Buckling and free vibration analysis of orthotropic plates by using exponential shear deformation theory

Abstract
In the present paper, an exponential shear deformation theory is used to determine the natural frequencies and critical buckling loads of orthotropic plates. The theory accounts for a parabolic distribution of the transverse shear strains across the thickness, and satisfies the zero traction boundary conditions on the top and bottom surfaces of the plate without using shear correction factors. The in-plane displacement field uses an exponential function in terms of thickness coordinate to include the effect of shear deformation and rotary inertia. Governing equations and boundary conditions are derived from the dynamic version of principle of virtual work. The Navier type solution is employed for solving the governing equations of simply supported square orthotropic plates. The results obtained using present higher order shear deformation theory are found to be agree well with those obtained by others several existing higher order theories for analyzing the buckling and free vibration behaviour of orthotropic plates.

Keywords
orthotropic plates, shear correction factor, shear deformation, natural frequencies, uniaxial, biaxial, critical buckling load.

1 INTRODUCTION
Orthotropic plates are widely used in structural applications because of their advantageous properties such as high stiffness and strength to weight ratios. Main failure mechanisms in orthotropic plates are bending, buckling and free vibration. Unlike any other isotropic plate, the buckling and free vibration analysis of orthotropic plate is more complicated due to inherently anisotropic. Thus, an accurate buckling and free vibration analysis of the orthotropic plates is an important part of the structural design.

Classical plate theory (CPT) which neglects the effect of transverse shear deformation, overestimates natural frequencies and critical buckling loads. The errors in natural frequencies and critical buckling loads are quite significant for plate made out of composite materials. The Reissner (1945) and Mindlin (1951) have developed first order shear deformation plate theories (FSDTs) considering the transverse shear and rotary inertia effects by the way of linear variation of in-plane displacements through the thickness of plate. Since these models violate the equilibrium conditions at the top and bottom surfaces of the plate, shear correction factors are required to correct the unrealistic variation of transverse shear stresses and shear strains through the thickness.

To overcome the limitations of classical plate theory and first order shear deformation theory, a many higher-order shear deformation plate theories which involve the higher-order terms in power series of the coordinate normal to the middle plane, have been proposed. A critical review of these higher-order shear deformation plate theories has been given by Vasil’ev (1992), Noor and

There exists another class of refined shear deformation theories, wherein use of an exponential function is made to take into account shear deformation effect. Karama et al. (2009) have used exponential function to predict the mechanical behavior multilayered laminated composite plates. Sayyad and Ghugul (2012a and 2012b) has carried out bending, buckling and free vibration analysis of isotropic plates using an exponential function in-terms of thickness coordinates to represent the effect of shear deformation. Sayyad (2013) also applied exponential shear deformation theory for the flexural analysis of orthotropic plates.

The purpose of the present study is to derive the analytical solutions of exponential shear deformation theory (Sayyad and Ghugul, 2012a and 2012b; Sayyad, 2013) for buckling and free vibration analysis of simply supported orthotropic plates. The displacement model contains exponential terms in addition to classical plate theory terms. The number of unknown variables is same as that of first order shear deformation theory. Governing equations and associated boundary conditions are derived from dynamic version of principle of virtual work. The Navier type solution is employed for solving the governing equations of simply supported rectangular orthotropic plates. The Navier type solution for orthotropic plate based on higher order shear deformation theory (HSDT) of Reddy (1984), trigonometric shear deformation theory (TSDT) of Ghugul and Sayyad (2011b), first order shear deformation theory (FSDT) of Mindlin (1951) and classical plate theory are generated for the verification purpose. The natural frequencies and critical buckling loads of orthotropic plates for various modular and aspect ratios are studied and discussed in detail. Exact elasticity solution for vibration of simply supported homogeneous thick rectangular plate is provided by Srinivas et al. (1970) whereas exact elasticity solution for buckling analysis of plates is not available in the literature.

2 Orthotropic Plate under Consideration

Consider a rectangular plate of sides ‘a’ and ‘b’ and a constant thickness of ‘h’. The plate is subjected to transverse load (q) and in-plane compressive forces (N_{xx}, N_{yy} and N_{xy}). The co-ordinate system (x, y, z) chosen and the coordinate parameters are such that, the plate occupies a region given by Eq. (1).

\[
0 \leq x \leq a; \quad 0 \leq y \leq b; \quad -h/2 \leq z \leq h/2
\]  

(1)

2.1 Assumptions made in theoretical formulation

Assumptions of the exponential shear deformation theory are as follows:

1. The displacements are small in comparison with the plate thickness ‘h’ and, therefore, strains involved are infinitesimal.
2. The in-plane displacement u in x direction as well as displacement v in y direction each consists of bending and shear components
\[ u = u_b + u_s; \quad v = v_b + v_s \]  

