Estimate for the Size of the Compactification Radius of a One Extra Dimension Universe

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In this work, we use the Casimir effect to probe the existence of one extra dimension. We begin by evaluating the Casimir pressure between two plates in a $M^4 \times S^1$ manifold, and then use an appropriate statistical analysis in order to compare the theoretical expression with a recent experimental data and set bounds for the compactification radius.

Keywords: Casimir effect; Kaluza-Klein theory

1. INTRODUCTION

In a broad sense, it is fair to say that the search for unification is the greatest enterprise of theoretical physics. It started a long time ago, when Sir Isaac Newton showed that celestial and terrestrial mechanics could be described by the same laws, and reached one of its highest peaks in the second half of the nineteenth century, when electricity, magnetism and optics were all gathered into Maxwell equations.

The quest for unification continued, and, in a historical paper T. Kaluza [1] managed to combine classical electromagnetism and gravitation into a single, very elegant scheme. The downside was that his theory required an extra spacial dimension, for which there was no evidence whatsoever. Some years later, O. Klein pushed the idea a little further [2], proposing, among other things, a circular topology of a very tiny radius for the extra dimension, maybe at the Planck scale region. Although it presented a great unification appeal, the Kaluza-Klein idea has been left aside for several decades. Only in the mid-seventies, due to the birth of supergravity theory [3], the extra dimensions came back to the theoretical physics scenario. As supergravity also had its own problems, it seemed that the subject would be washed out again, but, less than a decade later, the advent of string and superstring theories [4] made it a cornerstone in extremely high energy physics. Nowadays, with the development of M-theory [5] and some associated ideas, like the cosmology of branes [6], it might even be said that extra dimensions are almost a commonplace in modern high-energy physics.

1.1. The hierarchy problem

If our universe indeed has extra dimensions, then a lot of intriguing facts should readily come into play. A major development regarding this issue consists in the alternative approaches to the so called hierarchy problem, which stands unclear despite all the efforts carried out over the last thirty years [7]. In most of the extra-dimensional models, the additional dimensions are tightly curled up in a small volume, explaining thus how they have evaded our perception so far. Initially it was thought that 'small volume' should mean 'Planck-scale sized volume', but now it is conceded that some extra dimensions may be as large as a human cell, standing at the micrometer scale [8]. Well, how the hierarchy problem fits in this picture? In order to answer it, let us consider a N-dimensional space-time R in which 4 dimensions are large and $n = N - 4$ are compactified, containing in addition a small mass m at a given point P. For regions that are faraway from P, at least compared to the compactification radius r_c[1] everything should be as if the universe were four-dimensional, so the gravitational interaction is the observed newtonian field

$$\mathbf{g}_4 = G\frac{m}{r^2}\hat{\mathbf{r}} = \frac{1}{(M_4)^2}\frac{m}{r^2}\hat{\mathbf{r}} \ ; \quad r \gg r_c \ , \tag{1}$$

where we took $\hbar = c = 1$ and identified the Planck mass $M_4 \simeq 1.2 \cdot 10^{19} eV$. When we go to opposite limit ($r \ll r_c$) it is not possible to ignore the existence of the extra dimensions anymore, but, from the N-dimensional Gauss law and some dimensional analysis we get

$$\int_{S^{n+2}} \mathbf{g}_N \cdot \mathbf{dS} = 4\pi\frac{4\pi m}{(M_N)^{2+n}} \ . \tag{2}$$

It is then straightforward to deduce the behavior of the gravitational field

$$\mathbf{g}_N = \frac{1}{A_{S^{N-2}}\,(M_N)^{2+n}}\frac{m}{r^{2+n}}\hat{\mathbf{r}} \ ; \quad r \ll r_c \ , \tag{3}$$

where S^n is the appropriate n-sphere and A_{S^n} stands for its n-area. Let us notice that we had to choose a tiny n-sphere to

