) FAMILY OF DISTRIBUTIONS

Marshall and Olkin (1997) proposed a flexible semi-parametric family of distributions and defined a new sf $F^MO (x)$ by introducing an additional parameter $\alpha > 0$. Marshall and Olkin (1997) called $\alpha$ a tilt parameter and interpreted $\alpha$ in terms of the behavior of the hrfs of $F^MO$ and $G$. Their ratio is increasing in $t$ for $\alpha \geq 1$ and decreasing in $t$ for $0 < \alpha < 1$. Nanda and Das (2012) reinterpreted $\alpha$ as a tilt parameter since the hrfs of the underlying family is shifted below ($\alpha \geq 1$) or above ($0 < \alpha \leq 1$) the hrfs of the baseline distribution. Specifically, for all $t \geq 0$, $h^MO (t) \leq h(t)$ when $\alpha \geq 1$, and $h^MO (t) \geq h(t)$ when $0 < \alpha < 1$, where $h^MO (t)$ and $h(t)$ are the hrfs of the MOE and baseline distributions.

For any baseline pdf $g(t)$, cdf $G(t) = P (T \leq t)$ and sf $G^com(t) = P (T > t)$ of the baseline distribution, the sf $F^MO (t)$ of the MOE family of distributions is defined by

$$F^MO (t) = \frac{\alpha G(t)}{1 - \alpha G(t)} = \frac{\alpha G(t)}{G(t) + \alpha G(t)} = \frac{[1 - G(t)]}{\alpha + \alpha G(t)},$$

where $-\infty < t < \infty$, $\alpha > 0$ and $\overline{a} = 1 - \alpha$. The cdf and pdf associated with (3.1) are

$$F^MO (t) = \frac{\overline{G}(t)}{1 - \overline{a}G(t)} = \frac{\overline{G}(t)}{G(t) + \overline{a}G(t)} = \frac{1 - G(t)}{\alpha + \overline{a}G(t)},$$

and

$$f^MO (t) = \frac{ag(t)}{[1 - \overline{a}G(t)]^2} = \frac{ag(t)}{[\alpha + \overline{a}G(t)]^2},$$

where $-\infty < t < \infty$, $\alpha > 0$ and $\overline{a} = 1 - \alpha$. If $\alpha = 1$, then we have $F^MO (t) = \overline{G}(t)$. Other reliability measures like the hrf, rhrf and chrf associated with (3.1) are

$$h^MO (t) = \frac{f^MO (t)}{F^MO (t)} = \frac{g(t)}{\overline{G}(t)} \frac{1}{[1 - \overline{a}G(t)]} = \frac{h(t)}{1 - \overline{a}G(t)} = \frac{h(t)}{\alpha + \overline{a}G(t)},$$

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\[ r^{MO}(t) = \frac{f^{MO}(t)}{F^{MO}(t)} = a \frac{g(t)}{G(t)} \frac{1}{1 - \frac{aG(t)}{G(t)}} = \frac{ah(t)}{1 - \frac{aG(t)}{G(t)}} \text{ or } \frac{ah(t)}{a + aG(t)}, \]

and

\[ H^{MO}(t) = -\log \left[ \frac{aG(t)}{1 - \frac{aG(t)}{G(t)}} \right] \text{ or } -\log \left\{ \frac{a [1 - G(t)]}{a + aG(t)} \right\}, \]

where \( h(t) \) is the hrf of the baseline distribution.

Note that if we define

\[ F^{MO}(t) = \frac{G(t)}{1 - \frac{aG(t)}{G(t)}} \]

then

\[ F^{MO}(t) = \frac{aG(t)}{1 - \frac{aG(t)}{G(t)}} \text{ and } f^{MO}(t) = \frac{g(t)}{[1 - \frac{aG(t)}{G(t)}]^2}, \]

For more general results on the MOE family of distributions, the reader is referred to Barakat et al. (2009), Jose (2011), Krishna (2011), Barreto-Souza et al. (2013) and Cordeiro et al. (2014c).

**EXISTING GENERALIZED MOE FAMILY OF DISTRIBUTIONS**

In this section, we describe existing generalized Marshall-Olkin families of distributions.

Jayakumar and Mathew (2008) proposed a generalization of the Marshall and Olkin (1997) family of distributions (by using the LA1 approach) as

\[ F^{GMO}(t) = \left[ \frac{aG(t)}{1 - \frac{aG(t)}{G(t)}} \right]^\theta, \quad (3.2) \]

where \( -\infty < t < \infty, \alpha > 0, \) and \( \theta > 0 \) is an additional shape parameter. When \( \theta = 1, F^{GMO}(t) = F^{MO}(t). \)

The cdf and the pdf associated with (3.2) are

\[ F^{GMO}(t) = 1 - \left[ \frac{aG(t)}{1 - \frac{aG(t)}{G(t)}} \right]^\theta, \]

and

\[ f^{GMO}(t) = \theta \left[ \frac{aG(t)}{1 - \frac{aG(t)}{G(t)}} \right]^{\theta - 1} \left\{ \frac{ag(t)}{[1 - \frac{aG(t)}{G(t)}]^2} \right\}. \]

Other reliability measures like the hrf, rhrf and chrf associated with (3.2) are

\[ H^{GMO}(t) = -\log \left\{ 1 - \left[ \frac{aG(t)}{[1 - \frac{aG(t)}{G(t)}]} \right]^\theta \right\}, \]

where \( h(t) \) is the hrf of the baseline distribution.
A NEW GENERALIZED MOE FAMILY OF DISTRIBUTIONS

Here, we propose another generalization of the Marshall and Olkin (1997) family of distributions. Using the LA2 approach to the sf of the MOE family of distributions, we obtain

\[ F_{G_{2}^{MO}}(t) = 1 - \left[ 1 - \frac{aG(t)}{1 - aG(t)} \right]^\theta, \] (3.3)

where \(-\infty < t < \infty, \alpha > 0,\) and \(\theta > 0\) is the additional shape parameter. When \(\theta = 1, F_{G_{2}^{MO}}(t) = F_{MO}(t).\)

The cdf and the pdf associated with (3.3) are

\[ F_{G_{2}^{MO}}(t) = 1 - \frac{aG(t)}{1 - aG(t)} \theta, \]

and

\[ f_{G_{2}^{MO}}(t) = \theta \frac{a\bar{G}(t)G(t)^{\theta - 1}}{1 - a\bar{G}(t)} \] or \(\frac{\theta a\bar{G}(t)G(t)^{\theta - 1}}{a + a\bar{G}(t)}\).

