The Effective Charge Velocity of Spin-$\frac{1}{2}$ Superlattices

J. Silva-Valencia*, R. Franco†, and M. S. Figueira ‡

*Departamento de Física, Universidad Nacional de Colombia, A. A. 5997, Bogotá, Colombia
†Instituto de Física, Universidade Federal Fluminense (UFF), Avenida litorânea s/n, CEP: 24210-340, Caixa Postal: 100.093, Niterói, Rio de Janeiro, Brazil

Received on 8 December, 2005

We calculate the spin gap of homogeneous and inhomogeneous spin chains, using the White’s density matrix renormalization group technique. We found that the spin gap is related to the ration between the spin velocity and the correlation exponent. We consider a spin superlattice, which is composed of a repeated pattern of two spin-$\frac{1}{2}$ XXZ chains with different anisotropy parameters. The behavior of the charge velocity as a function of the anisotropy parameter and the relative size of sub-chains was investigated. We found reasonable agreement between the bosonization results and the numerical ones.

Keywords: Heisenberg; DMRG; Plateaus; Luttinger liquid

I. INTRODUCTION

The study of one dimensional spin systems has increased in the last decade, impelled by theoretical results such as the Haldane conjecture[1, 2], which affirms that the ground state of isotropic Heisenberg chains with integer spin are gapful, whereas half-integer spin ones are gapless. Also, the synthesis of new materials has been crucial, because interesting phenomena such as the magnetization plateaus were observed[3]. Oshikawa, Yamanaka and Affleck[4] derived the condition $p(S-m^2) = \text{integer}$, necessary for the appearance of the magnetization plateaus in 1D systems. Here, $p$ is the number of sites in the unit cell of the magnetic ground state. $S$ is the magnitude of the spin and $m^2$ is the magnetization per site (taken to be in the $z$-direction).

The low-energy properties of spin chains with spin $S$ in partially magnetized phases are described by the one-component Luttinger liquid theory. The first parameter of this theory is the location of the Fermi points $\pm k_F$, given by $2k_F = 2\pi(S-m)$, where $m$ is the magnetization. The second parameter, the spin velocity is just an energy scale, whereas the third parameter determines the universality class and the critical exponents. Spin chains with $S = 1/2$ can be solved exactly using the Bethe Ansatz, particularly for the anisotropic model Yang and Yang[5] found a Luttinger liquid (gapless) phase for $-1 < \Delta < 1$, where $\Delta$ is the anisotropy parameter. The validity of the Luttinger liquid theory has also been checked numerically for $S = 1[6]$, and in fact, it is conjectured in general for higher $S$[7].

Different Inhomogeneous spin chains has been studied in the last years[8–10]. These systems are obtained when we consider the spatial variation of the coupling constants or an inhomogeneous magnetic field. The special case of spin superlattice (SS) composed of a repeated pattern of two long and different spin-$\frac{1}{2}$ XXZ chains, was considered by one of us in a previous work[10]. We found that the magnetization curve presents a nontrivial plateau whose magnetization value depends on the relative size of sub-chains $\ell = L_2/L_1$ and is given by $M_\ell = 1/(1+\ell)$. For away from the plateaus gapless phases appears, which were described in terms of Luttinger liquid superlattice model parameterized by an effective velocity and an effective correlation exponent[11]. Here we extent the previous study of the gapless region using the White’s density matrix renormalization group technique[12, 13]. We considered lattice sizes up to 100 sites with up to $m = 600$ states per block. The truncation errors were below $10^{-9}$.

II. MODEL AND RESULTS

Consider a SS whose unit cell consists of two $S = 1/2$ XXZ chains with different anisotropy parameters $\Delta_k$ and sizes $L_k$ ($\lambda = 1, 2$) (but the same planar coupling) in the presence of a magnetic field $h$ applied along the anisotropy ($z$)-axis. Its Hamiltonian is

$$H = J \sum_{n=1}^{L} \left( S_n^z S_{n+1}^z + S_n^x S_{n+1}^x + \Delta_k S_n^y S_{n+1}^y \right) - h \sum_{n=1}^{L} S_n^z,$$

(1)

where $S^x$, $S^y$ and $S^z$ denote the spin-$\frac{1}{2}$ operators and $L = N_c (L_1 + L_2)$ is the superlattice size. Here, $N_c$ is the number of unit cells, each of which has a basis with $L_1 + L_2$ sites. We assume the chain is subjected to periodic boundary conditions. The homogeneous situation is recovered when $\Delta_k = \Delta$, independent of the position.

