Picard and Adomian methods for quadratic integral equation

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Abstract. We are concerning with two analytical methods; the classical method of successive approximations (Picard method) [14] which consists the construction of a sequence of functions such that the limit of this sequence of functions in the sense of uniform convergence is the solution of a quadratic integral equation, and Adomian method which gives the solution as a series see ([1-6], [12] and [13]). The existence and uniqueness of the solution and the convergence will be discussed for each method.

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1 Introduction

Quadratic integral equations (QIEs) are often applicable in the theory of radiative transfer, kinetic theory of gases, in the theory of neutron transport and in the traffic theory. The quadratic integral equations can be very often encountered in many applications.

The quadratic integral equations have been studied in several papers and monographs (see for examples [7-11] and [16-22]).

The Picard-Lindelof theorem for the initial value problem

$$x'(t) = f(t, x(t)), \quad x(0) = x_0$$

was proved in [14] and the solution can be shown as the limit of constructed sequence.

The existence of continuous solution of the nonlinear quadratic integral equation

$$x(t) = a(t) + g(t, x(t)) \int_0^t f(s, x(s)) \, ds \tag{1}$$

was proved in [18] by using Tychonov fixed point theorem where f and g satisfy Carathéodory condition.

In this work, we will prove the existence and the uniqueness of continuous solution for (1) by using the principle of contraction mapping. Also, we are concerning with the two methods; Picard method and Adomian method.

2   Main Theorem

Now, equation (1) will be investigated under the assumptions:

(i) $a : I \to R_+ = [0, +\infty)$ is continuous on I where $I = [0, 1]$;

(ii) $f, \, g : I \times D \subset R_+ \to R_+$ are continuous and there exist positive constants M_1 and M_2 such that $|g(t, x)| \le M_1$ and $|f(t, x)| \le M_2$ on D;

(iii) f, g satisfy Lipschitz condition with Lipschitz constants L_1 and L_2 such that,

$$
\begin{aligned}
|g(t, x) - g(t, y)| &\le L_1 |x - y|, \\
|f(t, x) - f(t, y)| &\le L_2 |x - y|.
\end{aligned}
$$

Let $C = C(I)$ be the space of all real valued functions which are continuous on I.

Define the operator F as

$$(Fx)(t) = a(t) + g(t, x(t)) \int_0^t f(s, x(s)) \, ds, \quad \forall x \in C.$$

Theorem 1. *Let the assumptions* (i)-(iii) *be satisfied. If* $h = (L_1 M_2 + L_2 M_1) < 1$, *then the nonlinear quadratic integral equation* (1) *has a unique positive solution* $x \in C$.

Proof. It is clear that the operator F maps C into C.

Now define a subset S of C as

$$S = \{x \in C : |x - a(t)| \le k\}, \quad k = M_1 M_2.$$

Then the operator F maps S into S, since for $x \in S$

$$|x(t) - a(t)| \le M_1 M_2 \int_0^t ds = M_1 M_2 \, t = M_1 M_2.$$

Moreover it is easy to see that S is a closed subset of C. In order to show that F is a contraction we compute

$$
\begin{aligned}
(Fx)(t) - (Fy)(t) &= g(t, x(t)) \int_0^t f(s, x(s)) \, ds \\
&\quad - g(t, y(t)) \int_0^t f(s, y(s)) \, ds + g(t, x(t)) \\
&\quad \times \int_0^t f(s, y(s)) \, ds - g(t, x(t)) \int_0^t f(s, y(s)) \, ds \\
&= [g(t, x(t)) - g(t, y(t))] \int_0^t f(s, y(s)) \, ds \\
&\quad + g(t, x(t)) \int_0^t [f(s, x(s)) - f(s, y(s))] \, ds \\
|(Fx)(t) - (Fy)(t)| &\le |g(t, x(t)) - g(t, y(t))| \int_0^t |f(s, y(s))| \, ds \\
&\quad + |g(t, x(t))| \int_0^t |f(s, x(s)) - f(s, y(s))| \, ds \\
&\le L_1 M_2 |x - y| + L_2 M_1 \int_0^t |x(s) - y(s)| \, ds \\
\|(Fx)(t) - (Fy)(t)\| &= \max_{t \in I} |(Fx)(t) - (Fy)(t)| \\
&\le L_1 M_2 \|x - y\| + L_2 M_1 \|x - y\| \\
&\le (L_1 M_2 + L_2 M_1) \|x - y\| \\
&\le h \|x - y\|.
\end{aligned}
$$