[1] We are tacitly assuming that all the curled up dimensions are roughly of the same "size", characterized by r_c

apply the Gauss law, or we would not be in the $r \ll r_c$ regime. But there is nothing fundamental about this choice, and we may as well use the Gauss law (2) in order to find the behavior of the gravitational field for large distances. We shall merely quote the result

$$\mathbf{g}_N = \frac{1}{(r_c)^n (M_N)^{2+n}} \frac{m}{r^2}\hat{\mathbf{r}} \ ; \quad r \gg r_c \ , \tag{4}$$

and refer the reader to the bibliography [9] for more details. Now, comparing (1) and (4), we find the following constraint relation

$$(M_4)^2 = (r_c)^n (M_{4+n})^{2+n} \implies r_c = \frac{1}{M_{4+n}} \left(\frac{M_4}{M_{4+n}} \right)^n , \tag{5}$$

which shows that we may look at M_4 as an effective Planck mass, depending fundamentally on the 'true' Planck mass M_{4+n} and the compactification radius r_c. That is a very interesting relation from the perspective of the hierarchy problem, because it allows for the effective mass M_4 that we observe to be huge even when the true mass M_{4+n} is not that big. Just to put some numbers, let us consider $r_c \simeq 1\,\mu m \Rightarrow (r_c)^{-1} \simeq 0.18\,eV$, which is just below the lower bound for the experimental validity of newtonian gravitation [10]. This automatically gives the following values for M_{4+n}

$$\begin{aligned} n = 1 \quad &\longrightarrow \quad M_5 \simeq 0.9 \cdot 10^7 \text{ TeV} \\ n = 2 \quad &\longrightarrow \quad M_6 \simeq 80 \text{ TeV} \\ n = 3 \quad &\longrightarrow \quad M_7 \simeq 110 \text{ GeV} \end{aligned} \tag{6}$$

and so on. Knowing that the electroweak scale M_{EW} is about 100 GeV, we conclude that an universe with just one extra dimension is definitely not the best case scenario. This does not mean that the reduction of the Planck mass by nine orders of magnitude is not quite something, but only that a ratio of $M_5/M_{EW} \simeq 10^8$ still leaves a great 'desert' ahead. However, despite this partial frustration in solving the hierarchy problem, we will proceed with just one extra dimension, for the plain reason that it is the simplest model to work with from both the theoretical and statistical perspectives. Last but not least, it is important to say that in more sophisticated models it is possible to deal with the hierarchy issue in a 5-dimensional picture, some of which are enjoying great success nowadays [11].

1.2. The Casimir effect

As the Casimir effect [12, 13] has a strong dependence with the space-time dimensionality, the Casimir force experiments [14, 15] may be a powerful tool to detect the existence of extra dimensions. In a recent paper, Poppenhaeger et al. [16] carried out a calculation in order to set bounds for the size of an hypothetic extra dimension. They conclude that their modified expression for the Casimir force with one extra dimension is with the experimental data of M. Sparnaay [17], as long as the upper limit for the compactification radius is at the nanometer range. Although we find these results very interesting, we would like to stress that the data present in [17] may be inadequate for such estimations, due to its lack of precision [2]. It is then a natural step to replace [17] for some more sophisticated experiments, which is precisely the purpose of this work.

We begin by evaluating the Casimir pressure between two plates in a hypothetical universe with a $M^4 \times S^1$ topology. We use the standard mode summation formula for the Casimir effect, and the calculations are carried out within the analytical regularization scheme, which is closely related to some generalized zeta functions. The result for the Casimir energy and pressure show an explicit dependence on the distance between the plates and on the S^1 radius, as they should. As our final task, we use some recent experimental data [15] and do a proper statistical analysis in order to set limits for the values of the compactification radius.