We then take advantage of the fact that each sub-chain is a LL connected at its ends to reservoirs (the rest of the lattice) to describe the low-energy properties of the SS in terms of a LL superlattice (LLSL)[11] with Hamiltonian

$$H = \frac{1}{2\pi} \int dx \left\{ u(x) K(x) \left( \partial_x \Theta \right)^2 + \frac{u(x)}{K(x)} \left( \partial_x \Phi \right)^2 \right\}.$$

(2)

Here, we have introduced the sub-chain-dependent parameters $u(x)$ and $K(x)$. For $x$ on the sublattice $\lambda$, one has $K(x) = K(J, \Delta_\lambda, h)$ and $u(x) = u(J, \Delta_\lambda, h)$, i.e., the usual uniform LL parameters for each sub-chain, which can be obtained directly from the Bethe Ansatz solution[14].

In the Hamiltonian (2), $\partial_x \Theta$ is the momentum field conjugate to $\Phi$: $\{ \Phi(x), \partial_x \Theta(y) \} = i\delta(x-y)$. $\Phi$ and $\Theta$ are dual fields,
subject to the same boundary conditions at the contacts as before, with \( \phi_p(x) \) replacing \( \Phi(x) \). The eigenvalues are given by

\[
\cos p(L_1 + L_2) = \cos \left( \frac{\omega_p L_2}{u_2} \right) \cos \left( \frac{\omega_p L_1}{u_1} \right) - \frac{\eta}{2} \sin \left( \frac{\omega_p L_2}{u_2} \right) \sin \left( \frac{\omega_p L_1}{u_1} \right),
\]

where \( \eta = K_1/K_2 + K_2/K_1 \). For \( p \ll \pi/(L_2 + L_1) \), \( \omega_p \approx |p| \), and the effective velocity for the SS is

\[
c = \frac{u_1 (1 + \ell)}{\sqrt{1 + \eta u_1 u_2 + (\ell u_1 u_2)^2}},
\]

where \( \ell = L_2/L_1 \). Clearly, \( c \to u_2 \) as \( \ell \to \infty \), and \( c \to u_1 \) as \( \ell \to 0 \). In terms of the bosonic fields, the spin operators read

\[
S^z = \frac{M}{2} - \frac{1}{\sqrt{2\pi}} \Phi + \frac{1}{\pi \alpha} \cos \left[ 2c \Phi(x) - 2\delta(x) \right],
\]

\[
S^+ = \frac{1}{\sqrt{2\pi \alpha}} e^{-i\Phi(x)} \left\{ 1 + e^{2\delta(x)} \cos \left[ 2c \Phi(x) \right] \right\},
\]

where \( \delta(x) = k_F x - \phi_0(x) \), the Fermi momentum \( k_F \) is related to the magnetization by \( k_F = (1 + M)/\pi/2 \) and \( \alpha \) is a cutoff parameter[16]. Thus, the correlation functions of the SS (for well separated \( x \) and \( y \)) are given by

\[
\langle S^z(y) S^z(x) \rangle \sim \frac{C}{2\pi^2 |x-y|^2} + A \frac{e^{2(\Phi(y) - \Phi(x))}}{m |x-y|^{2\alpha}},
\]

\[
\langle S^+(y) S^-(x) \rangle \sim \frac{B_1}{|x-y|^{2\alpha+2}} + B_2 \frac{e^{2(\Phi(y) - \Phi(x))}}{|x-y|^{2\alpha+2\alpha}},
\]

where the LLSL effective exponent is

\[
K^* = \sqrt{\frac{1}{K_1} + \frac{1}{K_2} + \frac{u_1}{u_2} \left( \frac{u_1}{u_2} \right)^2} \equiv f(K_1, K_2),
\]

FIG. 1: The charge velocity for a homogeneous spin chain as a function of the anisotropy parameter. The dashed line was obtained from the exact solution using the Bethe Ansatz.
observed a good agreement between the bosonization results obtained by Bethe Ansatz and the exact value, observed a slight discrepancy. But near the critical value we see that there is reasonable agreement, with slightly larger discrepancies at larger anisotropy parameter.

First, we will calculate the charge velocity for a homogeneous spin chain with a superlattice structure as a function of the anisotropy parameter. In Fig.1 we observe that the charge velocity increases with the anisotropy parameter and an excellent agreement between the predictions of the theory.

III. CONCLUSIONS

We found that using the scaling of the spin gap with the system size, the Tomonaga-Luttinger parameters for homogeneous and inhomogeneous spin chains can be estimated. For a spin chain with a superlattice structure the previous bosonization
tion results for the effective spin velocity and effective correlation exponent were recovered numerically using density matrix renormalization group for finite systems. The ration \( c/K^* \) decreases (increases) with \( \ell \) if \( \Delta_2/\Delta_1 < 1 \) (\( \Delta_2/\Delta_1 > 1 \)). Thus, the low energy properties of spin superlattices are well described in terms of Luttinger liquid superlattice theory.

Acknowledgments

We acknowledge useful discussion with A. L. Malvezzi, J. C. Xavier and E. Miranda. This work was supported by COLCIENCIAS (1101-05-13619 CT-033-2004) and DIB-UNAL (803954).