Since

$$h = (L_1 M_2 + L_2 M_1) < 1,$$

then F is a contraction and F has a unique fixed point in S, thus there exists a unique solution for (1). $\qquad\square$

3 Method of successive approximations (Picard method)

Applying Picard method to the quadratic integral equation (1), the solution is constructed by the sequence

$$
\begin{aligned}
x_n(t) &= a(t) + g(t, x_{n-1}(t)) \int_0^t f(s, x_{n-1}(s)) \, ds, \quad n = 1, 2, \ldots, \\
x_0(t) &= a(t).
\end{aligned}
\tag{2}
$$

All the functions $x_n(t)$ are continuous functions and x_n can be written as a sum of successive differences:

$$x_n = x_0 + \sum_{j=1}^{n} (x_j - x_{j-1}),$$

This means that convergence of the sequence x_n is equivalent to convergence of the infinite series $\sum (x_j - x_{j-1})$ and the solution will be,

$$x(t) = \lim_{n \to \infty} x_n(t),$$

i.e. if the infinite series $\sum (x_j - x_{j-1})$ converges, then the sequence $x_n(t)$ will converge to $x(t)$. To prove the uniform convergence of $\{x_n(t)\}$, we shall consider the associated series

$$\sum_{n=1}^{\infty} [x_n(t) - x_{n-1}(t)].$$

From (2) for $n = 1$, we get

$$x_1(t) - x_0(t) = g(t, x_0(t)) \int_0^t f(s, x_0(s)) \, ds$$

and

$$| x_1(t) - x_0(t) | \le M_1 M_2 \int_0^t ds = M_1 M_2 \, t.
\tag{3}$$

Now, we shall obtain an estimate for $x_n(t) - x_{n-1}(t)$, $n \geq 2$

$$x_n(t) - x_{n-1}(t) \leq g(t, x_{n-1}(t)) \int_0^t f(s, x_{n-1}(s)) \, ds$$

$$- g(t, x_{n-2}(t)) \int_0^t f(s, x_{n-2}(s)) \, ds + g(t, x_{n-1}(t))$$

$$\times \int_0^t f(s, x_{n-2}(s)) \, ds - g(t, x_{n-1}(t)) \int_0^t f(s, x_{n-2}(s)) \, ds$$

$$\leq g(t, x_{n-1}(t)) \int_0^t [f(s, x_{n-1}(s)) - f(s, x_{n-2}(s))] \, ds$$

$$+ [g(t, x_{n-1}(t)) - g(t, x_{n-2}(t))] \int_0^t f(s, x_{n-2}(s)) \, ds,$$

using assumptions (ii) and (iii), we get

$$|x_n(t) - x_{n-1}(t)| \leq M_1 L_2 \int_0^t |x_{n-1}(s) - x_{n-2}(s)| \, ds$$

$$+ M_2 L_1 |x_{n-1}(t) - x_{n-2}(t)| \int_0^t ds.$$

Putting $n = 2$, then using (3) we get

$$|x_2(t) - x_1(t)| \ \leq \ M_1 L_2 \int_0^t |x_1(s) - x_0(s)| ds + M_2 L_1 |x_1(t) - x_0(t)| t.$$

$$|x_2 - x_1| \ \leq \ M_1^2 M_2 L_2 \frac{t^2}{2} + M_1 M_2^2 L_1 t^2$$

$$\leq \ M_1 M_2 \left(\frac{1}{2} M_1 L_2 + M_2 L_1 \right) t^2$$

$$|x_3 - x_2| \ \leq \ M_1 L_2 \int_0^t |x_2(s) - x_1(s)| ds + M_2 L_1 |x_2(t) - x_1(t)| \int_0^t ds$$

$$\leq \ M_1 M_2 \left(\frac{1}{2} M_1 L_2 + M_2 L_1 \right) \left(\frac{1}{3} M_1 L_2 + M_2 L_1 \right) t^3.$$