Other reliability measures like the hrf, rhrf and chrf associated with (3.3) are

\[ h_{G_{2}^{MO}}(t) = \theta \frac{a\bar{G}(t)G(t)^{\theta - 1}}{a + a\bar{G}(t)} \left\{ 1 - \left[ 1 - \frac{a\bar{G}(t)}{1 - a\bar{G}(t)} \right]^\theta \right\}^{-1}, \]

\[ r_{G_{2}^{MO}}(t) = \theta a \frac{aG(t) - 1}{a + a\bar{G}(t)} = \frac{\theta a r(t)}{a + a\bar{G}(t)}, \]

and

\[ H_{G_{2}^{MO}}(t) = -\log \left\{ 1 - \left[ 1 - \frac{a\bar{G}(t)}{1 - a\bar{G}(t)} \right]^\theta \right\}, \]

where \(r(t)\) is the rhrf of the baseline distribution.

The construction in (3.3) is similar to that due to Jayakumar and Mathew (2008). But there is an important distinction. Suppose that a system consists of \(\theta\) independent components. Suppose too that each component has a lifetime with the sf given by \(a\bar{G}(t) / [1 - a\bar{G}(t)].\) Then (3.2) is the sf of the minimum of the lifetimes and (3.3) is the sf of the maximum of the lifetimes. So, (3.2) can be used to model the minimum of the lifetimes and (3.3) can be used to model the maximum of the lifetimes.

SEMI-TYPE PROCESSES BASED ON CHARACTERISTIC FUNCTION

In this section, we briefly discuss semi-Pareto, semi-Burr, semi-Laplace, semi-logistic and semi-Weibull distributions based on the characteristic function (cf) \(\psi(t)\) of the baseline distribution. The concept of semi-type distributions arose from the minification process. Tavares (1980) defined a minification process as observations in a process generated by

\[ X_n = k \min (X_{n-1}, \varepsilon_n), \] (3.4)

where \(n \geq 1, k > 1\) is a constant and \(\{\varepsilon_n\}\) is an innovation process of independent and identically distributed random variables. Here, \(\{X_n\}\) is called the first order autoregressive AR(1) minification process. There exists many modified minification processes.
Linnik (1963) introduced the $\alpha$-Laplace distribution, a symmetric distribution defined on $(-\infty, \infty)$. For $\alpha = 2$, the Linnik distribution reduces to the Laplace distribution. Pillai (1985) generalized the Linnik distribution and introduced the semi-$\alpha$-Laplace distribution.


Pillai (1985) proposed the semi-$\alpha$-Laplace distribution. Its sf is

$$F_{SLap}(t) = \frac{1}{1 + \psi(t)},$$

where $\psi(t)$ satisfies the functional equation

$$\psi(t) = \frac{1}{p} \psi \left(\frac{tp^{1/\alpha}}{\log p} \right),$$

(3.5)

where $\alpha > 0$ and $0 < p < 1$. The solution of (3.5) is $\psi(t) = |t|^\alpha \eta(t)$, where $\eta(t)$ is periodic in $\log |t|$. In the particular case $\eta(t) = c$, the semi-$\alpha$-Laplace distribution reduces to the Linnik distribution.

A random variable $T$ is said to have the semi-Pareto distribution if its sf is

$$F_{SP}(t) = \frac{1}{1 + \psi(t)},$$

where $t > 0$ and $\psi(t)$ satisfies the functional equation

$$\psi(t) = \frac{1}{p} \psi \left(pt^{1/\gamma} \right),$$

(3.6)

where $0 < p < 1$, $t > 0$ and $\gamma > 0$. The solution of (3.6) is $\psi(t) = t^\gamma \eta(t)$, where $\eta(t)$ is periodic in $\log t$ with period $\left(\frac{-2\pi}{\log p} \right)$. Further details are in Pillai (1991) and Pillai et al. (1995).

If $\psi(t) = t^\gamma$ (that is for $\eta(t) = 1$), we obtain the semi-Pareto distribution of type III having the sf

$$F_{SP3}(t) = \frac{1}{1 + t^\gamma},$$

where $t > 0$ and $\gamma > 0$. For details, see Chrapek et al. (1996), Balakrishna (1998) and Cifarelli et al. (2010).

A random variable $T$ is said to have the semi-Burr distribution if its sf is

$$F_{SB}(t) = \left[\frac{1}{1 + \psi(t)}\right]^\beta,$$

where $t > 0$, $\beta > 0$ and $\psi(t)$ satisfies the same functional as (3.6).

Cifarelli et al. (2010) expressed the sf of the semi-Burr distribution as

$$F_{SB}(t) = \frac{1}{\left[1 + \psi(t)\right]^{b+1}},$$

where $\psi(t)$ satisfies the same functional as (3.6) and $b > 0$.

According to Arnold (1992) and Jayakumar and Mathew (2005), a random variable $T$ is said to have the semi-logistic distribution if its sf is
where \( \psi(t) \) is a nondecreasing and right-continuous function satisfying
\[
\psi(t) = \frac{1}{p} \left( t + \frac{1}{\sigma} \log p \right),
\]
where \( 0 < p < 1 \), \( t > 0 \), and \( \sigma > 0 \).

According to Jose (1994) and Thomas and Jose (2005), a random variable \( T \) is said to have the semi-Weibull distribution if its sf is
\[
F_{SW}(t) = \exp \left[ -\psi(t) \right],
\]
where \( \psi(t) \) satisfies the functional equation
\[
p \psi(t) = \psi \left( \frac{1}{p} \frac{1}{\sigma} \log t \right), \tag{3.8}
\]
where \( \sigma > 0 \) and \( 0 < p < 1 \). Note that (3.8) yields the iterative solution
\[
p^n \psi(t) = \psi \left( \frac{1}{p^\gamma} \log t \right).
\]
Solving (3.8), we have \( \psi(t) = t^\gamma h(t) \), where \( h(t) \) is periodic in \( \log t \) with period \( \frac{-2\pi}{\log p} \).
More details are in Thomas and Jose (2005).

