On the generalized nonlinear ultra-hyperbolic heat equation related to the spectrum

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Abstract. In this paper, we study the nonlinear equation of the form
\[
\frac{\partial}{\partial t} u(x, t) - c^2 \Box^k u(x, t) = f(x, t, u(x, t))
\]
where $\Box^k$ is the ultra-hyperbolic operator iterated $k$-times, defined by
\[
\Box^k = \left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \frac{\partial^2}{\partial x_{p+2}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2} \right)^k,
\]
$p + q = n$ is the dimension of the Euclidean space $\mathbb{R}^n$, $(x, t) = (x_1, x_2, \ldots, x_n, t) \in \mathbb{R}^n \times (0, \infty)$, $k$ is a positive integer and $c$ is a positive constant.

On the suitable conditions for $f$, $u$ and for the spectrum of the heat kernel, we can find the unique solution in the compact subset of $\mathbb{R}^n \times (0, \infty)$. Moreover, if we put $k = 1$ and $q = 0$ we obtain the solution of nonlinear equation related to the heat equation.

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1 Introduction

It is well known that for the heat equation

$$\frac{\partial}{\partial t} u(x, t) = c^2 \Delta u(x, t) \tag{1.1}$$

with the initial condition

$$u(x, 0) = f(x)$$

where $\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ is the Laplace operator and $(x, t) = (x_1, x_2, \ldots, x_n, t) \in \mathbb{R}^n \times (0, \infty)$, and $f$ is a continuous function, we obtain the solution

$$u(x, t) = \frac{1}{(4c^2\pi t)^{n/2}} \int_{\mathbb{R}^n} \exp \left( -\frac{|x-y|^2}{4ct} \right) f(y) dy \tag{1.2}$$

as the solution of (1.1).

Now, (1.2) can be written as $u(x, t) = E(x, t) \ast f(x)$ where

$$E(x, t) = \frac{1}{(4c^2\pi t)^{n/2}} \exp \left[ -\frac{|x|^2}{4ct} \right]. \tag{1.3}$$

$E(x, t)$ is called the heat kernel, where $|x|^2 = x_1^2 + x_2^2 + \cdots + x_n^2$ and $t > 0$, see [1, p. 208–209].

Moreover, we obtain $E(x, t) \rightarrow \delta$ as $t \rightarrow 0$, where $\delta$ is the Dirac-delta distribution. We also have extended (1.1) to be the equation

$$\frac{\partial}{\partial t} u(x, t) = c^2 \Box u(x, t) \tag{1.4}$$

where $\Box$ is the ultra-hyperbolic operator, defined by

$$\Box = \left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \frac{\partial^2}{\partial x_{p+2}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2} \right).$$

We obtain the ultra-hyperbolic heat kernel

$$E(x, t) = \frac{(i)^q}{(4c^2\pi t)^{n/2}} \exp \left[ -\sum_{i=1}^{p} x_i^2 - \sum_{j=p+1}^{p+q} x_j^2 \right]$$

where $p + q = n$ is the dimension of the Euclidean space $\mathbb{R}^n$ and $i = \sqrt{-1}$. For finding the kernel $E(x, t)$ see [4].

In this paper, we extend (1.4) to be the general of the nonlinear form

$$\frac{\partial}{\partial t} u(x, t) - c^2 \Box^k u(x, t) = f(x, t, u(x, t))$$

(1.5)

for \((x, t) \in \mathbb{R}^n \times (0, \infty)\) and with the following conditions on \(u\) and \(f\) as follows,

1. \(u(x, t) \in C^{(2k)}(\mathbb{R}^n)\) for any \(t > 0\) where \(C^{(2k)}(\mathbb{R}^n)\) is the space of continuous function with \(2k\)-derivatives.

2. \(f\) satisfies the Lipchitz condition, that is

\[|f(x, t, u) - f(x, t, w)| \leq A|u - w|\]

where \(A\) is constant and \(0 < A < 1\).

3. \[
\int_0^\infty \int_{\mathbb{R}^n} \left| f(x, t, u(x, t)) \right| dx \, dt < \infty
\]

for \(x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n, t \in (0, \infty)\) and \(u(x, t)\) is continuous function on \(\mathbb{R}^n \times (0, \infty)\).