Repeating this technique, we obtain the general estimate for the terms of the series:

$$|x_n - x_{n-1}|$$

$$\leq M_1 M_2 \left(\frac{1}{2} M_1 L_2 + M_2 L_1 \right) \left(\frac{1}{3} M_1 L_2 + M_2 L_1 \right) \times \cdots \times \left(\frac{1}{n} M_1 L_2 + M_2 L_1 \right) t^n$$

$$\leq M_1 M_2 \left(\frac{1}{2} M_1 L_2 + M_2 L_1\right) \left(\frac{1}{3} M_1 L_2 + M_2 L_1\right) \times \cdots \times \left(\frac{1}{n} M_1 L_2 + M_2 L_1\right)$$

$$\leq M_1 M_2 (M_1 L_2 + M_2 L_1)(M_1 L_2 + M_2 L_1) \times \cdots \times (M_1 L_2 + M_2 L_1)$$

$$\leq (M_1 L_2 + M_2 L_1)^n.$$

Since $(L_1 M_2 + L_2 M_1) < 1$, then the uniform convergence of

$$\sum_{n=1}^{\infty} \left[x_n(t) - x_{n-1}(t)\right]$$

is proved and so the sequence $\{x_n(t)\}$ is uniformly convergent.

Since $f(t, x)$ and $g(t, x)$ are continuous in x, then

$$
\begin{aligned}
x(t) &= \lim_{n \to \infty} g(t, x_n(t)) \int_0^t f(s, x_n(s)) \, ds \\
&= g(t, x(t)) \int_0^t f(s, x(s)) \, ds.
\end{aligned}
$$

thus, the existence of a solution is proved.

To prove the uniqueness, let $y(t)$ be a continuous solution of (1). Then

$$y(t) = a(t) + g(t, y(t)) \int_0^t f(s, y(s)) \, ds \quad t \in [0, 1]$$

and

$$
\begin{aligned}
y(t) - x_n(t) &\leq g(t, y(t)) \int_0^t f(s, y(s)) \, ds - g(t, x_{n-1}(t)) \int_0^t f(s, x_{n-1}(s)) \, ds \\
&\quad + g(t, y(t)) \int_0^t f(s, x_{n-1}(s)) \, ds - g(t, y(t)) \int_0^t f(s, x_{n-1}(s)) \, ds \\
&= g(t, y(t)) \int_0^t [f(s, y(s)) - f(s, x_{n-1}(s))] \, ds \\
&\quad + [g(t, y(t)) - g(t, x_{n-1}(t))] \int_0^t f(s, x_{n-1}(s)) \, ds,
\end{aligned}
$$

using assumptions (ii) and (iii), we get

$$|y(t) - x_n(t)| \leq M_1 L_2$$

$$\times \int_0^t |y(s) - x_{n-1}(s)| \, ds + M_2 L_1 |y(t) - x_{n-1}(t)| \int_0^t \, ds. \tag{4}$$

But

$$|y(t) - a(t)| \leq M_1 M_2 t,$$

and using (4) then we get

$$|y(t) - x_n(t)| \leq (M_1 L_2 + M_2 L_1)^n.$$

Hence

$$\lim_{n \to \infty} x_n(t) = y(t) = x(t),$$

which completes the proof.   □

When $g(t, x) = 1$, then $M_1 = 1$ and $L_1 = 0$ and we obtain the original Picard theorem [14] and [15].

Corollary 1. *Let the assumptions of Theorem* 1 *(with* $g(t, x) = 1$*) be satisfied. If* $L_2 < 1$*, then the integral equation*

$$x(t) = x_0(t) + \int_0^t f(s, x(s)) \, ds$$

has a unique continuous solution.

4 Adomian Decomposition Method (ADM)

The Adomian decomposition method (ADM) is a nonnumerical method for solving a wide variety of functional equations and usually gets the solution in a series form.