**Semi-Type Marshall-Olkin Distributions Based on Characteristic Function**

Using (3.1), various authors have proposed *Marshall-Olkin semi-type* distributions from the baseline cf \( \psi(t) \).

Alice and Jose (2003) introduced the *Marshall-Olkin semi-Pareto* (MOSP) distribution with sf
\[
F_{MOSP}(t) = \frac{1}{1 + \frac{1}{a} \psi(t)},
\]
and established geometric extreme stability. Thomas and Jose (2005) and Alice and Jose (2005b) introduced the *Marshall-Olkin semi-Weibull* distribution with sf
\[
F_{MOSW}(t) = \frac{a}{e^{\psi(t)} - (1 - a)},
\]
where \( t > 0 \) and \( \alpha > 0 \). Jayakumar and Mathew (2008) proposed the *Marshall-Olkin semi-Burr* (GMOSB) distribution as that defined by the sf
\[
F_{GMOSB}(t) = \left[ \frac{a}{a + \psi(t)} \right]^\beta = \left[ \frac{1}{1 + \frac{1}{a} \psi(t)} \right]^\beta = (F_{MOSP}(t))^\beta,
\]
where \( \alpha > 0 \), \( \beta > 0 \) and \( \psi(t) \) satisfies the same functional as (3.7).

If \( 0 < \alpha < 1 \) and \( \varphi(t) \) is a valid cf then
\[
\psi(\varphi(t)) = \frac{\alpha \varphi(t)}{1 - (1 - \alpha) \varphi(t)}
\]
is also a valid cf. Using this fact, Krishna and Jose (2011) defined the *Marshall-Olkin generalized asymmetric Laplace* distribution as that having the cf

*An Acad Bras Cienc* (2015) 87 (2)
\[ \psi(t) = \frac{a}{(1 - \frac{\theta}{\lambda_1})^{-1} + (1 + \frac{\theta}{\lambda_2})^{-1} + a - 1}, \]

where \( i = \sqrt{-1}, \ 0 < \alpha < 1, \ \lambda_1 > 0, \ \lambda_2 > 0, \ \beta_1 > 0 \) and \( \beta_2 > 0 \). George and George (2013) defined the Marshall-Olkin Esscher transformed Laplace distribution as that having the cf

\[ \psi(t) = \left[ 1 + \frac{1}{a} \left( \frac{t^2}{1 - \theta^2} - \frac{2it\theta}{1 - \theta^2} \right) \right]^{-1} = \left[ 1 + \frac{1}{\lambda_2} \left[ \frac{t^2 - 2it\theta}{1} \right] \right]^{-1}, \]

where \( 0 < \alpha \leq 1, \ |\theta| < 1, \ \lambda = \sqrt{\alpha(1 - \theta^2)}, \ k = \frac{\lambda}{\theta + \frac{\lambda}{\theta}}, \ \lambda > 0 \) and \( k > 0 \). Jose and Uma (2009) defined the Marshall-Olkin Linnik and Mittag-Leffler distributions as those having the cfs

\[ \psi(t) = \frac{\beta}{(1 + |\theta|^\nu)^\nu + \beta - 1} \]

and

\[ \psi(t) = \frac{\beta}{\beta - s^\alpha} \]

respectively, where \( \nu > 0, \ 0 < \alpha \leq 2, \) and \( \beta > 0 \).

A list of papers on the MOE family is presented in Table II.

### TABLE II

Contributed work on the MOE family of distributions.

<table>
<thead>
<tr>
<th>S.No.</th>
<th>Pioneer year</th>
<th>Distribution</th>
<th>Author(s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1997</td>
<td>MOE exponential distribution</td>
<td>Marshall and Olkin (1997)</td>
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<td></td>
<td></td>
<td></td>
<td>Alice and Jose (2004b)</td>
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<td>Parikh et al. (2008)</td>
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<td>Salah et al. (2009)</td>
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<td>Bdair (2011)</td>
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<td>Gopal and Damodaran (2011)</td>
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<td>Krishna (2011)</td>
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<tr>
<td></td>
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<td></td>
<td>Rao et al. (2011)</td>
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<td></td>
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<td>Salah (2012)</td>
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<td></td>
<td>Pushkarna et al. (2013)</td>
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<td></td>
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<td></td>
<td>Jose and Alice (2001)</td>
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<td></td>
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<td></td>
<td>Hirose (2002)</td>
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<td></td>
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<td></td>
<td>Ghitany et al. (2005)</td>
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<td>Zhang and Xie (2007)</td>
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<td>Caroni (2010)</td>
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<td></td>
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<td>Srivastava and Kumar (2011)</td>
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<td></td>
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<td>Athar et al. (2012)</td>
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<td></td>
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<td></td>
<td>Cordeiro and Lemonte (2013)</td>
</tr>
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<td>5</td>
<td>2004</td>
<td>MOE Pareto-I distribution</td>
<td>Alice and Jose (2004a)</td>
</tr>
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<td></td>
<td></td>
<td></td>
<td>Ghitany (2005)</td>
</tr>
<tr>
<td>6</td>
<td>2005</td>
<td>MOE logistic distribution</td>
<td>Alice and Jose (2005a)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Kumar (2013)</td>
</tr>
<tr>
<td>7</td>
<td>2005</td>
<td>MO semi-logistic distribution</td>
<td>Alice and Jose (2005a)</td>
</tr>
</tbody>
</table>
Consider the cdf of a beta random variable of type 1 with two shape parameters $a$ and $b$ given by

$$F_{B1}(x) = P(X \leq x) = I_x(a, b) = \frac{B_x(a, b)}{B(a, b)} = \frac{1}{B(a, b)} \int_0^x t^{a-1} (1 - t)^{b-1} dt,$$

where $a > 0$, $b > 0$, $x \in (0, 1)$, $B_t(a, b) = \int_0^t t^{a-1} (1 - t)^{b-1} dt$ is the incomplete beta function, $I_t(a, b)$ is the incomplete beta function ratio and $B(a, b) = \int_0^1 t^{a-1} (1 - t)^{b-1} dt = \frac{\Gamma(a) \Gamma(b)}{\Gamma(a + b)}$ is the beta function. The pdf corresponding to (4.1) is

$$f_{B1}(x) = \frac{1}{B(a, b)} x^{a-1} (1 - x)^{b-1},$$

where $a > 0$, $b > 0$, $x \in (0, 1)$.  