Under such conditions of \(f, u\) and for the spectrum of \(E(x, t)\), we obtain the convolution

\[u(x, t) = E(x, t) \ast f(x, t, u(x, t))\]

as a unique solution in the compact subset of \(\mathbb{R}^n \times (0, \infty)\) and \(E(x, t)\) is an elementary solution defined by (2.5).

2 Preliminaries

**Definition 2.1.** Let \(f(x) \in L_1(\mathbb{R}^n)\)-the space of integrable function in \(\mathbb{R}^n\). The Fourier transform of \(f(x)\) is defined by

\[\hat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i(\xi, x)} f(x) \, dx\]

(2.1)

where \(\xi = (\xi_1, \xi_2, \ldots, \xi_n), x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n, (\xi, x) = \xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_n x_n\) is the usual inner product in \(\mathbb{R}^n\) and \(dx = dx_1 \, dx_2 \cdots dx_n\).

Also, the inverse of Fourier transform is defined by

\[f(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{i(\xi, x)} \hat{f}(\xi) \, d\xi,\]

(2.2)
**Definition 2.2.** The spectrum of the kernel $E(x, t)$ defined by (2.5) is the bounded support of the Fourier transform $\hat{E}(\xi, t)$ for any fixed $t > 0$.

**Definition 2.3.** Let $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$ be a point in $\mathbb{R}^n$ and we write

$$u = \xi_1^2 + \xi_2^2 + \ldots + \xi_p^2 - \xi_{p+1}^2 - \xi_{p+2}^2 - \ldots - \xi_{p+q}^2, \quad p + q = n.$$  

Denote by

$$\Gamma_+ = \{ \xi \in \mathbb{R}^n : \xi_1 > 0 \text{ and } u > 0 \}$$

the set of an interior of the forward cone, and $\overline{\Gamma}_+$ denotes the closure of $\Gamma_+$.

Let $\Omega$ be spectrum of $E(x, t)$ defined by Definition 2.2 for any fixed $t > 0$ and $\Omega \subset \overline{\Gamma}_+$. Let $E(\xi, t)$ be the Fourier transform of $E(x, t)$ and define

$$E(\hat{\xi}, t) = \begin{cases} \frac{1}{(2\pi)^n} \int \exp \left[ c^2 t \left( \sum_{j=p+1}^{p+q} \xi_j^2 - \sum_{i=1}^{p} \xi_i^2 \right) \right] d\xi & \text{for } \xi \in \Gamma_+, \\ 0 & \text{for } \xi \notin \Gamma_+. \end{cases}$$  

(2.3)

**Lemma 2.1.** Let $L$ be the operator defined by

$$L = \frac{\partial}{\partial t} - c^2 \Box^k$$  

where $\Box^k$ is the ultra-hyperbolic operator iterated $k$-times defined by

$$\Box^k = \left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \ldots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \frac{\partial^2}{\partial x_{p+2}^2} - \ldots - \frac{\partial^2}{\partial x_{p+q}^2} \right)^k,$$

$p + q = n$ is the dimension of $\mathbb{R}^n$, $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, $t \in (0, \infty)$, $k$ is a positive integer and $c$ is a positive constant. Then we obtain

$$E(x, t) = \frac{1}{(2\pi)^n} \int_{\Omega} \exp \left[ c^2 t \left( \sum_{j=p+1}^{p+q} \xi_j^2 - \sum_{i=1}^{p} \xi_i^2 \right) \right] d\xi$$  

(2.5)

as a elementary solution of (2.4) in the spectrum $\Omega \subset \mathbb{R}^n$ for $t > 0$. 

Proof. Let \( LE(x, t) = \delta(x, t) \) where \( E(x, t) \) is the kernel or the elementary solution of operator \( L \) and \( \delta \) is the Dirac-delta distribution. Thus

\[
\frac{\partial}{\partial t} E(x, t) - c^2 \Box^k E(x, t) = \delta(x)\delta(t).
\]

Take the Fourier transform defined by (2.1) to both sides of the equation, we obtain

\[
\frac{\partial}{\partial t} \hat{E}(\xi, t) - c^2 \left( \sum_{j=p+1}^{p+q} \xi_j^2 - \sum_{i=1}^{p} \xi_i^2 \right)^k \hat{E}(\xi, t) = \frac{1}{(2\pi)^{n/2}} \delta(t).
\]