2. THE CASIMIR EFFECT IN A $M^4 \times S^1$ SPACETIME

Let us begin by writing the line element of the $M^4 \times S^1$ universe

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2 - r^2\, d\theta^2, \tag{7}$$

where r is the S^1 radius. Due to the simplicity of this metric, the field equations in this manifold are essentially the same as the minkovskian ones. This holds in particular for the massless vectorial field, and so we have

$$\partial_\mu F^{\mu\nu} = 0, \quad \partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = 0, \tag{8}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{9}$$

In the radiation gauge we may write

$$A_0 = 0, \ \ \partial_\mu A^\mu = 0, \tag{10}$$

and so the field equation may be recast into

$$\Box A^\mu = 0 \tag{11}$$

Let us assume that the conducting plates are at the planes $x = 0$ and $x = a$. This setup leads to the following boundary conditions (BC)

$$F^{\mu\nu}|_{x=0} = F^{\mu\nu}|_{x=a} = 0 \quad \text{if } \mu \neq 1, \nu \neq 1. \tag{12}$$

The S^1 topology also imposes a periodicity condition for the electromagnetic field

$$A^\mu(x^4) = A^\mu(x^4 + 2\pi r). \tag{13}$$

Now we have to solve equation (11) constrained by conditions (12) e (13). That is a straightforward task, so we merely quote the eigenmodes and the eigenfrequencies

$$
\begin{aligned}
A_1 &= A_1^{(0)} \cos\left(\frac{m_1 \pi x}{a}\right) \mathrm{e}^{i(\vec{k}_\perp \cdot \vec{x}_\perp + n\theta - \omega t)}, \; m_1 = 0,1,2,... \\
A_j &= i A_j^{(0)} \sin\left(\frac{m_j \pi x}{a}\right) \mathrm{e}^{i(\vec{k}_\perp \cdot \vec{x}_\perp + n\theta - \omega t)}, \; m_j = 1,2,... \\
\omega^2_{\mathbf{k}\lambda} &= \omega^2_{mnk_\perp} = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n}{r}\right)^2 + k_\perp^2, \;\; j = 2,3,4; \;\; n \in \mathbb{Z}
\end{aligned} \tag{14}
$$

where the fields amplitudes are related by

$$
A_1^{(0)} \frac{m_1 \pi}{a} + \mathbf{A^{(0)}} \cdot \mathbf{k}_\perp + \frac{n}{r} A_4^{(0)} = 0\,, \tag{15}
$$

as a consequence of the gauge condition (10).

The Casimir energy of the electromagnetic field in a $M^4 \times S^1$ universe is given by the sum of allowed modes

$$
\begin{aligned}
\mathcal{E}(a,r) &= \frac{\hbar}{2} \sum_{\mathbf{k}\lambda} \omega_{\mathbf{k}\lambda} = \frac{\hbar c L^2}{8\pi^2} \int d^2 \mathbf{k}_\parallel \sum_{n=-\infty}^{\infty} \left[\sqrt{\left(\frac{n}{r}\right)^2 + k_\parallel^2} \right. \\
&+ \left. p \sum_{m=1}^{\infty} \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n}{r}\right)^2 + k_\parallel^2} \right],
\end{aligned} \tag{16}
$$

where p is the number of possible polarizations of the photon ($p = 3$ in this case). The previous expression is purely formal, since its r.h.s. is infinite. So, in order to proceed, we introduce a cut-off parameter s in (16). Then

$$
\begin{aligned}
\mathcal{E}_{reg}(a,r;s) &= \frac{L^2 \hbar c}{4\pi} \int_0^\infty k_\parallel dk_\parallel \sum_{m=-\infty}^{\infty} \left\{ p \sum_{n=1}^{\infty} \left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n}{r}\right)^2 + k_\parallel^2 \right]^{\frac{1-s}{2}} \right. \\
&+ \left. \left[\left(\frac{n}{r}\right)^2 + k_\parallel^2 \right]^{\frac{1-s}{2}} \right\}.
\end{aligned} \tag{17}
$$

Performing the integral in $k_\parallel$ we arrive at

$$
\begin{aligned}
\mathcal{E}_{reg}(a,r;s) &= \frac{\hbar c L^2 p}{4\pi(s-3)} \left(\frac{a}{\pi}\right)^{s-3} \left[\sum_{m=1}^{\infty} m^{3-s} + 2 \sum_{n,m=1}^{\infty} \left(m^2 + \left(\frac{na}{\pi r}\right)^2 \right)^{\frac{3-s}{2}} \right] \\
&+ \frac{\hbar c L^2}{2\pi(s-3)} r^{s-3} \sum_{n=1}^{\infty} n^{3-s}.
\end{aligned} \tag{18}
$$

