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Translation Hypersurfaces with Constant S_r Curvature in the Euclidean Space

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ABSTRACT

The main goal of this paper is to present a complete description of all translation hypersurfaces with constant r-curvature S_r, in the Euclidean space $\mathbb{R}^{n+1}$, where $3 \leq r \leq n-1$.

Key words: Euclidean space, Scherk's surface, Translation hypersurfaces, r-Curvature.

INTRODUCTION

It is well known that translation hypersurfaces are very important in Differential Geometry, providing an interesting class of constant mean curvature hypersurfaces and minimal hypersurfaces in a number of spaces endowed with good symmetries and even in certain applications in Microeconomics. There are many results about them, for instance, Chen et al. (2003), Dillen et al. (1991), Inoguchi et al. (2012), Lima et al. (2014), Liu (1999), López (2011), López and Moruz (2015), López and Munteanu (2012), Seo (2013) and Chen (2011), for an interesting application in Microeconomics.

Scherk (1835) obtained the following classical theorem: Let $M := \{(x, y, z) :\ z = f(x) + g(y)\}$ be a translation surface in $\mathbb{R}^3$, if is minimal then it must be a plane or the Scherk surface defined by

$$z(x, y) = \frac{1}{a} \ln \left| \frac{\cos(ay)}{\cos(ax)} \right|,$$

where a is a nonzero constant. In a different aspect, Liu (1999) considered the translation surfaces with constant mean curvature in 3-dimensional Euclidean space and Lorentz-Minkowski space and Inoguchi et al. (2012) characterized the minimal translation surfaces in the Heisenberg group Nil_3, and López and Munteanu, the minimal translation surfaces in Sol_3.

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The concept of translation surfaces was also generalized to hypersurfaces of $\mathbb{R}^{n+1}$ by Dillen et al. (1991), who obtained a classification of minimal translation hypersurfaces of the $(n+1)$-dimensional Euclidean space. A classification of the translation hypersurfaces with constant mean curvature in $(n+1)$-dimensional Euclidean space was made by Chen et al. (2003).

The absence of an affine structure in hyperbolic space does not permit to give an intrinsic concept of translation surface as in the Euclidean setting. Considering the half-space model of hyperbolic space, López (2011), introduced the concept of translation surface and presented a classification of the minimal translation surfaces. Seo (2013) has generalized the results obtained by Lopez to the case of translation hypersurfaces of the $(n+1)$-dimensional hyperbolic space.

Definition 1. *We say that a hypersurface M^n of the Euclidean space $\mathbb{R}^{n+1}$ is a translation hypersurface if it is the graph of a function given by*

$$F(x_1, \ldots, x_n) = f_1(x_1) + \ldots + f_n(x_n)$$

where $(x_1, \ldots, x_n)$ are cartesian coordinates and each f_i is a smooth function of one real variable for $i = 1, \ldots, n$.

Now, let $M^n \subset \mathbb{R}^{n+1}$ be an oriented hypersurface and $\lambda_1, \ldots, \lambda_n$ denote the principal curvatures of M^n. For each $r = 1, \ldots, n$, we can consider similar problems to the above ones, related with the r-th elementary symmetric polynomials, S_r, given by

$$S_r = \sum_{1 \leq i_1 < \cdots < i_r \leq n} \lambda_{i_1} \cdots \lambda_{i_r}$$

In particular, S_1 is the mean curvature, S_2 the scalar curvature and S_n the Gauss-Kronecker curvature, up to normalization factors. A very useful relationship involving the various S_r is given in the [Proposition 1, Caminha (2006)]. This result will play a central role along this paper.

Recently, some authors have studied the geometry of translational hypersurfaces under a condition in the S_r curvature, where $r > 1$. Namely, Leite (1991) gave a new example of a translation hypersurface of $\mathbb{R}^4$ with zero scalar curvature. Lima et al. 2014 presented a complete description of all translation hypersurfaces with zero scalar curvature in the Euclidean space $\mathbb{R}^{n+1}$ and Seo 2013 proved that if M is a translation hypersurface with constant Gauss-Kronecker curvature GK in $\mathbb{R}^{n+1}$, then M is congruent to a cylinder, and hence $GK = 0$.

In this paper, we obtain a complete classification of translation hypersurfaces of $\mathbb{R}^{n+1}$ with $S_r = 0$. We prove the following

Theorem 1. *Let M^n $(n \geq 3)$ be a translation hypersurface in $\mathbb{R}^{n+1}$. Then, for $2 < r < n$, M^n has zero S_r curvature if, and only if, it is congruent to the graph of the following functions*

- $F(x_1, \ldots, x_n) = \sum_{i=1}^{n-r+1} a_i x_i + \sum_{j=n-r+2}^{n} f_j(x_j) + b,$

on $\mathbb{R}^{n-r+1} \times J_{n-r+2} \times \cdots \times J_n$, for certain intervals $J_{n-r+2}, \ldots, J_n$, and arbitrary smooth functions $f_i : J_i \subset \mathbb{R} \to \mathbb{R}$. Which defines, after a suitable linear change of variables, a vertical cylinder, or

- *A generalized periodic Enneper hypersurface given by*

$$\begin{aligned} F(x_1,\ldots,x_n) &= \sum_{i=1}^{n-r-1} a_i x_i \\ &+ \sum_{k=n-r}^{n-1} \frac{\sqrt{\beta}}{a_k} \ln \left| \frac{\cos\left(-\dfrac{a_{n-r}\cdots a_{n-1}}{\sigma_{r-1}(a_{n-r},\ldots,a_{n-1})}\sqrt{\beta}x_n + b_n\right)}{\cos(a_k\sqrt{\beta}x_k + b_k)} \right| + c \end{aligned}$$

on $\mathbb{R}^{n-r-1} \times I_{n-r} \times \cdots \times I_n$, *with* $a_1,\ldots,a_{n-r},\ldots,a_{n-1},b_{n-r},\ldots,b_n$ *and* c *are real constants where* $a_{n-r},\ldots,a_{n-1}$ *and* $\sigma_{r-1}(a_{n-r},\ldots,a_{n-1})$ *nonzero,* $\beta = 1 + \sum_{i=1}^{n-r-1} a_i^2$, $I_k\,(n-r \le k \le n-1)$ *are open intervals defined by the conditions* $|a_k\sqrt{\beta}x_k + b_k| < \pi/2$ *while* I_n *is defined by* $\left|-\dfrac{a_{n-r}\cdots a_{n-1}}{\sigma_{r-1}(a_{n-r},\ldots,a_{n-1})}\sqrt{\beta}x_n + b_n\right| < \pi/2.$

Theorem 2. *Any translation hypersurface in* $\mathbb{R}^{n+1}$ $(n \ge 3)$ *with* S_r *constant, for* $2 < r < n$, *must have* $S_r = 0$.