(2)

a) The bending components \( u_b \) and \( v_b \) are assumed to be similar to the displacements given by the classical plate theory. Therefore, the expression for \( u_b \) and \( v_b \) can be given as.

\[ u_b = -z \frac{\partial w(x,y,t)}{\partial x}; \quad v_b = -z \frac{\partial w(x,y,t)}{\partial y} \]  

(3)

b) Shear components \( u_s \) and \( v_s \) are assumed to be exponential in nature with respect to thickness coordinate, such that the maximum shear stress occurs at neutral plane. Consequently, the expressions for \( u_s \) and \( v_s \) can be given as.

\[ u_s = z \exp\left[-2\left(\frac{z}{h}\right)^2\right] \phi(x,y,t); \quad v_s = z \exp\left[-2\left(\frac{z}{h}\right)^2\right] \psi(x,y,t) \]  

(4)

where, \( \phi \) and \( \psi \) are the unknown functions associated with the shear slopes.

3. The transverse displacement \( w \) in \( z \) direction is assumed to be a function of coordinates \( x \) and \( y \) and time \( t \).

\[ w(x,y,z,t) = w(x,y,t) \]  

(5)

2.2 Kinematics:

Based on the above mentioned assumptions, the displacement field can be obtained using Eqs. (2)–(5) as:

\[ u(x,y,z,t) = -z \frac{\partial w(x,y,t)}{\partial x} + z \exp\left[-2\left(\frac{z}{h}\right)^2\right] \phi(x,y,t) \]  

(6)

\[ v(x,y,z,t) = -z \frac{\partial w(x,y,t)}{\partial y} + z \exp\left[-2\left(\frac{z}{h}\right)^2\right] \psi(x,y,t) \]

\[ w(x,y,t) = w(x,y,t) \]

This displacement field accounts for zero traction boundary conditions on the top and bottom surfaces of the plate and the parabolic variation of transverse shear strains and stresses through the thickness of plates. The kinematic relations can be obtained as follows:

\[ \varepsilon_x = z k_x^b + f(z) k_x^s; \quad \varepsilon_y = z k_y^b + f(z) k_y^s; \quad \varepsilon_z = 0; \]  

(7)

\[ \gamma_{xy} = z k_{xy}^b + f(z) k_{xy}^s; \quad \gamma_{xz} = g(z) \phi; \quad \gamma_{yz} = g(z) \psi. \]

where,
\[
\begin{aligned}
    k^b_x &= -\frac{\partial^2 w}{\partial x^2}; \\
    k^s_x &= \frac{\partial \phi}{\partial x}; \\
    k^b_y &= -\frac{\partial^2 w}{\partial y^2}; \\
    k^s_y &= \frac{\partial \psi}{\partial y}; \\
    k^b_{xy} &= -2 \frac{\partial^2 w}{\partial x \partial y}; \\
    k^s_{xy} &= \frac{\partial \psi}{\partial y} + \frac{\partial \phi}{\partial x};
\end{aligned}
\]

\[
f(z) = z \exp \left[ -2 \left( \frac{z}{h} \right)^2 \right]; \\
g(z) = \exp \left[ -2 \left( \frac{z}{h} \right)^2 \right] \left[ 1 - 4 \left( \frac{z}{h} \right)^2 \right]
\]

### 2.3 Constitutive relations

The constitutive relations for orthotropic materials are as follows:

\[
\begin{pmatrix}
    \sigma_x \\
    \sigma_y \\
    \tau_{xy} \\
    \tau_{yz} \\
    \tau_{zx}
\end{pmatrix} =
\begin{pmatrix}
    \overline{Q}_{11} & \overline{Q}_{12} & 0 & 0 & 0 \\
    \overline{Q}_{12} & \overline{Q}_{22} & 0 & 0 & 0 \\
    0 & 0 & \overline{Q}_{66} & 0 & 0 \\
    0 & 0 & 0 & \overline{Q}_{44} & 0 \\
    0 & 0 & 0 & 0 & \overline{Q}_{55}
\end{pmatrix}
\begin{pmatrix}
    \varepsilon_x \\
    \varepsilon_y \\
    \gamma_{xy} \\
    \gamma_{yz} \\
    \gamma_{zx}
\end{pmatrix}
\]

where, \( \overline{Q}_{ij} \) are the plane stress reduced elastic constants in the material axes of the plate, and are defined as:

\[
\begin{aligned}
    \overline{Q}_{11} &= \frac{E_1}{1 - \mu_{12} \mu_{21}}; \\
    \overline{Q}_{12} &= \frac{\mu_{12} E_2}{1 - \mu_{12} \mu_{21}}; \\
    \overline{Q}_{22} &= \frac{E_2}{1 - \mu_{12} \mu_{21}}; \\
    \overline{Q}_{66} &= G_{12}; \\
    \overline{Q}_{55} &= G_{13}; \\
    \overline{Q}_{44} &= G_{23}.
\end{aligned}
\]

