Continuous Time Stochastic Models for Vehicular Traffic on Highways

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1 Introduction

Vehicular traffic flow is a subject, outside of the traditional areas of physics, that has been the object of investigation by the methods of statistical physics [1-7]. Stochastic nonequilibrium models defined on lattices are useful tools on the study of vehicular traffic from the microscopic point of view [6, 7]. In this approach, the traffic of vehicles is modeled by a system of interacting particles, each particle representing an individual vehicle. The evolution of the system is defined by dynamical irreversible rules implying that these systems should be studied within the framework of nonequilibrium statistical mechanics [8-13]. We will be concerned here with the so-called “particle-hoping” models. Originally, these models were defined as cellular automata, with deterministic rules, or as probabilistic cellular automata, with stochastic rules. In both cases the updating is synchronous. In this paper, however, we will examine the continuous time version of such models, with sequential updating or, more precisely, random sequential updating and whose time evolution of the probability distribution is governed by a master equation.

The stochastic microscopic models which we investigate here are defined by stochastic rules that are suitable for computer simulation. Numerical simulations are useful since analytical approaches for nonequilibrium systems are very difficult to implement. Even though, we will present here some analytical results coming from mean-field approximations and compare them with numerical results. We will focus our attention on a continuous time version of the Nagel-Schreckenberger cellular automaton model [14] of vehicular traffic on highways (model A) and a modification of it that takes into account the velocity of the vehicle moving ahead of another one (model B).

The main property we want to find is the relation between the flux and density, usually called the fundamental diagram of road traffic. For low densities, empirical data [15] shows a linear dependence of the flux on density. Increasing the density, the flux reaches a maximum (maximum capacity) and then drops. These properties are found for both models studied here. For model A, the flux vanishes when the density is the closed-packed density. However, for model B, the flux vanishes at a critical density which is strictly less than the closed-packed density.

2 Model A

Let us consider a system of \( N \) particles moving in a one dimensional lattice with \( L \) sites and periodic boundary conditions. Each site can be either empty or occupied by a particle (vehicle). The particles move forwardly and may have any velocity up to a maximum velocity. The position and the velocity of the \( j \) particle are denoted by \( x_j \) and \( v_j \), respectively. The distance between two consecutive sites is \( a \) so that \( x_j = a n_j \) where \( n_j \) can take the values \( 0, 1, \ldots, L - 1 \). The velocities of the particles take on discrete values \( v_j = u \nu \) where \( \nu = 0, 1, \ldots, \nu_m \), so that the maximum velocity is \( \nu_m = u \nu_m \). For simplicity we assume \( a = 1 \) and \( u = 1 \).

The state of the system is defined by the collection of the positions \( \{ x_j \} \) and velocities \( \{ v_j \} \) of the \( N \) particles. The time evolution of the system is defined as follows. At each time interval \( \tau / L \) one particle is chosen at random and its velocity and position are changed according to the following rules. The velocity is increased by one unit except if it has already the maximum velocity in which case it remains with the maximum velocity. The position of the particle is increased by an amount equal to the new velocity. In this model, however, there is no overtaking and two particles cannot occupy the same site. Therefore, one has to check whether there is enough space between the chosen particle and the particle moving ahead. If not, the new velocity will
We define the interval of time $\tau$ as the time it takes to make $N$ movements of particles or trials (a Monte Carlo step). The time step between two successive movement trials will be $\tau/N$. Since there are $N$ particles the average time between two movements of the same particle will be just $\tau$. Therefore, we must have the following relation $u = a\tau$.

The density of particles $\rho = N/(aL)$ is the number of particle per unit length. Since we are assuming $a = 1$ then $\rho = N/L$. The fundamental property we want to analyze is the flux $\phi$ of particles as a function of the density $\rho$. The flux is the average number of particles crossing a certain point per unit time. If $M$ is the number of particle crossing a specified point of the lattice in the time interval $\tau$ then $\phi = M/\tau$. If we denote by $v$ the average velocity of particles given by

$$v = \frac{1}{N} \sum_j (v_j).$$

then the flux is given by

$$\phi = \rho v.$$

The double-log plot of $\phi$ versus density of the system with lattice size $L = 10^5$. A straight line fitted to the data points at small densities has slope 0.42.