Finally, we observe that, when one considers the upper half-space model of the $(n+1)$-dimensional hyperbolic space $\mathbb{H}^{n+1}$, that is,

$$\mathbb{R}^{n+1}_+ = \{(x_1,\ldots,x_n,x_{n+1}) \in \mathbb{R}^{n+1} : x_{n+1} > 0\}$$

endowed with the hyperbolic metric $ds^2 = \dfrac{1}{x_{n+1}^2}\left(dx_1^2 + \ldots + dx_{n+1}^2\right)$ then, unlike in the Euclidean setting, the coordinates $x_1,\ldots,x_n$ are interchangeable, but the same does not happen with the coordinate x_{n+1} and, due to this observation, López 2011 and Seo 2013 considered two classes of translation hypersurfaces in $\mathbb{H}^{n+1}$:

A hypersurface $M \subset \mathbb{H}^{n+1}$ *is called a translation hypersurface of* **type I** (*respectively,* **type II**) *if it is given by an immersion* $X : U \subset \mathbb{R}^n \to \mathbb{H}^{n+1}$ *satisfying*

$$X(x_1,\ldots,x_n) = (x_1,\ldots,x_n,f_1(x_1)+\ldots+f_n(x_n))$$

where each f_i *is a smooth function of a single variable. Respectively, in case of* **type II**,

$$X(x_1,\ldots,x_n) = (x_1,\ldots,x_{n-1},f_1(x_1)+\ldots+f_n(x_n),x_n)$$

Seo proved

Theorem 3 (Theorem 3.2, Seo 2013)**.** *There is no minimal translation hypersurface of* **type I** *in* $\mathbb{H}^{n+1}$.

and with respect to **type II** surfaces he proved

Theorem 4 (Theorem 3.3, Seo 2013)**.** *Let* $M \subset \mathbb{H}^3$ *be a minimal translation surface of* **type II** *given by the parametrization* $X(x,z) = (x, f(x)+g(z), z)$. *Then the functions* f *and* g *are as follows:*

$$\begin{aligned} f(x) &= ax + b, \\ g(z) &= \sqrt{1+a^2}\int \frac{cz^2}{\sqrt{1-c^2z^4}}\,dz, \end{aligned}$$

where a, b, and c are constants.

We emphasize that the result proved by Seo, Theorem 3.2 of Seo 2013, implies that our result (Theorem 2) is not valid in the hyperbolic space context.

PRELIMINARIES AND BASIC RESULTS

Let $\overline{M}^{n+1}$ be a connected Riemannian manifold. In the remainder of this paper, we will be concerned with isometric immersions, $\Psi : M^n \to \overline{M}^{n+1}$, from a connected, n-dimensional orientable Riemannian manifold, M^n, into $\overline{M}^{n+1}$. We fix an orientation of M^n, by choosing a globally defined unit normal vector field, ξ, on M. Denote by A, the corresponding shape operator. At each $p \in M$, A restricts to a self-adjoint linear map $A_p : T_pM \to T_pM$. For each $1 \leq r \leq n$, let $S_r : M^n \to \mathbb{R}$ be the smooth function such that $S_r(p)$ denotes the r-th elementary symmetric function on the eigenvalues of A_p, which can be defined by the identity

$$\det(A_p - \lambda I) = \sum_{k=0}^{n} (-1)^{n-k} S_k(p) \lambda^{n-k}. \tag{1}$$

where $S_0 = 1$ by definition. If $p \in M^n$ and $\{e_l\}$ is a basis of T_pM, given by eigenvectors of A_p, with corresponding eigenvalues $\{\lambda_l\}$, one immediately sees that

$$S_r = \sigma_r(\lambda_1, \ldots, \lambda_n),$$

where $\sigma_r \in \mathbb{R}[X_1, \ldots, X_n]$ is the r-th elementary symmetric polynomial on $X_1, \ldots, X_n$. Consequently,

$$S_r = \sum_{1 \leq i_1 < \cdots < i_r \leq n} \lambda_{i_1} \cdots \lambda_{i_r}, \quad \text{where} \quad r = 1, \ldots, n.$$

In the next result we present an expression for the curvature S_r of a translation hypersurface in the Euclidean space. This expression will play an essential role in this paper.

Proposition 1. *Let $F : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ be a smooth function, defined as $F(x_1, \ldots, x_n) = \sum_{i=1}^{n} f_i(x_i)$, where each f_i is a smooth function of one real variable. Let M^n be the graphic of F, given in coordinates by*

$$\varphi(x_1, \ldots, x_n) = \sum_{i=1}^{n} x_i e_i + F(x_1, \ldots, x_n) e_{n+1}. \tag{2}$$

The S_r curvature of M^n is given by

$$S_r = \frac{1}{W^{r+2}} \cdot \sum_{1 \leq i_1 < \ldots < i_r \leq n}^{n} \ddot{f}_{i_1} \ldots \ddot{f}_{i_r} (1 + \sum_{\substack{1 \leq m \leq n \\ m \neq i_1 \ldots i_r}} \dot{f}_m^2), \tag{3}$$

where the dot represents derivative with respect to the corresponding variable, that is, for each $j = 1, \ldots, n$, one has $\dot{f}_j = \dfrac{df_j}{dx_j}(x_j) = \dfrac{\partial F}{\partial x_j}(x_1, \ldots, x_n)$ and $W^2 = 1 + |\nabla F|^2$