Let us now recall the definition of the Epstein functions, and, as a particular case, the Riemann zeta function [19]

$$
E_N(s; a_1, ..., a_N) = \sum_{n_1,...,n_N=1}^{\infty} \left[a_1 n_1^2 + ... + a_N n_N^2 \right]^{-s}, \; \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \,. \tag{19}
$$

By using these definitions, we may recast expression (18) into

$$
\begin{aligned}
\mathcal{E}_{reg}(a,r;s) &= \frac{\hbar c L^2 p}{4\pi(s-3)} \left(\frac{a}{\pi}\right)^{s-3} \left[\zeta(s-3) + 2 E_2\left(\frac{s-3}{2}; 1, \frac{a^2}{\pi^2 r^2}\right) \right] \\
&+ \frac{\hbar c L^2 r^{s-3}}{2\pi(s-3)} \zeta(s-3)\,.
\end{aligned} \tag{20}
$$

The Epstein functions have a well known analytical continuation, which were thoroughly studied in [19], among other references. As a more detailed discussion of that matter would take us too far afield, let us merely quote the analytic continuation of the Epstein function $E_2(s; a_1, a_2)$

$$
\begin{aligned}
E_2\left(s; a_1^2, a_2^2\right) &= -\frac{a_1^{-2s}}{2} \zeta(2s) + \frac{\sqrt{\pi}}{2a_2} \frac{\Gamma(s-1/2)}{\Gamma(s)} a_1^{1-2s} \zeta(2s-1) \\
&+ \frac{2\pi^s}{\Gamma(s)} \sum_{n_1,n_2=1}^{\infty} a_2^{-s-1/2} \left(\frac{n_1}{a_1 n_2}\right)^{s-1/2} K_{s-1/2}\left(\frac{2\pi a_1 n_1 n_2}{a_2}\right),
\end{aligned} \tag{21}
$$

where $K_\nu(x)$ stands for the modified Bessel function. The reflection formula for the Riemann zeta function will also be very useful

$$\zeta(s) = \pi^{s-\frac{1}{2}} \frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} \zeta(1-s)\,. \tag{22}$$

It is now a straightforward matter put (20) into the form

$$\begin{aligned}
\mathcal{E}_{reg}(a,r;s) &= p\frac{\hbar c L^2}{4\pi(s-3)}\frac{1}{\Gamma\left(\frac{s-3}{2}\right)}\left[\frac{a^{s-3}}{\sqrt{\pi}}\Gamma\left(2-\frac{s}{2}\right)\zeta(4-s)\right. \\
&+ \frac{ar^{s-4}}{\pi^{5-s}}\Gamma\left(\frac{5-s}{2}\right)\zeta(5-s) \\
&+ \left.\frac{4a^{\frac{s}{2}-1}}{\sqrt{\pi}}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{mr}{n}\right)^{\frac{s}{2}-2}K_{\frac{s}{2}-2}\left(\frac{2mna}{r}\right)\right] \\
&+ (p-2)\frac{\hbar c L^2}{4(s-3)}\frac{\Gamma\left(2-\frac{s}{2}\right)}{\Gamma\left(\frac{s-3}{2}\right)}\frac{r^{s-3}}{\pi^{\frac{9}{2}-s}}\zeta(4-s)\,,
\end{aligned} \tag{23}$$

and, in the limit of $s \to 0$, we get

$$\begin{aligned}
\mathcal{E}(a,r) = &-p\frac{\hbar c L^2\pi^2}{1440a^3} - (p-2)\frac{\hbar c L^2}{1440\pi r^3} - 2p\pi r L^2\frac{3\hbar c}{128\pi^7}\frac{a}{r^5}\zeta(5) \\
&-p\frac{\hbar c L^2}{4\pi^2 r^2 a}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{n}{m}\right)^2 K_2\left(\frac{2mna}{r}\right)
\end{aligned} \tag{24}$$