Since the beginning of the 1980s, Adomian ([1-6] and [12-13]) has presented and developed a so-called decomposition method for solving algebraic, differential, integro-differential, differential-delay, and partial differential equations. The solution is found as an infinite series which converges rapidly to accurate solutions. The method has many advantages over the classical techniques, mainly, it makes unnecessary the linearization, perturbation and other restrictive methods and assumptions which may change the problem being solved, sometimes seriously. In recent decades, there has been a great deal of interest in the Adomian decomposition method. The method was successfully applied to

a large amount of applications in applied sciences. For more details about the method and its application, see ([1-6] and [12-13]).

In this section, we shall study Adomian decomposition method (ADM) for the quadratic integral equation (1).

The solution algorithm of the quadratic integral equation (1) using ADM is,

$$x_0(t) \ = \ a(t), \tag{5}$$

$$x_i(t) \ = \ A_{(i-1)}(t) \int_0^t B_{(i-1)}(s)\, ds. \tag{6}$$

where A_i and B_i are Adomian polynomials of the nonlinear terms $g(t, x)$ and $f(s, x)$ respectively, which have the form

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[f\left(t, \sum_{i=0}^{\infty}\lambda^i x_i\right)\right]_{\lambda=0}, \tag{7}$$

$$B_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[g\left(t, \sum_{i=0}^{\infty}\lambda^i x_i\right)\right]_{\lambda=0} \tag{8}$$

and the solution will be,

$$x(t) = \sum_{i=0}^{\infty} x_i(t) \tag{9}$$

4.1   Convergence analysis

Theorem 2.   *Let the solution of the QIE (1) exists. If $|x_1(t)| < l, l$ is a positive constant, then the series solution (9) of the QIE (1) using ADM converges.*

Proof.   Define the sequence $\{S_p\}$ such that,

$$S_p = \sum_{i=0}^{p} x_i(t)$$

is the sequence of partial sums from the series solution $\sum_{i=0}^{\infty} x_i(t)$, and we have

$$g(t, x) \ = \ \sum_{i=0}^{\infty} A_i,$$

$$f(s, x) \ = \ \sum_{i=0}^{\infty} B_i.$$

Let S_p and S_q be two arbitrary partial sums with $p > q$. Now, we are going to prove that $\{S_p\}$ is a Cauchy sequence in this Banach space E.

$$S_p - S_q = \sum_{i=0}^{p} x_i - \sum_{i=0}^{q} x_i$$

$$= \sum_{i=0}^{p} A_{(i-1)}(t) \int_0^t \sum_{i=0}^{p} B_{(i-1)}(s) \, ds - \sum_{i=0}^{q} A_{(i-1)}(t) \int_0^t \sum_{i=0}^{q} B_{(i-1)}(s) \, ds$$

$$= \sum_{i=0}^{p} A_{(i-1)}(t) \int_0^t \sum_{i=0}^{p} B_{(i-1)}(s) \, ds - \sum_{i=0}^{q} A_{(i-1)}(t) \int_0^t \sum_{i=0}^{p} B_{(i-1)}(s) \, ds$$

$$+ \sum_{i=0}^{q} A_{(i-1)}(t) \int_0^t \sum_{i=0}^{p} B_{(i-1)}(s) \, ds - \sum_{i=0}^{q} A_{(i-1)}(t) \int_0^t \sum_{i=0}^{q} B_{(i-1)}(s) \, ds$$

$$= \left[\sum_{i=0}^{p} A_{(i-1)}(t) - \sum_{i=0}^{q} A_{(i-1)}(t) \right] \int_0^t \sum_{i=0}^{p} B_{(i-1)}(s) \, ds$$

$$+ \sum_{i=0}^{q} A_{(i-1)}(t) \int_0^t \left[\sum_{i=0}^{p} B_{(i-1)}(s) - \sum_{i=0}^{q} B_{(i-1)}(s) \right] ds$$

$$\| S_p - S_q \| \leq \max_{t \in I} \left| \sum_{i=q+1}^{p} A_{(i-1)}(t) \int_0^t \sum_{i=0}^{p} B_{(i-1)}(s) \, ds \right|$$

$$+ \max_{t \in I} \left| \sum_{i=0}^{q} A_{(i-1)}(t) \int_0^t \sum_{i=q+1}^{p} B_{(i-1)}(s) \, ds \right|$$