Using rules (1) and (2) we have simulated a system of $N$ particles with periodic boundary conditions with lattices of size $L$ sites. Fig. 1 shows the flux of particles as a function of the density for $L = 1000$ and for several values of the maximum velocity $v_m$. The flux increases linearly at small densities, reaches a maximum and then decreases at high densities. For small densities the flux increases linearly according to

$$\phi = \rho v_m,$$

as long as $v_m$ is finite. If the maximum velocity is infinite, that is, if the velocity of a particle may increase without bounds, the behavior (5) is no longer valid. We assume then the following power law behavior for small densities

$$\phi \sim \rho^\alpha.$$  

The mean-field approximation given by (10) gives the general behavior expected for the flux and is correct at low and high densities as can be seen in Fig. 3.
3 Model B

In this section we set up a model to describe more precisely the interaction between a vehicle, say vehicle \( i \), and the vehicle which is moving ahead, vehicle \( i + 1 \). In model A the velocity of the former does not depend on the velocity of the latter. In model B, if the velocity of the vehicle \( i + 1 \) is zero and the gap between vehicles \( i \) and \( i + 1 \) is 1 then the vehicle \( i \) is not allowed to occupy the gap and it remains in its place with zero velocity. Therefore the rules are the same as those given by equations (1) and (2) except when \( \ell_i = 1 \) in which case the new velocity is given by

\[
  v'_j = 0 \quad \text{if} \quad v_{i+1} = 0, \quad (11)
\]

\[
  v'_j = 1 \quad \text{if} \quad v_{i+1} \neq 0. \quad (12)
\]

We have simulated model B for several values of the maximum velocity. The flux as a function of the density is shown in Fig. 4. At small densities the behavior is the same as the model A. However, at high density the behavior is entirely distinct from model A. As shown in Fig. 4, there is a critical density \( \rho_c \) above which the flux vanishes. Table I shows the values of the critical density for several values of the maximum velocity \( v_m \). As one increases the density of vehicles, model B displays, therefore, a nonequilibrium phase transition from a state with nonzero flux to a jammed state with zero flux. The critical behavior of the flux near the transition is

\[
  \phi \sim (\rho_c - \rho)^\beta, \quad (13)
\]

with \( \beta = 1 \).

![Figure 3. Flux \( \phi \) versus density of vehicles \( \rho \) for model A obtained from a mean-field approximation. The curves correspond, from bottom to top, to \( v_m = 1, 2, 3, \) and 7, respectively.](image)

![Figure 4. Flux \( \phi \) versus density of vehicles \( \rho \) for model B for several values of the maximum velocity \( v_m \) obtained from simulation of system with lattice size \( L = 1000 \).](image)

<table>
<thead>
<tr>
<th>( v_m )</th>
<th>( \rho_c )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.582(1)</td>
</tr>
<tr>
<td>2</td>
<td>0.679(1)</td>
</tr>
<tr>
<td>3</td>
<td>0.768(1)</td>
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<tr>
<td>4</td>
<td>0.760(1)</td>
</tr>
<tr>
<td>7</td>
<td>0.768(1)</td>
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<tr>
<td>15</td>
<td>0.768(1)</td>
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<tr>
<td>( \infty )</td>
<td>0.768(1)</td>
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</tbody>
</table>

![Table 1: Critical density \( \rho_c \) for several values of the maximum velocity of vehicles \( v_m \) for model B.](image)
move. If after a certain number of Monte Carlo steps $T$, previously defined, the vehicles are still moving, we insert a vehicle into the system. If, on the other hand, the flux vanishes, before the maximum time $T$, we remove a vehicle, chosen at random. In other words, we remove a vehicle whenever the system falls into a jammed state and insert one if the flux is nonzero. At the beginning the density of vehicles $\rho$ in the system decreases linearly from the maximum value $\rho = 1$ until it reaches a value where it begins to fluctuates around a certain value $\rho^*$ of the density as can be seen in Fig. 5 for the case of system size $L = 1000$ and maximum number of Monte Carlos steps $T = 1000$. The inset of Fig. 5 shows that the density fluctuates around the value $\rho^* = 0.731$.

The average density $\rho^*$ depends on the system size $L$ and on the maximum number of Monte Carlo steps $T$, and is expected to approach $\rho_c$ as $L \to \infty$ and $T \to \infty$.

![Figure 5](image-url)  
**Figure 5.** Density of particles for model B with infinity maximum velocity as a function of $t$, a quantity that increases by one unit whenever a particle is removed or added to the system. The inset shows a histogram of the densities with average $\rho^* = 0.731$. The data are taken from a system with $L = 1000$ and $T = 1000$.

## 4 Conclusion

We have simulated two stochastic models for vehicular traffic on highways. In our approach, the traffic of vehicles is modeled by a system of interacting particles, each particle representing an individual vehicle. The first model is a continuous time version of the Nagel-Schreckenberg model. In the second we introduce an interaction between a vehicle and the vehicle moving ahead. We found that this second model has a phase transition from a state with nonzero flux to a jammed state which occurs for a density of particles which is strictly less than the closed-packed density.

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**References**