Due to renormalization issues, we now have to evaluate the Casimir energy of the region defined by the plates, but with no plates whatsoever. This calculation is analogous to the one leading to (24), so we merely state the result

$$\mathcal{E}_{ED}(a,r) = -2p\pi r L^2\frac{3\hbar c}{128\pi^7}\frac{a}{r^5}\zeta(5)\,. \tag{25}$$

Then, subtracting this term from (24), we finally obtain the Casimir energy for the $M^4 \times S^1$ with Dirichlet plates

$$\begin{aligned}
\mathcal{E}_{Cas}(a,r) = &-p\frac{\hbar c L^2\pi^2}{1440a^3} - (p-2)\frac{\hbar c L^2}{1440\pi r^3} \\
&-p\frac{\hbar c L^2}{4\pi^2 r^2 a}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{n}{m}\right)^2 K_2\left(\frac{2mna}{r}\right)\,.
\end{aligned} \tag{26}$$

The first thing to be stressed about the previous result is that it precisely coincides with the expression found on [16], although we have derived it in a more clear and pedagogic way. Now, if we want to make some comparison with the experiments, we need an expression for the Casimir pressure. Fortunately, the relation between the Casimir energy and pressure is a simple one

$$\begin{aligned}
\mathcal{P}(a,r) = -\frac{1}{L^2}\frac{\partial\mathcal{E}_{Cas}}{\partial a} = &-p\frac{\pi^2\hbar c}{480a^4} - p\frac{\hbar c}{4\pi^2 r^2 a^2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left[3\left(\frac{n}{m}\right)^2 K_2\left(\frac{2mna}{r}\right)\right. \\
&\left.+2\frac{n^3 a}{mr}K_1\left(\frac{2mna}{r}\right)\right]\,,
\end{aligned} \tag{27}$$

where we used some recurrence relations between the modified Bessel functions [20]. If we now make $p = 2$ in expressions (26) and (27) and take the limiting case of $r \to 0$, we will get respectively the standard Casimir energy and pressure obtained in [12].

3. ESTIMATE OF THE COMPACTIFICATION RADIUS

The plane geometry is by far the simplest to work with in theoretical calculations, but unfortunately the situation is not so friendly from the experimental point of view. A good measurement of the Casimir force between two plates requires, among other things, a high degree of parallelism between the plates, which is very difficult to sustain throughout the course of the experiment. Due to these parallelism problems, the most popular setup nowadays for measuring the Casimir effect is the sphere-plate configuration [14], for which very precise measurements were reported. There is, however, at least one modern experiment designed to detect the Casimir force between parallel plates [15], and due to its relevance for us we feel that it is important to describe it a little further.

The apparatus itself used in that experiment is very interesting. The two parallel plates are simulated by the opposing faces two silicon beams. One of these beams is rigidly connected to a frame, in a such a way to provide an accurate control of the distance between the two beams. The other beam is a thin cantilever that plays the part of a resonator, since it is free to oscillate around its clamping point. The apparatus is designed to measure the square plates oscillating frequency shift ($\Delta\nu^2$), that is related to the Casimir pressure in the following way [15]

$$\Delta\nu^2 = \nu^2 - \nu_0^2 = -\frac{L^2}{4\pi^2 m_{eff}}\frac{\partial \mathcal{P}}{\partial a}, \tag{28}$$

where m_{eff} is the effective mass of the resonator.