**TABLE II (continuation)**

<table>
<thead>
<tr>
<th>S.No.</th>
<th>Pioneer year</th>
<th>Distribution</th>
<th>Author(s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>2005</td>
<td>MO semi-Weibull distribution</td>
<td>Alice and Jose (2005b)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Thomas and Jose (2005)</td>
</tr>
<tr>
<td>9</td>
<td>2005</td>
<td>MOE Fréchet distribution</td>
<td>Jose and Alice (2005)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Krishna (2011)</td>
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<tr>
<td></td>
<td></td>
<td></td>
<td>Krishna et al. (2013)</td>
</tr>
<tr>
<td>10</td>
<td>2007</td>
<td>MOE Lomax distribution</td>
<td>Ghitany et al. (2007)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Gupta et al. (2010)</td>
</tr>
<tr>
<td>12</td>
<td>2007</td>
<td>MOE gamma distribution</td>
<td>Ristić et al. (2007)</td>
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<td></td>
<td></td>
<td>Jose (2009)</td>
</tr>
<tr>
<td>13</td>
<td>2008</td>
<td>MOE q-Weibull distribution</td>
<td>Naik et al. (2008)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Jose et al. (2010)</td>
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<tr>
<td></td>
<td></td>
<td></td>
<td>El-Bassiouny and Abdo (2010)</td>
</tr>
<tr>
<td>15</td>
<td>2008</td>
<td>MOE semi-Burr distribution†</td>
<td>Jayakumar and Mathew (2008)</td>
</tr>
<tr>
<td>16</td>
<td>2008</td>
<td>MOE semi-Pareto III distribution†</td>
<td>Jayakumar and Mathew (2008)</td>
</tr>
<tr>
<td>17</td>
<td>2009</td>
<td>MOE Linnik distribution†</td>
<td>Jose and Uma (2009)</td>
</tr>
<tr>
<td>18</td>
<td>2009</td>
<td>MOE Mittag-Leffler distribution†</td>
<td>Jose and Uma (2009)</td>
</tr>
<tr>
<td>19</td>
<td>2009</td>
<td>MOE beta distribution</td>
<td>Jose et al. (2009)</td>
</tr>
<tr>
<td>20</td>
<td>2011</td>
<td>MOE uniform distribution</td>
<td>Krishna (2011)</td>
</tr>
<tr>
<td>21</td>
<td>2011</td>
<td>MOE Gumbel distribution</td>
<td>Jose (2011)</td>
</tr>
<tr>
<td>22</td>
<td>2011</td>
<td>MOE generalized asymmetric Laplace distribution†</td>
<td>Krishna (2011)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Krishna and Jose (2011)</td>
</tr>
<tr>
<td>24</td>
<td>2013</td>
<td>MOE Zipf distribution</td>
<td>P´erez-Casany and Casellas (2013)</td>
</tr>
<tr>
<td>25</td>
<td>2013</td>
<td>MOE power log-normal distribution</td>
<td>Gui (2013a)</td>
</tr>
<tr>
<td>26</td>
<td>2013</td>
<td>MOE log-logistic distribution</td>
<td>Gui (2013b)</td>
</tr>
<tr>
<td>27</td>
<td>2013</td>
<td>MOE quasi-Lindley distribution</td>
<td>Gui (2013c)</td>
</tr>
<tr>
<td>28</td>
<td>2013</td>
<td>MOE Esscher transformed Laplace distribution†</td>
<td>George and George (2013)</td>
</tr>
</tbody>
</table>

† MO extension based on cf.
Similarly, the cdf of a beta random variable of type 2 with parameters \(a\) and \(b\) is

\[
F^{B2}(y) = P(Y \leq y) = I_{2}\gamma(a, b) = \frac{B_{2}\gamma(a, b)}{B2(a, b)} = \frac{1}{B2(a, b)} \int_{0}^{y} \frac{y^{a-1}}{(1+y)^{a+b}} \, dy,
\]  \hspace{1cm} (4.2)

where \(a > 0, b > 0, y > 0, B_{2}(a, b)\int_{0}^{t} t^{a-1} (1+t)^{-(a+b)} \, dt \) is the incomplete beta function, \(I_{2}\gamma(a, b)\) is the incomplete beta function ratio and \(B2(a, b) = \int_{0}^{\infty} t^{a-1} (1+t)^{-(a+b)} \, dt = \frac{\Gamma(a) \Gamma(b)}{\Gamma(a+b)}\) is the beta function.

The pdf corresponding to (4.2) is

\[
f^{B2}(y) = \frac{1}{B2(a, b)} \frac{y^{a-1}}{(1+y)^{(a+b)}},
\]

where \(a > 0, b > 0,\) and \(y > 0\). The beta type 2 distribution is also known as inverted beta distribution as it can be obtained from (4.1) by the transformation \(Y = \frac{x}{1-x}\).

Cardeño et al. (2005) introduced the beta type 3 distribution by transforming \(Z = \frac{Y}{2-Y}\) in (4.1). The cdf of a beta random variable of type 3 with parameters \(a\) and \(b\) is

\[
F^{B3}(z) = P(Z \leq z) = I_{3}\gamma(a, b) = \frac{B_{3}\gamma(a, b)}{B3(a, b)} = \frac{1}{B3(a, b)} \int_{0}^{z} \frac{z^{a-1}(1-z)^{b-1}}{(1+z)^{(a+b)}} \, dz,
\]  \hspace{1cm} (4.3)

where \(a > 0, b > 0, z \in (0, 1), B_{3}(a, b) = \int_{0}^{t} t^{a-1} (1-t)^{b-1} (1+t)^{-(a+b)} \, dt \) is the incomplete beta function, \(I_{3}\gamma(a, b)\) is the incomplete beta function ratio and \(B3(a, b) = \int_{0}^{\infty} t^{a-1} (1-t)^{b-1} (1+t)^{-(a+b)} \, dt = \frac{\Gamma(a) \Gamma(b)}{\Gamma(a+b)}\) is the beta function. The pdf corresponding to (4.3) is

\[
f^{B3}(z) = \frac{2^{a}}{B3(a, b)} \frac{z^{a-1}(1-z)^{b-1}}{(1+z)^{(a+b)}},
\]

where \(a > 0, b > 0,\) and \(z \in (0, 1)\).