Proof. Let F be as stated in the Proposition, denote by $\nabla F = \sum_{i=1}^{n} \dfrac{\partial F}{\partial x^i} \, e_i$ the Euclidean gradient of F and $\langle \, , \, \rangle$ the standard Euclidean inner product. Then, we have

$$\nabla F = \sum_{i=1}^{n} \dot{f}_i \; e_i$$

and the coordinate vector fields associated to the parametrization given in (2) have the following form

$$\frac{\partial \varphi}{\partial x_m} = e_m + \dot{f}_m e_{n+1}, \quad m = 1, \ldots, n.$$

Hence, the elements G_{ij} of the metric of M^n are given by

$$G_{ij} = \left\langle \frac{\partial \varphi}{\partial x^i}, \frac{\partial \varphi}{\partial x^j} \right\rangle = \delta_{ij} + \dot{f}_i \dot{f}_j,$$

implying that the matrix of the metric G has the following form

$$G = I_n + (\nabla F)^t \; \nabla F,$$

where I_n is the identity matrix of order n. Note that the i-th column of G, which will be denoted by G^i, has the expression given by the column vector

$$G^i = e_i + \dot{f}_i \nabla F. \tag{4}$$

An easy calculation shows that the unitary normal vector field ξ of M^n satisfies

$$W\xi = e_{n+1} - \nabla F,$$

where $W^2 = 1 + |\nabla F|^2$. Thus, the second fundamental form B_{ij} of M^n satisfies

$$WB_{ij} = \left\langle W\xi, \frac{\partial^2 \varphi}{\partial x^i \partial x^j} \right\rangle = \left\langle e_{n+1} - \nabla F, \delta_{ij} \ddot{f}_i e_{n+1} \right\rangle = \delta_{ij} \ddot{f}_i,$$

implying that the matrix of B is diagonal

$$B = \frac{1}{W} \cdot \mathrm{diag}(\ddot{f}_1, \ldots, \ddot{f}_n),$$

with i-th column given by the column vector

$$B^i = \frac{\ddot{f}_i}{W} \; e_i. \tag{5}$$

If A denotes the matrix of the Weingarten mapping, then $A = G^{-1}B$. In (1), changing λ by λ^{-1} gives

$$\det(\lambda A - I) = \sum_{i=1}^{n} (-1)^{n-i} S_i \lambda^i.$$

Thus, we conclude that the expression for curvature S_r can be found by the following calculation

$$(-1)^{n-r} S_r = \frac{1}{r!} \frac{d^r}{d\lambda^r}_{\big|\lambda=0} \det(\lambda A - I).$$

Note that

$$(-1)^{n-r} \det G \cdot S_r = \frac{1}{r!} \det G \cdot \frac{d^r}{d\lambda^r}_{\big|\lambda=0} \det(\lambda A - I) = \frac{1}{r!} \frac{d^r}{d\lambda^r}_{\big|\lambda=0} \det(\lambda B - G).$$

Due to the multilinearity of function det, on its n column vectors, it follows immediately that

$$\frac{d}{d\lambda}\Big|_{\lambda=0} \det\,[\lambda B^1 - G^1, \cdots, \lambda B^n - G^n] = \sum_{i=1}^{n} (-1)^{n-1} \det\,[G^1, \cdots, \underbrace{B^i}_{\text{i-th term}}, \cdots, G^n],$$

leading to the conclusion

$$\frac{d^r}{d\lambda^r}\Big|_{\lambda=0} \det\,(\lambda B - G) = r! \sum_{1\le i_1<\ldots<i_r\le n}^{n} (-1)^{n-r} \det\,[G^1, \cdots, B^{i_1}, \cdots, B^{i_r}, \cdots, G^n]$$

and thus

$$S_r = \frac{1}{\det G} \sum_{1\le i_1<\cdots<i_r\le n}^{n} \det\,[G^1, \cdots, B^{i_1}, \cdots, B^{i_r}, \cdots, G^n]. \tag{6}$$

Now, applying the expressions (4) and (5) in (6) we reach to the expression

$$S_r = \frac{1}{\det G \cdot W^r} \sum_{1\le i_1<\ldots<i_r\le n}^{n} \ddot{f}_{i_1} \ldots \ddot{f}_{i_r} \det\,[e_1 + \dot{f}_1 \nabla F, \ldots, e_{i_1}, \ldots, e_{i_r}, \ldots, e_n + \dot{f}_n \nabla F]. \tag{7}$$

Calculating the determinant on the right in the equality above, we get

$$\begin{aligned} \det\,[e_1 + \dot{f}_1\nabla F, \quad \ldots \quad &, e_{i_1}, \ldots, e_{i_r}, \ldots, e_n + \dot{f}_n \nabla F] = \\ &= 1 + \sum_{i\neq i_1,\ldots,i_r} \dot{f}_i \det\,[e_1, \ldots, e_{i_1}, \ldots, \underbrace{\nabla F}_{i\text{-th term}}, \ldots, e_{i_r}, \ldots, e_n] \\ &= 1 + \sum_{\substack{1\le i\le n \\ i\neq i_1,\ldots,i_r}} \dot{f}_i^2. \end{aligned}$$

Consequently, the expression for S_r in (7) assumes the following form

$$S_r = \frac{1}{\det G \cdot W^r} \sum_{1\le i_1<\ldots<i_r\le n}^{n} \ddot{f}_{i_1} \ldots \ddot{f}_{i_r} (1 + \sum_{\substack{1\le i\le n \\ i\neq i_1,\ldots,i_r}} \dot{f}_i^2).$$

Finally, using that $\det G = W^2$ we obtain the desired expression

$$S_r = \frac{1}{W^{r+2}} \sum_{1\le i_1<\ldots<i_r\le n}^{n} \ddot{f}_{i_1} \ldots \ddot{f}_{i_r} (1 + \sum_{\substack{1\le i\le n \\ i\neq i_1,\ldots,i_r}} \dot{f}_i^2).$$

□

RESULTS

In order to prove Theorem 1 we need the following lemma.