$$\leq \max_{t \in I} \left| \sum_{i=q}^{p-1} A_i(t) \right| \int_0^t \left| \sum_{i=0}^{p} B_{(i-1)}(s) \, ds \right| ds$$

$$+ \max_{t \in I} \left| \sum_{i=0}^{q} A_{(i-1)}(t) \right| \int_0^t \left| \sum_{i=q}^{p-1} B_i(s) \right| ds$$

$$\leq \max_{t \in I} \left| g(t, S_{p-1}) - g(t, S_{q-1}) \right| \int_0^t \left| f(t, S_p) \right| ds$$

$$+ \max_{t \in I} \left| g(t, S_q) \right| \int_0^t \left| f(t, S_{p-1}) - f(t, S_{q-1}) \right| ds$$

$$\leq L_1 M_2 \max_{t \in I} \left| S_{p-1} - S_{q-1} \right| + L_2 M_1 \max_{t \in I} \left| S_{p-1} - S_{q-1} \right|$$

$$\leq h \; \| S_{p-1} - S_{q-1} \|$$

Let $p = q + 1$ then,

$$\left\| S_{q+1} - S_q \right\| \leq h \left\| S_q - S_{q-1} \right\| \leq h^2 \left\| S_{q-1} - S_{q-2} \right\| \leq \cdots \leq h^q \left\| S_1 - S_0 \right\|$$

From the triangle inequality we have,

$$
\begin{aligned}
\left\| S_p - S_q \right\| &\leq \left\| S_{q+1} - S_q \right\| + \left\| S_{q+2} - S_{q+1} \right\| + \cdots + \left\| S_p - S_{p-1} \right\| \\
&\leq \left[h^q + h^{q+1} + \cdots + h^{p-1} \right] \left\| S_1 - S_0 \right\| \\
&\leq h^q \left[1 + h + \cdots + h^{p-q-1} \right] \left\| S_1 - S_0 \right\| \\
&\leq h^q \left[\frac{1 - h^{p-q}}{1 - h} \right] \left\| x_1(t) \right\|
\end{aligned}
$$

Now $0 < h < 1$, and $p > q$ implies that $(1 - h^{p-q}) \leq 1$. Consequently,

$$
\begin{aligned}
\left\| S_p - S_q \right\| &\leq \frac{h^q}{1 - h} \left\| x_1(t) \right\| \\
&\leq \frac{h^q}{1 - h} \max_{t \in I} |x_1(t)|
\end{aligned}
$$

but, $|x_1(t)| < l$ and as $q \to \infty$ then, $\left\| S_p - S_q \right\| \to 0$ and hence, $\{ S_p \}$ is a Cauchy sequence in this Banach space E and the series $\sum_{i=0}^{\infty} x_i(t)$ converges. $\square$

4.2   *Error analysis*

Theorem 3.   *The maximum absolute truncation error of the series solution* (9) *to the problem* (1) *is estimated to be,*

$$\max_{t \in I} \left| x(t) - \sum_{i=0}^{q} x_i(t) \right| \leq \frac{h^q}{1 - h} \max_{t \in I} |x_1(t)|$$

Proof.   From Theorem 2 we have,

$$\left\| S_p - S_q \right\| \leq \frac{h^q}{1 - h} \max_{t \in I} |x_1(t)|$$

but, $S_p = \sum_{i=0}^{p} x_i(t)$ as $p \to \infty$ then $S_p \to x(t)$ so,

$$\left\| x(t) - S_q \right\| \leq \frac{h^q}{1 - h} \max_{t \in I} |x_1(t)|$$

so, the maximum absolute truncation error in the interval I is,

$$\max_{t \in I} \left| x(t) - \sum_{i=0}^{q} x_i(t) \right| \leq \frac{h^q}{1-h} \max_{t \in I} |x_1(t)|$$

and this completes the proof.   □

5   Numerical Examples

In this section, we shall study some numerical examples and applying Picard and ADM methods, then comparing the results.