Eugene et al. (2002) and Jones (2004a) replaced the upper limit \(x\) of the integral in (4.1) with \(G(x)\). The resulting cdf of beta \(G\) family of distributions is

\[
F^{BG}(x) = I_{G\gamma}(a, b) = \frac{B_{G\gamma}(a, b)}{B(a, b)} = \frac{1}{B(a, b)} \int_{0}^{G(x)} \omega^{a-1} (1-\omega)^{b-1} \, d\omega.
\]  \hspace{1cm} (4.4)

The pdf corresponding to (4.4) is

\[
f^{BG}(x) = \frac{1}{B(a, b)} g(x) G(x)^{a-1} [1-G(x)]^{b-1},
\]  \hspace{1cm} (4.5)

where \(g(x) = dG(x)/dx\) denotes the pdf. The beta \(G\) family of distributions is also known as the beta logit family. For any lifetime random variable \(t\), the sf, hrf, hrf and chr are associated with (4.4) and (4.5) are

\[
\bar{T}(t) = 1 - I_{G\gamma}(a, b) = \frac{B(a, b) - B_{G\gamma}(a, b)}{B(a, b)},
\]

\[
h(t) = \frac{g(t)G(t)^{a-1} [1-G(t)]^{b-1}}{B(a, b) [I_{G\gamma}(a, b)]} = \frac{g(t)G(t)^{a-1} [1-G(t)]^{b-1}}{B_{G\gamma}(a, b)},
\]
\[ r(t) = \frac{g(t)G(t)^{a-1} \left[ 1 - G(t) \right]^{b-1}}{B(a, b) \left[ 1 - I_G(t)(a, b) \right]} = \frac{g(t)G(t)^{a-1} \left[ 1 - G(t) \right]^{b-1}}{B(a, b) - B_G(t)(a, b)}, \]

and

\[ H(t) = -\log \left[ \frac{B(a, b) - B_G(t)(a, b)}{B(a, b)} \right]. \]

A list of papers on the beta $G$ family of distributions is given in Table III.

**TABLE III**
Contributed work on the beta $G$ family of distributions.

<table>
<thead>
<tr>
<th>S.No.</th>
<th>Pioneer year</th>
<th>Distribution</th>
<th>Author(s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2002</td>
<td>Beta normal distribution</td>
<td>Eugene et al. (2002)</td>
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<td>Famoye et al. (2002)</td>
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<td>Eugene (2004)</td>
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<td>Gupta and Nadarajah (2004)</td>
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<td>Jones (2004b)</td>
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<td>Ręgo et al. (2012)</td>
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<td>Nadarajah and Kotz (2006b)</td>
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<td>3</td>
<td>2004</td>
<td>Beta gamma distribution</td>
<td>Kong (2004), Kong et al. (2007)</td>
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<td>6</td>
<td>2005</td>
<td>Beta Weibull distribution</td>
<td>Barreto-Souza et al. (2011)</td>
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<td>Lee et al. (2007)</td>
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<td>Sun (2011), Mdziniso (2012)</td>
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<td>Mahmoud and Mandouh (2012a, b, c)</td>
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<td>2006</td>
<td>Beta Bessel distribution</td>
<td>Gupta and Nadarajah (2006)</td>
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<td>Beta Pareto distribution</td>
<td>Akinsete et al. (2008)</td>
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<td>Beta Rayleigh distribution</td>
<td>Akinsete and Lowe (2009)</td>
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<td>Beta generalized logistic-IV distribution</td>
<td>Morais (2009)</td>
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<td>Morais et al. (2013)</td>
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<td>2010</td>
<td>Beta modified Weibull distribution</td>
<td>Silva et al. (2010)</td>
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<td>Nadarajah et al. (2012b)</td>
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<tr>
<td>13</td>
<td>2010</td>
<td>Beta generalized half-normal distribution</td>
<td>Pescim et al. (2010)</td>
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<tr>
<td>14</td>
<td>2010</td>
<td>Beta generalized exponential distribution</td>
<td>Barreto-Souza et al. (2010)</td>
</tr>
<tr>
<td>15</td>
<td>2010</td>
<td>Beta Maxwell distribution</td>
<td>Amusan (2010)</td>
</tr>
<tr>
<td>16</td>
<td>2010</td>
<td>Beta hyperbolic secant distribution</td>
<td>Fischer and Vaughan (2010)</td>
</tr>
<tr>
<td>17</td>
<td>2010</td>
<td>Beta inverse Weibull distribution</td>
<td>Kersey (2010)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Hanook et al. (2013)</td>
</tr>
<tr>
<td>18</td>
<td>2011</td>
<td>Beta Cauchy distribution</td>
<td>Alshawarbeh (2011)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Alshawarbeh et al. (2012)</td>
</tr>
<tr>
<td>19</td>
<td>2011</td>
<td>Beta half-Cauchy distribution</td>
<td>Cordeiro and Lemonte (2011b)</td>
</tr>
<tr>
<td>20</td>
<td>2011</td>
<td>Beta Burr XII distribution</td>
<td>Paranaíba et al. (2011)</td>
</tr>
<tr>
<td>21</td>
<td>2011</td>
<td>Beta generalized Pareto distribution</td>
<td>Mahmoudi (2011)</td>
</tr>
</tbody>
</table>
MCDONALD DISTRIBUTIONS AND MCDONALD G FAMILIES OF DISTRIBUTIONS

McDonald (1984) replaced the upper limit \( x \) of the integral in (4.1) with \( x^c \), where \( c \) is an additional (third) shape parameter. The resulting cdf of the McDonald type (Mc) distribution is

\[
F(x) = I_c(a, b) = \frac{B_c(a, b)}{B(a, b)} = \frac{1}{B(a, b)} \int_0^{x^c} x^{a-1} (1-x)^{b-1} \, dx,
\]

where \( a > 0, b > 0 \) and \( c > 0 \) are the three shape parameters. The Mc distribution includes as special cases the beta type 1 distribution \((c = 1)\) and the Kumaraswamy distribution \((a = 1)\). The pdf corresponding to (5.1) is

\[
f(x) = \frac{c}{B(a, b)} x^{ac-1} (1-x^c)^{b-1},
\]

where \( 0 < x < 1 \).