Lemma 1. *Let $f_1, \ldots, f_r$ be smooth functions of one real variable satisfying the differential equation*

$$\sum_{k=1}^{r} \ddot{f}_1(x_1) \ldots \widehat{\ddot{f}_k(x_k)} \ldots \ddot{f}_r(x_r)(\beta + \dot{f}_k^{\,2}(x_k)) = 0, \tag{8}$$

where β is a positive real constant and the big hat means an omitted term. If $\ddot{f}_i \neq 0$, for each $i = 1, \ldots r$ then

$$\sum_{k=1}^{r} f_k(x_k) = \sum_{k=1}^{r-1} \frac{\sqrt{\beta}}{a_k} \ln \left| \frac{\cos\left(-\frac{a_1 \ldots a_{r-1}}{\sigma_{r-2}(a_1, \ldots, a_{r-1})}\sqrt{\beta}x_r + b_r\right)}{\cos(a_k\sqrt{\beta}x_k + b_k)} \right| + c$$

where a_i, b_i, c, $i = 1, \ldots r$ are real constants with $a_i, \sigma_{r-2}(a_1, \ldots, a_{r-1}) \neq 0$.

Proof. Since the derivatives $\ddot{f}_i \neq 0$ it follows that $\ddot{f}_1(x_1) \ldots \ddot{f}_r(x_n) \neq 0$. Thus dividing (8) by this product we get the equivalent equation:

$$\sum_{k=1}^{r} \frac{\beta + \dot{f}_k^{\,2}(x_k)}{\ddot{f}_k(x_k)} = 0,$$

which implies, after taking derivative with respect to x_l for each $l = 1, \ldots r$, that $\left(\frac{\beta + \dot{f}_l^{\,2}(x_l)}{\ddot{f}_l(x_l)}\right)' = 0$, thus $\frac{\beta + \dot{f}_l^{\,2}(x_l)}{\ddot{f}_l(x_l)} = \tilde{a}_l$ for some non null constant $\tilde{a}_l$. Thus, setting $a_l = \frac{1}{\tilde{a}_l}$

$$\frac{\ddot{f}_l(x_l)}{\beta + \dot{f}_l^{\,2}(x_l)} = a_l \quad \text{for each } l = 1, \ldots, r$$

which can be easily solved to give:

$$\arctan\left(\frac{\dot{f}_l(x_l)}{\sqrt{\beta}}\right) = a_l\sqrt{\beta}x + b_l \quad \text{for some constant } b_l$$

and consequently

$$f_l(x_l) = -\frac{1}{a_l}\sqrt{\beta}\ln|\cos(a_l\sqrt{\beta}x_l + b_l)| + c_l, \quad l = 1, \ldots, r. \tag{9}$$

Now, since $\sum_{k=1}^{r} \frac{1}{a_k} = 0$ it implies that $\frac{1}{a_r} = -\frac{\sigma_{r-2}(a_1, \ldots, a_{r-1})}{a_1 \ldots a_{r-1}}$, from (9) it follows that

$$f_r(x_r) = \frac{\sigma_{r-2}(a_1, \ldots, a_{r-1})}{a_1 \ldots a_{r-1}}\sqrt{\beta}\ln|\cos(a_r\sqrt{\beta}x_r + b_r)| + c_r.$$

Consequently

$$\sum_{k=1}^{r} f_k(x_k) = \sum_{k=1}^{r-1} \frac{1}{a_k}\sqrt{\beta} \ln \left| \frac{\cos\left(-\frac{a_1 \ldots a_{r-1}}{\sigma_{r-2}(a_1, \ldots, a_{r-1})}\sqrt{\beta}x_r + b_r\right)}{\cos(a_k\sqrt{\beta}x_k + b_k)} \right| + c,$$

where $c = c_1 + \ldots + c_r$. □

With this lemma at hand we can go to the proof of Theorem 1.

Proof of the Theorem 1. From Proposition 1, we have that M^n has zero S_r curvature if, and only if,

$$\sum_{1\leq i_1<\ldots<i_r\leq n} \ddot{f}_{i_1}\ldots\ddot{f}_{i_r}\Big(1+\sum_{\substack{1\leq k\leq n\\ k\notin\{i_1,\ldots i_r\}}}\dot{f}_k^2\Big)=0. \tag{10}$$

In order to ease the analysis, we divide the proof in four cases.

Case 1: Suppose $\ddot{f}_i(x_i)=0, \forall\, i=1,\ldots,n-r+1$. In this case, we have no restrictions on the functions $f_{n-r+2},\ldots,f_n$. Thus

$$\Psi(x_1,\ldots,x_n)=(x_1,\ldots,x_n,\sum_{i=1}^{n-r+1}a_ix_i+\sum_{j=n-r+2}^{n}f_j(x_j)+b)$$

where $a_i, b\in\mathbb{R}$ and for $l=n-r+2,\ldots,n$, the functions $f_l: I_l\subset\mathbb{R}\to\mathbb{R}$ are arbitrary smooth functions of one real variable. Note that the parametrization obtained comprise hyperplanes.

Case 2: Suppose $\ddot{f}_i(x_i)=0, \forall\, i=1,\ldots,n-r$, then, there are constants α_i such that $\dot{f}_i=\alpha_i$, for $i=1,\ldots,n-r$. From (10) we have

$$\ddot{f}_{n-r+1}\ldots\ddot{f}_n(1+\alpha_1^2+\cdots+\alpha_{n-r}^2)=0,$$

from which we conclude that $\ddot{f}_k=0$ for some $k\in\{n-r+1,\ldots n\}$ and thus, this case is contained in the Case 1.