Thus

\[
\hat{E}(\xi, t) = \frac{H(t)}{(2\pi)^{n/2}} \exp \left[ c^2 t \left( \sum_{j=p+1}^{p+q} \xi_j^2 - \sum_{i=1}^{p} \xi_i^2 \right)^k \right]
\]

where \( H(t) \) is the Heaviside function. Since \( H(t) = 1 \) for \( t > 0 \). Therefore,

\[
\hat{E}(\xi, t) = \frac{1}{(2\pi)^{n/2}} \exp \left[ c^2 t \left( \sum_{j=p+1}^{p+q} \xi_j^2 - \sum_{i=1}^{p} \xi_i^2 \right)^k \right]
\]

which has been already defined by (2.3). Thus

\[
E(x, t) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{i(\xi, x)} \hat{E}(\xi, t) d\xi = \frac{1}{(2\pi)^{n/2}} \int_{\Omega} e^{i(\xi, x)} \hat{E}(\xi, t) d\xi
\]

where \( \Omega \) is the spectrum of \( E(x, t) \). Thus from (2.3)

\[
E(x, t) = \frac{1}{(2\pi)^{n}} \int_{\Omega} \exp \left[ c^2 t \left( \sum_{j=p+1}^{p+q} \xi_j^2 - \sum_{i=1}^{p} \xi_i^2 \right)^k + i(\xi, x) \right] d\xi \quad \text{for} \quad t > 0.
\]

Definition 2.4. Let us extend \( E(x, t) \) to \( \mathbb{R}^n \times \mathbb{R} \) by setting

\[
E(x, t) = \begin{cases} 
\frac{1}{(2\pi)^{n}} \int_{\Omega} \exp \left[ c^2 t \left( \sum_{j=p+1}^{p+q} \xi_j^2 - \sum_{i=1}^{p} \xi_i^2 \right)^k + i(\xi, x) \right] d\xi & \text{for } t > 0, \\
0 & \text{for } t \leq 0,
\end{cases}
\]

3 Main Results

Theorem 3.1. The kernel $E(x, t)$ defined by (2.5) have the following properties:

1. $E(x, t) \in C^\infty$ - the space infinitely differentiable.
2. $(\frac{\partial}{\partial t} - c^2 \Box^k) E(x, t) = 0$ for $t > 0$.
3. $|E(x, t)| \leq \frac{2^{2-n}}{\pi^{n/2}} \frac{M(t)}{\Gamma\left(\frac{n}{2}\right) \Gamma\left(\frac{2}{2}\right)}$, for $t > 0$, where $M(t)$ is a function of $t$ in the spectrum $\Omega$ and $\Gamma$ denote the Gamma function. Thus $E(x, t)$ is bounded for any fixed $t > 0$.
4. $\lim_{t \to 0} E(x, t) = \delta$.

Proof.

1. From (2.5), since

$$\frac{\partial^n}{\partial x^n} E(x, t) = \frac{1}{(2\pi)^n} \int_{\Omega} \frac{\partial^n}{\partial x^n} \exp \left[ c^2 t \left( \sum_{j=p+1}^{p+q} \xi_j^2 - \sum_{i=1}^{p} \xi_i^2 \right) + i(\xi, x) \right] d\xi.$$ 

Thus $E(x, t) \in C^\infty$ for $x \in \mathbb{R}^n, t > 0$.

2. By computing directly, we obtain

$$\left( \frac{\partial}{\partial t} - c^2 \Box^k \right) E(x, t) = 0.$$

3. We have

$$E(x, t) = \frac{1}{(2\pi)^n} \int_{\Omega} \exp \left[ c^2 t \left( \sum_{j=p+1}^{p+q} \xi_j^2 - \sum_{i=1}^{p} \xi_i^2 \right) + i(\xi, x) \right] d\xi,$$

$$|E(x, t)| \leq \frac{1}{(2\pi)^n} \int_{\Omega} \exp \left[ c^2 t \left( \sum_{j=p+1}^{p+q} \xi_j^2 - \sum_{i=1}^{p} \xi_i^2 \right) \right] d\xi.$$
By changing to bipolar coordinates