EXISTING MCDONALD G FAMILY OF DISTRIBUTIONS

For any baseline cdf \( G(x) \), Alexander et al. (2012) replaced the upper limit \( x^c \) of the integral in (5.1) with \( G(x)^\gamma \). Lemonte and Cordeiro (2013) stated that this simple transformation facilitates the computation of several properties of the \( G \) family of distributions.

### TABLE III (continuation)

<table>
<thead>
<tr>
<th>S.No.</th>
<th>Pioneer year</th>
<th>Distribution</th>
<th>Author(s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>22</td>
<td>2011</td>
<td>Beta Birnbaum-Sanders distribution</td>
<td>Nassar and Nada (2011)</td>
</tr>
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<td>23</td>
<td>2012</td>
<td>Beta skew-normal distribution</td>
<td>Cordeiro and Lemonte (2011c)</td>
</tr>
<tr>
<td>24</td>
<td>2012</td>
<td>Beta exponential-geometric distribution</td>
<td>Mameli (2012)</td>
</tr>
<tr>
<td>26</td>
<td>2012</td>
<td>Beta generalized Weibull distribution</td>
<td>Cordeiro et al. (2012d)</td>
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<tr>
<td>27</td>
<td>2012</td>
<td>Beta exponentiated Pareto distribution</td>
<td>Singla et al. (2012)</td>
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<td>28</td>
<td>2012</td>
<td>Beta power distribution</td>
<td>Zea et al. (2012)</td>
</tr>
<tr>
<td>29</td>
<td>2012</td>
<td>Beta linear failure rate distribution</td>
<td>Jafari and Mahmoudi (2012)</td>
</tr>
<tr>
<td>30</td>
<td>2012</td>
<td>Beta extended Weibull distribution</td>
<td>Cordeiro et al. (2012f)</td>
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<td>31</td>
<td>2012</td>
<td>Beta truncated Pareto distribution</td>
<td>Lourenzutti et al. (2012)</td>
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<tr>
<td>32</td>
<td>2013</td>
<td>Beta Weibull-geometric distribution</td>
<td>Cordeiro et al. (2013f)</td>
</tr>
<tr>
<td>33</td>
<td>2013</td>
<td>Beta generalized gamma distribution</td>
<td>Bidram et al. (2013)</td>
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<td>2013</td>
<td>Beta log-normal distribution</td>
<td>Castellars et al. (2013)</td>
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<td>35</td>
<td>2013</td>
<td>Beta generalized Rayleigh distribution</td>
<td>Cordeiro et al. (2013b)</td>
</tr>
<tr>
<td>36</td>
<td>2013</td>
<td>Beta generalized logistic distribution</td>
<td>Morais et al. (2013)</td>
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<tr>
<td>37</td>
<td>2013</td>
<td>Beta exponentiated Weibull distribution</td>
<td>Cordeiro et al. (2013c)</td>
</tr>
<tr>
<td>38</td>
<td>2013</td>
<td>Beta Nakagami distribution</td>
<td>Shittu and Adepoju (2013)</td>
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<tr>
<td>39</td>
<td>2013</td>
<td>Beta Burr III distribution</td>
<td>Gomes et al. (2013)</td>
</tr>
<tr>
<td>40</td>
<td>2013</td>
<td>Beta Dagum distribution</td>
<td>Domma and Condino (2013)</td>
</tr>
<tr>
<td>41</td>
<td>2013</td>
<td>Beta Stoppa distribution</td>
<td>Mansoor (2013)</td>
</tr>
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<td>42</td>
<td>2013</td>
<td>Beta inverse Rayleigh distribution</td>
<td>Le’ao et al. (2013)</td>
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<td>43</td>
<td>2014</td>
<td>Beta generalized inverse Weibull distribution</td>
<td>Baharith et al. (2014)</td>
</tr>
<tr>
<td>44</td>
<td>2014</td>
<td>Beta extended half-normal distribution</td>
<td>Cordeiro et al. (2014f)</td>
</tr>
<tr>
<td>45</td>
<td>2014</td>
<td>Beta log-logistic distribution</td>
<td>Lemonte (2014)</td>
</tr>
</tbody>
</table>

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The resulting cdf $F(x)$ of the Mc-generalized family of distributions (Mc $G$) is

$$F(x) = I_{G(a,b)}(a, b) = \frac{B_{G(a,b)}(a,b)}{B(a,b)} = \frac{1}{B(a,b)} \int_{0}^{G(x)} \omega^{a-1} (1 - \omega)^{b-1} d\omega, \quad (5.2)$$

where $I_{G(a,b)}(a, b)$ denotes the incomplete beta function ratio. The pdf corresponding to (5.2) is

$$f(x) = \frac{c}{B(a,b)} g(x) G(x)^{ac-1} [1 - G(x)^c]^{b-1}, \quad (5.3)$$

where $a > 0$, $b > 0$ and $c > 0$ are the three shape parameters. For a lifetime random variable $t$, the sf, hrf, rhrf and chrf associated with (5.2) and (5.3) are

$$F(t) = 1 - I_{G(a,b)}(a, b) = \frac{B(a,b) - B_{G(a,b)}(a,b)}{B(a,b)},$$

$$h(t) = \frac{c g(t) G(t)^{ac-1} [1 - G(t)^c]^{b-1}}{B(a,b) [I_{G(a,b)}(a,b)]} = \frac{c g(t) G(t)^{ac-1} [1 - G(t)^c]^{b-1}}{B_{G(a,b)}(a,b)},$$

$$r(t) = \frac{c g(t) G(t)^{ac-1} [1 - G(t)^c]^{b-1}}{B(a,b) [1 - I_{G(a,b)}(a,b)]} = \frac{c g(t) G(t)^{ac-1} [1 - G(t)^c]^{b-1}}{[B(a,b) - B_{G(a,b)}(a,b)]},$$

and

$$H(t) = -\log \left[ \frac{B(a,b) - B_{G(a,b)}(a,b)}{B(a,b)} \right].$$

NOTES ON EXISTING Mc $G$ FAMILIES OF DISTRIBUTIONS

The three shape parameters $a$, $b$ and $c$ introduce skewness, kurtosis, and vary tail weights. The parameters control skewness and kurtosis through altering the tail entropy (Alexander et al. 2012). They also control skewness and kurtosis through adding entropy to the center of the baseline distribution (Alexander et al. 2012). Cordeiro et al. (2014b) mentioned that $a$ and $b$ are skewness parameters that control relative tail weights but not the peak, but $c$ provides the control over the peak.