Case 3: Now suppose $\ddot{f}_i(x_i)=0, \forall\, i=1,\ldots,n-r-1$ and $\ddot{f}_k(x_k)\neq 0$, for every $k=n-r,\ldots,n$. Observe that if we had $\ddot{f}_k(x_k)=0$ for some $k=n-r,\ldots,n$ the analysis would reduce to the Cases 1 and 2. In this case, there are constants α_i such that $\dot{f}_i=\alpha_i$ for any $1\leq i\leq n-r-1$. From (10) we have

$$\sum_{k=n-r}^{n}\ddot{f}_{n-r}\ldots\widehat{\ddot{f}_k}\ldots\ddot{f}_n(\beta+\dot{f}_k^2)=0$$

where $\beta=1+\sum_{k=1}^{n-r-1}\alpha_k^2$ and the hat means an omitted term. Then, from Lemma 1 we have that

$$\sum_{k=n-r}^{n}f_k(x_k)=\sum_{k=n-r}^{n-1}\frac{\sqrt{\beta}}{a_k}\ln\left|\frac{\cos\left(-\dfrac{a_{n-r}\ldots a_{n-1}}{\sigma_{r-1}(a_{n-r},\ldots,a_{n-1})}\sqrt{\beta}x_n+b_n\right)}{\cos(a_k\sqrt{\beta}x_k+b_k)}\right|+c$$

where $a_{n-r},\ldots,a_{n-1},b_{n-r},\ldots,b_n$ and c are real constants, and $a_{n-r},\ldots,a_{n-1}$, and $\sigma_{r-1}(a_{n-r},\ldots,a_{n-1})$ are nonzero.

Case 4: Finally, suppose $\ddot{f}_i(x_i) = 0$, where $1 \leq i \leq k$ and $n - k \geq r + 2$, and $\ddot{f}_i(x_i) \neq 0$ for any $i > k$. We will show that this case cannot occur. In fact, note that for any fixed $l \geq k + 1$

$$\begin{aligned}\sum_{k+1\leq i_1<\ldots<i_r\leq n} \ddot{f}_{i_1}\ldots\ddot{f}_{i_r}\Big(1 &+ \sum_{\substack{1\leq m\leq n\\ m\neq i_1,\ldots i_r}} \dot{f}_m^2\Big)\\ &= \ddot{f}_l \sum_{\substack{k+1\leq i_1<\ldots<i_{r-1}\leq n\\ i_1,\ldots,i_{r-1}\neq l}} \ddot{f}_{i_1}\ldots\ddot{f}_{i_{r-1}}\Big(1+\sum_{\substack{1\leq m\leq n\\ m\neq l,i_1,\ldots,i_{r-1}}} \dot{f}_m^2\Big)\\ &+ \sum_{\substack{k+1\leq i_1<\ldots i_r\leq n\\ i_1,\ldots,i_r\neq l}} \ddot{f}_{i_1}\ldots\ddot{f}_{i_r}\Big(1+\sum_{\substack{1\leq m\leq n\\ m\neq i_1,\ldots i_r}} \dot{f}_m^2\Big)\end{aligned}$$

Derivative with respect to the variable x_l $(l \geq k+1)$, in the above equality, gives

$$\begin{aligned}\dddot{f}_l \sum_{\substack{k+1\leq i_1<\ldots<i_{r-1}\leq n\\ i_1,\ldots,i_{r-1}\neq l}} \ddot{f}_{i_1}\ldots\ddot{f}_{i_{r-1}}\Big(1 &+ \sum_{\substack{1\leq m\leq n\\ m\neq l,i_1,\ldots,i_{r-1}}} \dot{f}_m^2\Big)\\ &+ 2\dot{f}_l\ddot{f}_l \sum_{\substack{k+1\leq i_1<\ldots<i_r\leq n\\ i_1,\ldots,i_r\neq l}} \ddot{f}_{i_1}\ldots\ddot{f}_{i_r} = 0.\end{aligned} \tag{11}$$

That is, if we set

$$\begin{aligned}A_l &= \sum_{\substack{k+1\leq i_1<\ldots<i_{r-1}\leq n\\ i_1,\ldots,i_{r-1}\neq l}} \ddot{f}_{i_1}\ldots\ddot{f}_{i_{r-1}}\Big(1+\sum_{\substack{1\leq m\leq n\\ m\neq l,i_1,\ldots,i_{r-1}}} \dot{f}_m^2\Big) \quad \text{and}\\ B_l &= \sum_{\substack{k+1\leq i_1<\ldots<i_r\leq n\\ i_1,\ldots,i_r\neq l}} \ddot{f}_{i_1}\ldots\ddot{f}_{i_r}\end{aligned}$$

then, it follows that A_l, B_l do not depend on the variable x_l and we can write

$$A_l\dddot{f}_l + 2B_l\dot{f}_l\ddot{f}_l = 0. \tag{12}$$

We have two possible situations to take into account: **Case I.** $A_l \neq 0$, $\forall\, l \geq k+1$, and **Case II.** there is an $l \geq k+1$ such that $A_l = 0$.

Case I. $A_l \neq 0$**:** Under this assumption, there are constants α_l $(l = k+1, \ldots, n)$ such that equation (12) becomes $\dddot{f}_l + 2\alpha_l\dot{f}_l\ddot{f}_l = 0$. Furthermore, it can be shown that for $\{l_1, \ldots, l_{r+1}\} \subset \{k+1, \ldots, n\}$

$$\frac{\partial^{r+1}G_r(f_1,\ldots,f_n)}{\partial x_{l_1}\cdots\partial x_{l_{r+1}}} = 2\sum_{k=1}^{r+1}\left(\dot{f}_{l_k}\,\ddot{f}_{l_k}\prod_{\substack{m=1\\ m\neq k}}^{r+1}\dddot{f}_{l_m}\right) \tag{13}$$

where

$$G_r(f_{k+1},\ldots,f_n) := W^{r+2}S_r = \sum_{k+1\leq i_1<\ldots<i_r\leq n}^{n} \ddot{f}_{i_1}\ldots\ddot{f}_{i_r}(1+\sum_{\substack{1\leq i\leq n\\ i\neq i_1,\ldots,i_r}} \dot{f}_i^2).$$