Substituting (27) in the previous expression, we get

$$\begin{aligned}\Delta\nu^2(a,r) &= -p\frac{\hbar c L^2}{4\pi^2 m_{eff}}\left\{\frac{\pi^2}{120a^5}\right. \\ &+ \frac{1}{\pi^2 ar}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left[\left(3\frac{n}{m^3a^3}+\frac{5}{2}\frac{n^3}{mar^2}\right)K_1\left(\frac{2mna}{r}\right)\right. \\ &+ \left.\left.\left(3\frac{n^2}{m^2ra^2}+\frac{n^4}{r^3}\right)K_0\left(\frac{2mna}{r}\right)\right]\right\}. \end{aligned}\tag{29}$$

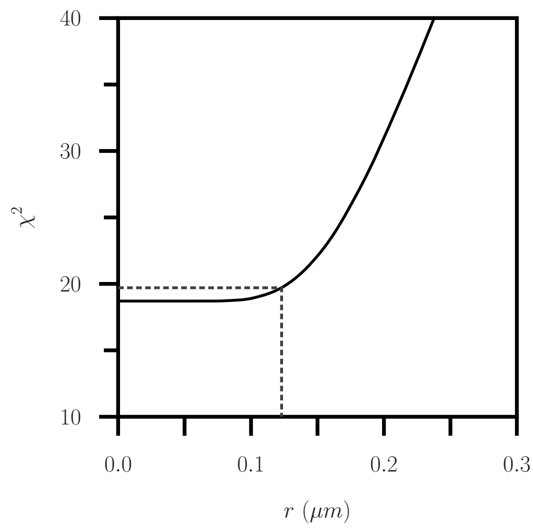

FIG. 1: Graph of χ^2 *versus* r. The vertical dashed line indicates the value $r = 123$ nm, where the function χ^2 hits the value 19.6 (see footnote 3).

Now that we have a theoretical expression of $\Delta\nu^2$ as a function of a and r, we will fit r using the least square method and the experimental data of [15]. As we are fitting just one parameter, we can estimate the best value for r from the graph on Fig. 1 just by looking for the value of r that leads to a minimum value of χ^2.

Our fit for the compactification radius produced the value of 0^{+123}_{-0} *nm*, and the uncertainties on r give the upper and lower bounds for this radius[3]. In a successful fit, the minimum value of χ^2 should coincide, approximately, with the number of degrees of freedom of the fit. As in this case we have 8 degrees of freedom[4] and the minimum for χ^2 turned out to be 18.6, we can state that no good agreement was obtained between the theoretical model and the experimental data.

4. CONCLUSION

In this article, we have used the Casimir effect to probe the existence of one extra dimension. We started by evaluating the Casimir pressure between two perfect conducting plates living in a 4+1 universe, given in (27), where the extra dimension is compactified in a S^1 topology. In order to set bounds for the compactification radius, we proceeded to the comparison of this result with the experimental data of [15], and, after an appropriate statistical analysis, this procedure showed that the best value for the compactification radius is below approximately 120nm.

We know that the results for the Minkowski space-time are in close agreement with the experimental data. In order to be consistent with this picture, the extra compactified di-

[3] As usual, the uncertainties were obtained by searching the two values of r that produce $\chi^2 = 19.6$ (minimum value plus one). Since a radius can not be negative, we imposed a vanishing lower bound.

[4] The degree of freedom of a fit is defined as being the subtraction of the number of experimental points used in the fit by the number of adjusted variables. In this case, we have 9 experimental points and one adjusted variable, which gives the degree of freedom aforementioned.

mension should contribute as a small perturbation to the four-dimensional result, but, as we have seen, this is not the case. Among other things, the extra dimension led to a new polarization degree for the electromagnetic field, which essentially bumped the M^4 result by a factor of approx. $3/2$, that is not small. It is important to say that this new polarization freedom does not allow the $r \to 0$ limit to be taken carelessly, for it represents the transition from $M^4 \times S^1$ to M^4, in which a polarization degree is discontinuously lost.

We finish by saying that there are other corrections to the Casimir effect, such as finite conductivity and finite temperature contributions [13, 21], that we have not taken into account and may completely overwhelm any extra dimensional effects. Besides that, there is the roughness of the plate material [22] and possibly some edge effects [23], which, if necessary, should also be considered. Hence, in a more rigorous approach, these influences should be taken into account, and the comparison should be made with very accurate experiments.

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