Alexander et al. (2012), Marciano et al. (2012), Cordeiro and Lemonte (2012, 2014), Cordeiro et al. (2012a, b, 2013d, 2014b), Lemonte and Cordeiro (2013) and Gomes et al. (2013a) used Mc $G$ distributions for developing McDonald normal, McDonald (extended) exponential, McDonald gamma, McDonald inverted beta, McDonald arcsine, McDonald Weibull, McDonald Birnbaum-Sanders (fatigue life), McDonald Lomax, McDonald Burr XII and McDonald Burr III distributions. These authors believe that the Mc $G$ family of distributions can fit skew data better than existing distributions. The Mc $G$ family of distributions is most applicable when $G(x)$ and $g(x)$ take simple analytical forms.

The Mc $G$ family of distributions reduces to the beta $G$ family of distribution for $c = 1$ and to the Kw $G$ family of distribution for $a = c$. Further, the Mc $G$ family of distributions for $G(x) = x$ contains as particular cases the beta type 1 distribution ($c = 1$) and the Kumaraswamy distribution ($a = c$).

Zografos (2011) studied a family of distributions based on McDonald and Xu (1995)’s generalized beta distribution. This family was called the family of generalized beta generated (GBG) distributions.

A list of papers on the Mc $G$ family of distributions is given in Table IV.
Kumaraswamy (1980) argued that the beta distribution does not fairly fit hydrological random variables like rainfall, daily stream flow, etc. Jones (2009) commented that “beta distribution is fairly tractable, but in some ways not fabulously so. In particular its distribution function is an incomplete beta function ratio and its quantile function the inverse thereof”. The Kumaraswamy (Kw) distribution is relatively much appreciated in comparison to the beta distribution, and has a simple form which can be unimodal, increasing, decreasing or constant, depending on the parameter values.

In this section, we give functional forms of Kw distributions. We also propose Kumaraswamy generalized families of distributions.

**EXISTING KUMARASWAMY DISTRIBUTIONS**

The Kw distribution has the cdf and the pdf specified by

\[ F(x) = 1 - (1 - x^a)^b, \]  

(6.1)

and

\[ f(x) = a b x^{a-1} (1 - x^a)^{b-1}, \]  

(6.2)

respectively, where \( 0 < x < 1 \) and \( a > 0, b > 0 \) are both shape parameters.

**EXISTING KUMARASWAMY G FAMILY OF DISTRIBUTIONS**

For a baseline cdf \( G(x) \) with pdf \( g(x) \), Cordeiro and de Castro (2011) defined the Kw G distribution specified by the cdf and the pdf

\[ F(x) = 1 - [1 - G(x)]^a, \]  

(6.3)

and
where $x > 0$, $g(x) = dG(x) = dx$ and $a > 0$, $b > 0$ are shape parameters in addition to those in the baseline distribution. They partly govern skewness and vary tail weights. For a lifetime random variable $t$, the sf, hrf, rhfr and chrfr associated with (6.3) and (6.4) are

\[
F(t) = [1 - G(t)]^b,
\]
\[
h(t) = a \cdot b \cdot g(t) \cdot G(t)^{a-1} \cdot [1 - G(t)]^{-1}
\]
\[
r(t) = a \cdot b \cdot g(t) \cdot G(t)^{a-1} \cdot [1 - G(t)]^{b-1} \cdot \{1 - [1 - G(x)]^b\}^{-1},
\]
and

\[
H(t) = -b \log [1 - G(x)]^b.
\]

**NOTES ON KUMARASWAMY G FAMILIES OF DISTRIBUTIONS**

Equations (6.3) and (6.4) do not involve any special function like the beta function, incomplete beta function, incomplete beta ratio, gamma function, incomplete gamma function or the incomplete gamma ratio. Therefore, the generalization in (6.3) and (6.4) is computationally more efficient compared to beta $G$ and $Mc$ $G$ families of distributions.

The Kw $G$ families of distributions are more flexible than the baseline distribution in the sense that the families allow for greater flexibility of tail properties. Their second benefit is their ability to fit skew data that cannot be properly fitted by existing distributions.

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A list of papers on the Kw $G$ family of distributions is given in Table V.

**TABLE V**

<table>
<thead>
<tr>
<th>S.No.</th>
<th>Pioneer year</th>
<th>Distribution</th>
<th>Author(s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2010</td>
<td>Kumaraswamy Weibull distribution</td>
<td>Cordeiro et al. (2010)</td>
</tr>
<tr>
<td>2</td>
<td>2011</td>
<td>Kumaraswamy generalized gamma distribution</td>
<td>de Pascoa et al. (2011)</td>
</tr>
<tr>
<td>6</td>
<td>2012</td>
<td>Kumaraswamy Gumbel distribution</td>
<td>Cordeiro et al. (2012c)</td>
</tr>
<tr>
<td>7</td>
<td>2012</td>
<td>Kumaraswamy Birnbaum-Sanders distribution</td>
<td>Saulo et al. (2012)</td>
</tr>
</tbody>
</table>
PARAMETER INDUCTION IN DISTRIBUTIONS

Setting \( X = 1 - Y \) in (6.1) and (6.2), we obtain a distribution specified by the cdf and the pdf

\[
F(x) = 1 - \left[ 1 - (1 - x)^a \right]^b
\]

and

\[
f(x) = a b (1 - x)^{a-1} \left[ 1 - (1 - x)^a \right]^{b-1},
\]

where \( 0 < x < 1 \) and \( a > 0, b > 0 \) are the shape parameters.