Since $S_r = 0$ it follows that $G_r = 0$, and using that $\prod_{k=1}^{r+1} \dot{f}_{l_k} \ddot{f}_{l_k} \neq 0$ we obtain

$$\sum_{k=1}^{r+1} \left(\prod_{\substack{m=1 \\ m\neq k}}^{r+1} \frac{\dddot{f}_{l_m}}{\dot{f}_{l_m} \ddot{f}_{l_m}} \right) = \frac{\sum_{s=1}^{r+1} \left(\dot{f}_{l_s} \ddot{f}_{l_s} \prod_{\substack{m=1 \\ m\neq s}}^{r+1} \dddot{f}_{l_m} \right)}{\prod_{k=1}^{r+1} \dot{f}_{l_k} \ddot{f}_{l_k}} = 0. \tag{14}$$

Now, for $l = l_1, \dots l_{r+1}$, substitute $\dddot{f}_l + 2\alpha_l \dot{f}_l \ddot{f}_l = 0$ in (14) to obtain the identity

$$\sigma_r(\alpha_{l_1}, \dots, \alpha_{l_r}, \alpha_{l_{r+1}}) = 0 \tag{15}$$

for any $l_1, \dots, l_r, l_{r+1} \in \{k+1, \dots, n\}$. Hence we conclude that,

$$\begin{aligned} \sigma_r(\alpha_{k+1}, \dots, \alpha_n) &= 0 \\ \sigma_{r+1}(\alpha_{k+1}, \dots, \alpha_n) &= 0. \end{aligned}$$

These equalities, from [Proposition 1, Caminha (2006)], imply that at most $r-1$ of the constants α_l $(l \geq k+1)$ are nonzero. If $\alpha_{l_1} \neq 0, \dots, \alpha_{l_m} \neq 0$ with $m \leq r-1$, in the expression obtained for B_l, making $l \neq l_1, \dots, l_m$ and taking derivatives with respect to the variables $x_{l_1}, \dots, x_{l_m}$ we get

$$\prod_{j=l_1}^{l_m} \dddot{f}_j \cdot \sigma_{r-m}(\ddot{f}_{k+1}, \dots, \widehat{\ddot{f}_l}, \dots, \widehat{\ddot{f}}_{l_1}, \dots, \widehat{\ddot{f}}_{l_m}, \dots, \ddot{f}_n) = 0$$

for all $l \in \{k+1, \dots, n\} \setminus \{l_1, \dots, l_m\}$. As $\dddot{f}_j \neq 0$ for all $j \in \{l_1 \dots, l_m\}$, we obtain that

$$\sigma_{r-m}(\ddot{f}_{k+1}, \dots, \widehat{\ddot{f}_l}, \dots, \widehat{\ddot{f}}_{l_1}, \dots, \widehat{\ddot{f}}_{l_m}, \dots, \ddot{f}_n) = 0$$

for all $l \in \{k+1, \dots, n\} \setminus \{l_1, \dots, l_m\}$. Consequently,

$$\begin{aligned} \sigma_{r-m}(\ddot{f}_{k+1}, , \dots, \widehat{\ddot{f}}_{l_1}, \dots, \widehat{\ddot{f}}_{l_m}, \dots, , \ddot{f}_n) &= 0 \\ \sigma_{r-m+1}(\ddot{f}_{k+1}, , \dots, \widehat{\ddot{f}}_{l_1}, \dots, \widehat{\ddot{f}}_{l_m}, \dots, , \ddot{f}_n) &= 0. \end{aligned}$$

Since $(n-k-m)-(r-m) = n-k-r \geq 2$, at most $r-m-1$ of the functions $\ddot{f}_l$ are nonzero, for $k+1 \leq l \leq n$ and $l \neq l_1, \dots, l_m$, leading to a contradiction. So, $\alpha_j = 0$ for all $j \in \{l_1 \dots, l_{r-1}\}$, which implies that $\ddot{f}_l$ is constant for all $l \in \{k+1, \dots, n\}$. Now, again from equation (11) we get

$$\sum_{\substack{k+1 \leq i_1 < \dots < i_r \leq n \\ i_1, \dots, i_r \neq l}} \ddot{f}_{i_1} \dots \ddot{f}_{i_r} = 0, \qquad \text{for any } l \in \{k+1, \dots, n\}.$$

From which, we conclude that

$$\begin{aligned} \sigma_r(\ddot{f}_{k+1}, \dots, \ddot{f}_n) &= 0 \\ \sigma_{r+1}(\ddot{f}_{k+1}, \dots, \ddot{f}_n) &= 0. \end{aligned}$$

Therefore, at most $r-1$ of the functions $\ddot{f}_l$ $(k+1 \leq l \leq n)$ are nonzero, leading to a contradiction. Thus, it follows that Case 4 cannot occur, if $A_l \neq 0$ for every l.

Case $A_l = 0$: In this case, we have $B_l \dot{f}_l \ddot{f}_l = 0$ implying

$$\begin{aligned} A_l &= \sum_{\substack{k+1 \leq i_1 < \ldots < i_{r-1} \leq n \\ i_1, \ldots, i_{r-1} \neq l}} \ddot{f}_{i_1} \ldots \ddot{f}_{i_{r-1}} \Big(1 + \sum_{\substack{1 \leq m \leq n \\ m \neq l, i_1, \ldots, i_{r-1}}} \dot{f}_m^2 \Big) = 0 \quad \text{and} \\ B_l &= \sum_{\substack{k+1 \leq i_1 < \ldots < i_r \leq n \\ i_1, \ldots, i_r \neq l}} \ddot{f}_{i_1} \ldots \ddot{f}_{i_r} = 0. \end{aligned}$$