\[\xi_1 = r\omega_1, \xi_2 = r\omega_2, \ldots, \xi_p = r\omega_p \quad \text{and} \quad \xi_{p+1} = s\omega_{p+1}, \xi_{p+2} = s\omega_{p+2}, \ldots, \xi_{p+q} = s\omega_{p+q}\]

where \(\sum_{i=1}^{p} \omega_i^2 = 1\) and \(\sum_{j=p+1}^{p+q} \omega_j^2 = 1\). Thus

\[|E(x, t)| \leq \frac{1}{(2\pi)^n} \int_{\Omega} \exp\left[c^2 t (s^2 - r^2)^k\right] r^{p-1} s^{q-1} dr ds d\Omega_p d\Omega_q\]

where \(d\xi = r^{p-1} s^{q-1} dr ds d\Omega_p d\Omega_q\) and \(d\Omega_p\) and \(d\Omega_q\) are the elements of surface area of the unit sphere in \(\mathbb{R}^p\) and \(\mathbb{R}^q\) respectively. Since \(\Omega \subset \mathbb{R}^n\) is the spectrum of \(E(x, t)\) and we suppose \(0 \leq r \leq R\) and \(0 \leq s \leq L\) where \(R\) and \(L\) are constants. Thus we obtain

\[|E(x, t)| \leq \frac{\Omega_p \Omega_q}{(2\pi)^n} \int_0^R \int_0^L \exp\left[c^2 t (s^2 - r^2)^k\right] r^{p-1} s^{q-1} ds dr\]

\[= \frac{\Omega_p \Omega_q}{(2\pi)^n} M(t) \quad \text{for any fixed } t > 0 \quad \text{in the spectrum } \Omega\]

\[= \frac{2^{2-n}}{\pi^{n/2} \Gamma(\frac{p}{2}) \Gamma(\frac{q}{2})} M(t) \quad (3.1)\]

where

\[M(t) = \int_0^R \int_0^L \exp\left[c^2 t (s^2 - r^2)^k\right] r^{p-1} s^{q-1} ds dr \quad (3.2)\]

is a function of

\[t > 0, \quad \Omega_p = \frac{2\pi^{p/2}}{\Gamma(\frac{p}{2})} \quad \text{and} \quad \Omega_q = \frac{2\pi^{q/2}}{\Gamma(\frac{q}{2})}.\]

Thus, for any fixed \(t > 0\), \(E(x, t)\) is bounded.

(4) By (2.5), we have

\[E(x, t) = \frac{1}{(2\pi)^n} \int_{\Omega} \exp\left[c^2 t \left(\sum_{j=p+1}^{p+q} \xi_j^2 - \sum_{i=1}^{p} \xi_i^2\right) + i(\xi, x)\right] d\xi.\]
Since $E(x, t)$ exists, then
\[
\lim_{t \to 0} E(x, t) = \frac{1}{(2\pi)^n} \int_{\Omega} e^{i(\xi, x)} d\xi = \frac{1}{(2\pi)^n} \int_{\Omega} e^{i(\xi, x)} d\xi = \delta(x), \quad \text{for} \ x \in \mathbb{R}^n.
\]

See [3, p. 396, Eq. (10.2.19b)].

**Theorem 3.2.** Given the nonlinear equation
\[
\frac{\partial}{\partial t} u(x, t) - c^2 \square^k u(x, t) = f(x, t, u(x, t)) \tag{3.3}
\]
for $(x, t) \in \mathbb{R}^n \times (0, \infty)$, $k$ is positive number and with the following conditions on $u$ and $f$ as follows,

1. $u(x, t) \in C^{(2k)}(\mathbb{R}^n)$ for any $t > 0$ where $C^{(2k)}(\mathbb{R}^n)$ is the space of continuous function with $2k$-derivatives.

2. $f$ satisfies the Lipchitz condition, that is
\[
|f(x, t, u) - f(x, t, w)| \leq A|u - w|
\]
where $A$ is constant and $0 < A < 1$.

3. \[
\int_0^\infty \int_{\mathbb{R}^n} |f(x, t, u(x, t))| \, dx \, dt < \infty
\]
for $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, $t \in (0, \infty)$ and $u(x, t)$ is continuous function on $\mathbb{R}^n \times (0, \infty)$.