OTHER KW G FAMILIES OF DISTRIBUTIONS

Replacing \( x \) with \( G(x) \) in (6.5), we obtain a Kw G distribution specified by the cdf

\[
F(x) = 1 - \{ 1 - [1 - G(x)]^a \}^b,
\]

where \( a > 0 \) and \( b > 0 \) are both shape parameters. The pdf corresponding to (6.7) is

\[
f(x) = a b g(x) \left[ 1 - G(x) \right]^{a-1} \{ 1 - [1 - G(x)]^a \}^{b-1}.
\]

Equations (6.6) and (6.7) are the cdf and the pdf of the Exp G family of distributions recently proposed by Cordeiro et al. (2013e). For a lifetime random variable \( t \), the sf, hrf, rhrf and chrf associated with (6.6) and (6.7) are

\[
F(t) = \{ 1 - [1 - G(t)]^a \}^b,
\]

\[
h(t) = a b g(t) \left[ 1 - G(t) \right]^{a-1} \{ 1 - [1 - G(t)]^a \}^{-1},
\]

\[
r(t) = a b g(t) \left[ 1 - G(t) \right]^{a-1} \{ 1 - [1 - G(t)]^a \}^{b-1} \left[ 1 - \{ 1 - [1 - G(t)]^a \}^b \right]^{-1},
\]

and

\[
H(t) = - b \log \{ 1 - [1 - G(t)]^a \}.
\]
CONCLUSIONS

We first refer to some important surveys on the developments of continuous univariate distributions: Kotz and Vicari (2005) surveyed the developments in the theory of skewed continuous distributions; Gupta and Kundu (2009) described six different methods for the induction of shape and/or skewness parameter(s) in univariate probability distributions; Chakraborty and Hazarika (2011) surveyed the theoretical developments of the univariate skew-normal distribution, its extensions and generalizations; Lee et al. (2013) surveyed recent methods for generating families of univariate continuous distributions. They discussed five general methods for generating $G$ families of distributions: (1) method for generating skewed distributions, (2) method for adding parameters (e.g., exponentiation), (3) beta $G$, (4) transformed-transformer (T-X) family, and (5) composite method. Recently, Nadarajah (2015a, 2015b) introduced the R package Newdistns which computes the pdf, cdf, quantiles and random numbers for nineteen general families of distributions.

In this paper, we have discussed the well-established and widely used $G$ families of distributions: the EF of distributions, the MOE distributions, the beta $G$ distributions, the Mc $G$ distributions, the Kw $G$ distributions and the Exp $G$ distributions. We have provided exhaustive lists of papers on these families of distributions. We have cited 28 papers on the EF of distributions, 28 papers on the MOE distributions, 45 papers on the beta $G$ distributions, 16 papers on the Mc $G$ distributions, 21 papers on the Kw $G$ distributions and 2 papers on the Exp $G$ distributions. The literature review in Lee et al. (2013) appears less detailed.

We have introduced several new families of distributions relating to the MOE distributions and the Kw $G$ distributions. Of course, this is not an attempt to increase the frequency of articles on new families of distributions but rather to effectively explore real life phenomena through data sets available from different fields. We have noted that contributors (practitioners) have used different model selection criteria: the maximized log-likelihood $\ell(\hat{\theta})$, the Akaike Information Criterion (AIC), the Bayesian Information Criterion (BIC), the Consistent Akaike Information Criterion (CAIC), the Hannan-Quinn Information Criterion (HQIC), the Cramer-von-Mises ($W^*$), the Anderson-Darling ($A^*$), the Wald ($W$) statistic, the Kolmogorov-Smirnov (K-S) test and graphical inspection of the proximity of histograms to the fitted pdfs.

Tractability and effectiveness for modeling censored data require, among other things, closed form expressions for the cdf. So, the Kw $G$ distributions can be tractable and effective models for censored data. The EF and MOE distributions can also be tractable and effective models for censored data, provided $G$ is in closed form. However, beta $G$ and Mc $G$ distributions may not be tractable or effective models for censored data since their cdfs involve the incomplete beta function.

It is very appreciating that the contributors have expanded the horizon of applications with efficient statistical modeling. In this regard, the acknowledgements and appreciation go to Professors M. C. Jones, Narayanaswamy Balakrishnan, Kostas Zografos, Felix Famoye, Carl M.-S. Lee, Ramesh C. Gupta, Arjun Kumar Gupta, Rameshwar D. Gupta, Debasis Kundu, Mohamad E. Ghitany, and K. K. Jose. Special acknowledgements and appreciation go to the Brazilian Statisticians Group headed by Professor Gauss M. Cordeiro for introducing the Mc $G$, Kw $G$, Exp $G$, beta extended $G$, Weibull $G$ families and exploring their properties. We note that 58 of the listed papers in the References section belong to Professor Cordeiro.

ACKNOWLEDGMENTS

Both authors would like to thank Professors Felix Famoye, Carl Lee, Kostas Zografos, Miroslav Ristić, Edwin Ortega, Antonio Gomes, Devendra Kumar, Shola Adeyemi, Ibrahim Elbatal and Wenhao Gui for...
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PARAMETER INDUCTION IN DISTRIBUTIONS

RESUMO

O método de adicionar parâmetros a uma distribuição especificada tem sido bastante adotado nos últimos anos. A adição de um ou mais parâmetros de forma torna a distribuição gerada mais flexível especialmente no estudo de suas propriedades. Esse método tem se mostrado eficaz, também, na melhoria das estatísticas de adequação do ajuste da nova distribuição. Desde 1985, muitas famílias de distribuições continuas geradas por esse método têm sido investigadas. Neste artigo, as famílias geradoras mais conhecidas de distribuições como a família estendida de Marshall-Olkin, a família beta, as famílias generalizadas de McDonald e Kumaraswamy e as famílias exponentencializadas são discutidas. Apresentam-se as referências mais importantes dessas famílias. Algumas formas mais amplas da família estendida de Marshall-Olkin e da família generalizada de Kumaraswamy são propostas.

Palavras-chave: Distribuição beta, família exponentencializada, distribuição de Kumaraswamy, família de Marshall-Olkin, distribuição de McDonald, propriedades da confiabilidade.

REFERENCES


PARAMETER INDUCTION IN DISTRIBUTIONS


CORDEIRO GM, ORTEGA EMM, HAMEDANI GG AND GRACIA DA. IN PRESS. The McDonald Burr XII model: Properties and applications.


JONES MC. 2004b. The moments of the beta-normal distribution with integer parameters are the moments of order statistics from the normal distribution (Letter to the editor). Commun Statist Theor Meth 33: 2869-2870.


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PARAMETER INDUCTION IN DISTRIBUTIONS


MUHAMMAD H. TAHIR and SARALEES NADARAJAH


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Shaw WT and Buckley IRC. 2007. The alchemy of probability distributions: Beyond Gram-Charlier expansions, and a skew-kurtotic-normal distribution from a rank transmutation map. Research report, Department of Mathematics, King's College London, UK.