Example 1.   Consider the following nonlinear QIE,

$$x(t) = \left(t^2 - \frac{t^{10}}{35} \right) + \frac{t}{5} x(t) \int_0^t s^2 x^2(s) \, ds, \tag{10}$$

and has the exact solution $x(t) = t^2$.

Applying Picard method to equation (10), we get

$$x_n(t) = \left(t^2 - \frac{t^{10}}{35} \right) + \frac{t}{5} x_{n-1}(t) \int_0^t s^2 \, x_{n-1}^2(s) \, ds, \quad n = 1, 2, \ldots,$$

$$x_0(t) = \left(t^2 - \frac{t^{10}}{35} \right).$$

and the solution will be,

$$x(t) = \lim_{n \to \infty} x_n(t).$$

Applying ADM to equation (10), we get

$$x_0(t) = \left(t^2 - \frac{t^{10}}{35} \right),$$

$$x_i(t) = \frac{t}{5} x_{i-1}(t) \int_0^t s^2 A_{i-1}(s) \, ds, \quad i \geq 1.$$

where A_i are Adomian polynomials of the nonlinear term x^2, and the solution will be,

$$x(t) = \sum_{i=0}^{q} x_i(t)$$

Table 1 shows a comparison between the absolute error of Picard (when $n = 5$) and ADM solutions (when $q = 5$).

t	$\lvert x_{\text{Exact}} - x_{\text{Picard}} \rvert$	$\lvert x_{\text{Exact}} - x_{\text{ADM}} \rvert$
0.1	0	1.57823×10^{-19}
0.2	0	1.03431×10^{-14}
0.3	2.94109×10^{-22}	6.79375×10^{-12}
0.4	0	6.77831×10^{-10}
0.5	0	2.40789×10^{-8}
0.6	1.50584×10^{-19}	4.44997×10^{-7}
0.7	1.57984×10^{-16}	5.23525×10^{-6}
0.8	9.63707×10^{-14}	0.0000441862
0.9	2.73899×10^{-11}	0.000288498
1	4.26772×10^{-9}	0.00153008

Table 1 – Absolute error.

Example 2. Consider the following nonlinear QIE,

$$x(t) = \left(t^3 - \frac{t^{19}}{100} - \frac{t^{20}}{110} \right) + \frac{t^3}{10} x^2(t) \int_0^t (s+1) x^3(s)\, ds, \qquad (11)$$

and has the exact solution $x(t) = t^3$.

Applying Picard method to equation (11), we get

$$x_n(t) = \left(t^3 - \frac{t^{19}}{100} - \frac{t^{20}}{110} \right) + \frac{t^3}{10} x_{n-1}^2(t) \int_0^t (s+1) x_{n-1}^3(s)\, ds, \quad n = 1, 2, \ldots,$$

$$x_0(t) = \left(t^3 - \frac{t^{19}}{100} - \frac{t^{20}}{110} \right).$$

and the solution will be,

$$x(t) = \lim_{n \to \infty} x_n(t).$$

Applying ADM to equation (11), we get

$$x_0(t) = \left(t^3 - \frac{t^{19}}{100} - \frac{t^{20}}{110} \right),$$

$$x_i(t) = \frac{t^3}{10} A_{i-1}(t) \int_0^t (s+1) B_{i-1}(s)\, ds, \quad i \geq 1.$$

where A_i and B_i are Adomian polynomials of the nonlinear terms x^2 and x^3 respectively, and the solution will be,

$$x(t) = \sum_{i=0}^{q} x_i(t)$$

Table 2 shows a comparison between the absolute error of Picard (when $n = 3$) and ADM solutions (when $q = 3$).

t	$|x_{\text{Exact}} - x_{\text{Picard}}|$	$|x_{\text{Exact}} - x_{\text{ADM}}|$
0.1	1.88079×10^{-34}	1.88079×10^{-34}
0.2	2.46519×10^{-29}	1.89203×10^{-26}
0.3	2.24387×10^{-26}	9.48288×10^{-21}
0.4	1.61559×10^{-24}	1.08415×10^{-16}
0.5	0	1.55749×10^{-13}
0.6	5.39199×10^{-21}	6.01199×10^{-11}
0.7	1.90012×10^{-16}	9.35496×10^{-9}
0.8	1.21415×10^{-12}	7.47272×10^{-7}
0.9	2.79233×10^{-9}	0.0000356866
1	2.82633×10^{-6}	0.0011119

Table 2 – Absolute error.