Derivative of A_l with respect to variable x_s, for $s = k+1, \ldots, n$ and $s \neq l$, gives

$$\begin{aligned} &\dddot{f}_s \sum_{\substack{k+1 \leq i_1 < \ldots < i_{r-2} \leq n \\ i_1, \ldots, i_{r-2} \neq l, s}} \ddot{f}_{i_1} \ldots \ddot{f}_{i_{r-2}} \Big(1 + \sum_{\substack{k+1 \leq m \leq n \\ m \neq l, s, i_1, \ldots, i_{r-2}}} \dot{f}_m^2 \Big) \\ &+ 2 \dot{f}_s \ddot{f}_s \sum_{\substack{k+1 \leq i_1 < \ldots < i_{r-1} \leq n \\ i_1, \ldots, i_{r-1} \neq l, s}} \ddot{f}_{i_1} \ldots \ddot{f}_{i_{r-1}} = 0. \end{aligned} \tag{16}$$

Now, for $i_1, \ldots, i_r \in \{k+1, \ldots, n\}$ with $i_1, \ldots, i_r, l$ distinct indices, taking the derivatives of B_l with respect to $x_{i_1}, \ldots, x_{i_r}$ gives

$$\dddot{f}_{i_1} \ldots \dddot{f}_{i_r} = 0.$$

Consequently, for at most $r-1$ indices, say $i_1, \ldots, i_{r-1}$, we can have $\dddot{f}_{i_m} \neq 0$, $(m = 1, \ldots, r-1)$, and $\dddot{f}_j = 0$ for every $j = k+1, \ldots, n$, with $j \neq l, i_1, \ldots, i_{r-1}$. Thus $\dddot{f}_{i_m} \neq 0$, with $i_m \neq l$, together with equation $\frac{\partial B_l}{\partial x_{i_m}} = 0$ implies that the sum

$$\sum_{\substack{k+1 \leq i_1 < \ldots < i_{r-1} \leq n \\ i_1, \ldots, i_{r-1} \neq l, i_m}} \ddot{f}_{i_1} \ldots \ddot{f}_{i_{r-1}} = 0.$$

Now, if $\dddot{f}_j = 0$ we have by equation (16) that

$$\sum_{\substack{k+1 \leq i_1 < \ldots < i_{r-1} \leq n \\ i_1, \ldots, i_{r-1} \neq l, j}} \ddot{f}_{i_1} \ldots \ddot{f}_{i_{r-1}} = 0.$$

Therefore,

$$\sum_{\substack{k+1 \leq i_1 < \ldots < i_{r-1} \leq n \\ i_1, \ldots, i_{r-1} \neq l, j}} \ddot{f}_{i_1} \ldots \ddot{f}_{i_{r-1}} = 0, \quad j = k+1, \ldots, n \text{ and } j \neq l$$

From which, we conclude that

$$\begin{aligned} \sigma_{r-1}(\ddot{f}_{k+1}, \ldots, \widehat{\ddot{f}_l}, \ldots, \ddot{f}_n) &= 0 \\ \sigma_r(\ddot{f}_{k+1}, \ldots, \widehat{\ddot{f}_l}, \ldots, \ddot{f}_n) &= 0. \end{aligned}$$

Thus, for at most $r-2\,(r \geq 3)$ indices we must have $\ddot{f}_j \neq 0$, for every $j = k+1, \ldots, n$, and $j \neq l$. This contradicts the hypothesis assumed in Case 4. Hence, $A_l = 0$ cannot occur. Since the case $A_l \neq 0$, cannot occur as well, it follows that Case 4 is not possible. This completes the proof of the theorem. □

Proof of the Theorem 2. Let $M^n \subset \mathbb{R}^{n+1}$ be a translation hypersurface with constant S_r curvature. First, note that

$$\frac{\partial^m W^{r+2}}{\partial x_{i_1} \cdots \partial x_{i_m}} = \prod_{j=1}^{m}(r+4-2j) \cdot \prod_{k=1}^{m} \dot{f}_{i_k}\, \ddot{f}_{i_k} \cdot W^{r+2-2m}. \tag{17}$$

We have as a consequence of the proof of Theorem 1, see (13), the identity

$$\frac{\partial^{r+1} G_r(f_1,\ldots,f_n)}{\partial x_{l_1} \cdots \partial x_{l_{r+1}}} = 2\sum_{k=1}^{r+1}\left(\dot{f}_{l_k}\, \ddot{f}_{l_k} \prod_{\substack{m=1\\ m\neq k}}^{r+1} \dddot{f}_{l_m}\right)$$

where $G_r(f_1,\ldots,f_n) = \displaystyle\sum_{1\le i_1<\ldots<i_r\le n}^{n} \ddot{f}_{i_1}\ldots\ddot{f}_{i_r}(1+\sum_{\substack{1\le i\le n\\ i\neq i_1,\ldots,i_r}} \dot{f}_i^2)$. With this we conclude, by Proposition 1, that

$$\begin{aligned}\prod_{j=1}^{r+1}(r+4-2j)\cdot\prod_{k=1}^{r+1}\dot{f}_{l_k}\,\ddot{f}_{l_k}\cdot W^{-r}S_r &= \frac{\partial^{r+1}\left(W^{r+2}S_r\right)}{\partial x_{l_1}\cdots\partial x_{l_{r+1}}}\\ &= 2\sum_{k=1}^{r+1}\left(\dot{f}_{l_k}\,\ddot{f}_{l_k}\prod_{\substack{m=1\\ m\neq k}}^{r+1}\dddot{f}_{l_m}\right).\end{aligned} \tag{18}$$

Now, we have two cases to consider: r odd and r even.