Then, for the spectrum of $E(x, t)$ we obtain the convolution
\[
u(x, t) = E(x, t) * f(x, t, u(x, t)) \tag{3.4}
\]
as a unique solution of (3.3) for $x \in \Omega_0$ where $\Omega_0$ is an compact subset of $\mathbb{R}^n$, $0 \leq t \leq T$ with $T$ is constant and $E(x, t)$ is an elementary solution defined by (2.5) and also $u(x, t)$ is bounded.

In particular, if we put $k = 1$ and $q = 0$ in (3.3) then (3.3) reduces to the nonlinear heat equation.

Proof. Convolving both sides of (3.3) with $E(x,t)$ and then we obtain the solution

$$u(x,t) = E(x,t) * f(x,t,u(x,t))$$

or

$$u(x,t) = \int_{-\infty}^{\infty} \int_{\mathbb{R}^n} E(r,s) f(x-r,t-s,u(x-r,t-s)) \, dr \, ds$$

where $E(r,s)$ is given by Definition 2.4.

We next show that $u(x,t)$ is bounded on $\mathbb{R}^n \times (0, \infty)$. We have

$$|u(x,t)| \leq \int_{-\infty}^{\infty} \int_{\mathbb{R}^n} |E(r,s)||f(x-r,t-s,u(x-r,t-s))| \, dr \, ds$$

$$\leq \frac{2^{2-n}N.M(t)}{\pi^{n/2} \Gamma(\frac{n}{2}) \Gamma(\frac{n}{2})}$$

by the condition (3) and (3.1) where

$$N = \int_0^\infty \int_{\mathbb{R}^n} |f(x,t,u(x,t))| \, dx \, dt.$$ 

Thus $u(x,t)$ is bounded on $\mathbb{R}^n \times (0, \infty)$.

To show that $u(x,t)$ is unique, suppose there is another solution $w(x,t)$ of equation (3.3). Let the operator

$$L = \frac{\partial}{\partial t} - c^2 \square^k$$

then (3.3) can be written in the form

$$L u(x,t) = f(x,t,u(x,t)).$$

Thus

$$L u(x,t) - L w(x,t) = f(x,t,u(x,t)) - f(x,t,w(x,t)).$$

By the condition (2) of the Theorem,

$$|L u(x,t) - L w(x,t)| \leq A|u(x,t) - w(x,t)|.$$ (3.5)

Let $\Omega_0 \times (0, T]$ be compact subset of $\mathbb{R}^n \times (0, \infty)$ and $L : C^{(2k)}(\Omega_0) \rightarrow C^{(2k)}(\Omega_0)$ for $0 \leq t \leq T$. 

Now \( \left( C^{(2k)}(\Omega_0), \| \cdot \| \right) \) is a Banach space where \( u(x,t) \in C^{(2k)}(\Omega_0) \) for \( 0 \leq t \leq T \), \( \| \cdot \| \) given by
\[
\| u(x,t) \| = \sup_{x \in \Omega_0} |u(x,t)|.
\]
Then, from (3.5) with \( 0 < A < 1 \), the operator \( L \) is a contraction mapping on \( C^{(2k)}(\Omega_0) \). Since \( \left( C^{(2k)}(\Omega_0), \| \cdot \| \right) \) is a Banach space and \( L: C^{(2k)}(\Omega_0) \rightarrow C^{(2k)}(\Omega_0) \) is a contraction mapping on \( C^{(2k)}(\Omega_0) \), by Contraction Theorem, see [3, p. 300], we obtain the operator \( L \) has a fixed point and has uniqueness property. Thus \( u(x,t) = w(x,t) \). It follows that the solution \( u(x,t) \) of (3.3) is unique for \( (x,t) \in \Omega_0 \times (0,T] \) where \( u(x,t) \) is defined by (3.4).

In particular, if we put \( k = 1 \) and \( q = 0 \) in (3.3) then (3.3) reduces to the nonlinear heat equation
\[
\frac{\partial}{\partial t} u(x,t) - c^2 \Delta u(x,t) = f(x,t,u(x,t))
\]
which has solution
\[
u(x,t) = E(x,t) \ast f(x,t,u(x,t))
\]
where \( E(x,t) \) is defined by (2.5) with \( k = 1 \) and \( q = 0 \). That is complete of proof. \( \square \)

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