Example 3.   Consider the following nonlinear QIE [9],

$$x(t) = t^3 + \left(\frac{1}{4} x(t) + \frac{1}{4} \right) \left(\int_0^t t + \cos\left(\frac{x(s)}{1 + x^2(s)} \right) \right) ds, \qquad (12)$$

Applying Picard method to equation (12), we get

$$x_n(t) = t^3 + \left(\frac{1}{4} x_{n-1}(t) + \frac{1}{4} \right) \left(\int_0^t t + \cos\left(\frac{x_{n-1}(s)}{1 + x_{n-1}^2(s)} \right) \right) ds, \quad n = 1, 2, \ldots,$$

$$x_0(t) = t^3.$$

and the solution will be,

$$x(t) = x_n(t).$$

Applying ADM to equation (12), we get

$$x_0(t) = t^3,$$

$$x_{n+1}(t) = \left(\frac{1}{4}x_n(t) + \frac{1}{4}\right)\left(\int_0^t t + A_{i-1}(s)\right)ds, \quad i \geq 1.$$

where A_i are Adomian polynomials of the nonlinear term $\cos\left(\frac{x(s)}{1+x^2(s)}\right)$ and the solution will be,

$$x(t) = \sum_{i=0}^{q} x_i(t).$$

Table 3 shows a caparison between ADM and Picard solutions.

t	ADM solution	Picard solution	$|x_{\text{ADM}} - x_{\text{Picard}}|$
0.1	0.0285275	0.0272762	0.00125125
0.2	0.0684798	0.0634398	0.00504
0.3	0.127128	0.115575	0.0115538
0.4	0.212929	0.191649	0.02128
0.5	0.335783	0.300627	0.0351563
0.6	0.507262	0.452542	0.05472
0.7	0.740798	0.65854	0.0822587
0.8	1.05195	0.930992	0.12096
0.9	1.45895	1.28389	0.175061
1	1.98343	1.73343	0.25

Table 3 – Absolute error.

Example 4.   Consider the following nonlinear QIE [8],

$$x(t) = e^{-t} + x(t)\int_0^t \frac{t^2 \ln(1 + s|x(s)|)}{2e^{(t+s)}}ds, \quad 0 < t \leq 2. \qquad (13)$$

Applying Picard method to equation (13), we get

$$x_n(t) = e^{-t} + x_{n-1}(t)\int_0^t \frac{t^2 \ln(1 + s|x_{n-1}(s)|)}{2e^{(t+s)}}ds, \quad n = 1, 2, \ldots, x_0(t)$$

$$= e^{-t}.$$

and the solution will be,

$$x(t) = x_n(t).$$

Applying ADM to equation (13), we get

$$x_0(t) = e^{-t},$$
$$x_{n+1}(t) = x_n(t) \int_0^t \frac{t^2 A_{i-1}(s)}{2e^{(t+s)}} ds, \quad i \geq 1.$$

where A_i are Adomian polynomials of the nonlinear term $\ln(1 + s|x(s)|)$ and the solution will be,

$$x(t) = \sum_{i=0}^q x_i(t).$$

Table 4 shows a caparison between ADM and Picard solutions.

t	ADM solution	Picard solution	$|x_{\text{ADM}} - x_{\text{Picard}}|$
0.2	0.818926	0.818926	1.11022×10^{-16}
0.4	0.671897	0.671897	0
0.6	0.552921	0.552921	0
0.8	0.456136	0.456136	0
1	0.376724	0.376724	0
1.2	0.31109	0.31109	0
1.4	0.256612	0.256612	2.77556×10^{-17}
1.6	0.211338	0.211338	0
1.8	0.173748	0.173748	0
2	0.142602	0.142602	0

Table 4 – Absolute error.

6   Conclusion

We used two analytical methods to solve QIEs; Picard method and ADM, from the results in the tables we see that Picard method gives more accurate solution than ADM.

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