Case r odd: Suppose that there are $l_1,\ldots,l_{r+1}$ such that $\displaystyle\prod_{k=1}^{r+1}\dot{f}_{l_k}\,\ddot{f}_{l_k} \neq 0$. Then,

$$\begin{aligned} Q_r := \prod_{j=1}^{r+1}(r+4-2j)\cdot W^{-r}S_r &= 2\frac{\displaystyle\sum_{s=1}^{r+1}\left(\dot{f}_{l_s}\,\ddot{f}_{l_s}\prod_{\substack{m=1\\ m\neq s}}^{r+1}\dddot{f}_{l_m}\right)}{\displaystyle\prod_{k=1}^{r+1}\dot{f}_{l_k}\,\ddot{f}_{l_k}}\\ &= 2\sum_{k=1}^{r+1}\left(\prod_{\substack{m=1\\ m\neq k}}^{r+1}\frac{\dddot{f}_{l_m}}{\dot{f}_{l_m}\ddot{f}_{l_m}}\right).\end{aligned}$$

Therefore,

$$\frac{\partial^{r+1}Q_r}{\partial x_{l_1}\cdots\partial x_{l_{r+1}}} = 0.$$

On the other hand, using (17) we obtain

$$\frac{\partial^{r+1}Q_r}{\partial x_{l_1}\cdots\partial x_{l_{r+1}}} = \prod_{j=1}^{r+1}(r+4-2j)\prod_{i=1}^{r+1}(-r+2-2i)\prod_{k=1}^{r+1}\dot{f}_{l_k}\,\ddot{f}_{l_k}W^{-3r-2}S_r.$$

Since r is odd, we conclude that $r+4-2j \neq 0$ and $-r+2-2j \neq 0$, for any $j \in \mathbb{N}$ and, therefore, $S_r = 0$. Now, if for at most r indices we have $\ddot{f}_j \neq 0$ for example $j = l_1, \ldots, l_r$ then

$$W^{r+2} S_r = \ddot{f}_{l_1} \cdots \ddot{f}_{l_r} \alpha,$$

for some constant $\alpha \neq 0$. Thus,

$$(r+2) W^r \dot{f}_{l_1} \ddot{f}_{l_1} S_r = \dddot{f}_{l_1} \ddot{f}_{l_2} \cdots \ddot{f}_{l_r} \alpha.$$

If $\dddot{f}_{l_1} = 0$, then $S_r = 0$. Otherwise,

$$(r+2) W^{r+2} \dot{f}_{l_1} \ddot{f}_{l_1} S_r = \dddot{f}_{l_1} \ddot{f}_{l_2} \cdots \ddot{f}_{l_r} W^2 \alpha \quad \Rightarrow \quad (r+2) \dot{f}_{l_1} (\ddot{f}_{l_1})^2 = \dddot{f}_{l_1} W^2.$$

As $r > 1$ implies that W does not depend on the variables $x_{l_2}, \ldots, x_{l_n}$, it follows that $\ddot{f}_{l_2} = \cdots = \ddot{f}_{l_n} = 0$ leading to a contradiction.

Case r even: In this case, there is a natural $q \geq 2$ such that $r = 2q$. Then $r+1 \geq q+2$ and consequently

$$\prod_{k=1}^{r+1} (r+4-2k) = 0.$$

Therefore, by (18) we get

$$\sum_{k=1}^{r+1} \left(\dot{f}_{l_k} \ddot{f}_{l_k} \prod_{\substack{m=1 \\ m \neq k}}^{r+1} \dddot{f}_{l_m} \right) = 0.$$

Suppose that there are $l_1, \ldots, l_{r+1}$ such that $\prod\limits_{k=1}^{r+1} \ddot{f}_{l_k} \neq 0$. In this case,

$$\sum_{k=1}^{r+1} \left(\prod_{\substack{m=1 \\ m \neq k}}^{r+1} \frac{\dddot{f}_{l_m}}{\dot{f}_{l_m} \ddot{f}_{l_m}} \right) = 0.$$

We conclude that for each l_i there is a constant α_{l_i} such that $\dddot{f}_{l_i} = \alpha_{l_i} \dot{f}_{l_i} \ddot{f}_{l_i}$. Now, it is easy to verify (see (11)) that

$$\begin{aligned} (r+2) \dot{f}_{l_{r+1}} \ddot{f}_{l_{r+1}} W^r S_r &= \frac{\partial G_r(f_1, \ldots, f_n)}{\partial x_{l_{r+1}}} \\ &= \dddot{f}_{l_{r+1}} G_{r-1}(f_1, \ldots, \widehat{f_l}, \ldots, f_n) \\ &+ 2 \dot{f}_{l_{r+1}} \ddot{f}_{l_{r+1}} \sum_{\substack{1 \leq i_1 < \ldots < i_r \leq n \\ i_1, \ldots, i_r \neq l_{r+1}}} \ddot{f}_{i_1} \ldots \ddot{f}_{i_r}. \end{aligned}$$

Therefore,

$$(r+2) W^r S_r = \alpha_{l_{r+1}} G_{r-1}(f_1, \ldots, \widehat{f}_{l_{r+1}}, \ldots, f_n) + 2 \sum_{\substack{1 \leq i_1 < \ldots < i_r \leq n \\ i_1, \ldots, i_r \neq l_{r+1}}} \ddot{f}_{i_1} \ldots \ddot{f}_{i_r}.$$

Differentiating this identity with respect to the variable $x_{l_{r+1}}$, gives

$$(r+2)\, r\, \dot{f}_{l_{r+1}} \ddot{f}_{l_{r+1}} W^{r-2} S_r = 0 \quad \text{implying that} \quad S_r = 0.$$

Finally, suppose that for any $(r+1)$-tuple of indices, say $l_1, \ldots, l_{r+1}$ it holds that $\prod_{k=1}^{r+1} \ddot{f}_{l_k} = 0$. Then,

$$\begin{aligned} \sigma_{r+1}(\ddot{f}_1, \ldots, \ddot{f}_n) &= 0 \\ \sigma_{r+2}(\ddot{f}_1, \ldots, \ddot{f}_n) &= 0. \end{aligned}$$

Implying that at least $n-r$ derivatives $\ddot{f}_l$ vanish, i.e., there are at most r functions such that $\ddot{f}_j \neq 0$ for example $j = l_1, \ldots, l_r$. Thus, by Proposition 1

$$W^{r+2} S_r = \ddot{f}_{l_1} \cdots \ddot{f}_{l_r} \alpha$$

for some constant $\alpha \neq 0$. We conclude that $S_r = 0$ analogously to the way it was presented for the case r odd. □

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