### 3 Derivation of Governing Equations and Boundary Conditions

The governing equations and boundary conditions are derived using principle of virtual work. Let \( \delta \) be the arbitrary variations

\[
\int \int \int_V (\delta U - \delta W + \delta T) = 0
\]

where the virtual strain energy \( \delta U \), virtual potential energy \( \delta W \) due to the transverse load \( q(x, y) \) and constant inplane compressive and shear forces \( (N^0_{xx}, N^0_{yy} \text{ and } N^0_{xy}) \) and the virtual kinetic energy \( \delta T \) are given by
\[
\delta U = \int_{-h/2}^{h/2} \int_0^a \int_0^b \left( \sigma_x \delta e_x + \sigma_y \delta e_y + \tau_{yx} \delta y_{yz} + \tau_{zy} \delta y_{zy} + \tau_{xy} \delta y_{xy} \right) dx dy dz
\]  

(12)

\[
\delta W = \int_0^a \int_0^b q(x, y) \delta w dx dy + \frac{1}{2} \int_{-h/2}^{h/2} \int_0^a \int_0^b \delta \left[ N_{xx}^0 \left( \frac{\partial w}{\partial x} \right)^2 + N_{yy}^0 \left( \frac{\partial w}{\partial y} \right)^2 + 2N_{xy}^0 \left( \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \right) \right] dx dy dz
\]

\[
\delta T = \rho \int_{-h/2}^{h/2} \int_0^a \int_0^b \left[ \frac{\partial^2 u}{\partial t^2} \delta u + \frac{\partial^2 v}{\partial t^2} \delta v + \frac{\partial^2 w}{\partial t^2} \delta w \right] dx dy dz
\]

Substituting Eq. (12) in Eq. (11)

\[
\int_{-h/2}^{h/2} \int_0^a \int_0^b \left( \sigma_x \delta e_x + \sigma_y \delta e_y + \tau_{yx} \delta y_{yz} + \tau_{zy} \delta y_{zy} + \tau_{xy} \delta y_{xy} \right) dx dy dz
\]

(13)

\[
-\int_0^a \int_0^b q(x, y) \delta w dx dy - \frac{1}{2} \int_{-h/2}^{h/2} \int_0^a \int_0^b \delta \left[ N_{xx}^0 \left( \frac{\partial w}{\partial x} \right)^2 + N_{yy}^0 \left( \frac{\partial w}{\partial y} \right)^2 + 2N_{xy}^0 \left( \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \right) \right] dx dy dz
\]

\[
+ \rho \int_{-h/2}^{h/2} \int_0^a \int_0^b \left[ \frac{\partial^2 u}{\partial t^2} \delta u + \frac{\partial^2 v}{\partial t^2} \delta v + \frac{\partial^2 w}{\partial t^2} \delta w \right] dx dy dz = 0
\]

Substituting Eqs. (6) - (10) into the Eq. (13) and integrating through the thickness of the plate, the principle of virtual work of the plate can be rewritten as

\[
\int_{0}^{a} \int_{0}^{b} \left[ \begin{array}{c}
M_x \frac{\partial^2 w}{\partial x^2} - N_{xx} \frac{\partial \delta \phi}{\partial x} + M_y \frac{\partial^2 w}{\partial y^2} - N_{yy} \frac{\partial \delta \psi}{\partial y} + 2M_{xy} \frac{\partial^2 w}{\partial x \partial y} - N_{xy} \frac{\partial \delta \phi}{\partial y} \\
- N_{xx} \frac{\partial \delta \psi}{\partial x} - N_{xy} \frac{\partial \delta \psi}{\partial y} - N_{xy} \frac{\partial \delta \phi}{\partial y}
\end{array} \right] dx dy
\]

(14)

\[
+ \int_{0}^{a} \int_{0}^{b} \left[ q(x, y) \delta w + N_{xx}^0 \frac{\partial \delta w}{\partial x} + N_{yy}^0 \frac{\partial \delta w}{\partial y} + 2N_{xy}^0 \frac{\partial \delta w}{\partial x \partial y} \right] dx dy dz
\]

\[
\int_{0}^{a} \int_{0}^{b} \left[ \begin{array}{c}
I_1 \frac{\partial^2 w}{\partial t^2} \delta w + I_2 \frac{\partial^3 w}{\partial x^2 \partial t} \delta w - I_3 \frac{\partial^3 w}{\partial x \partial y \partial t} \delta \phi - I_4 \frac{\partial^2 \phi}{\partial t^2} \delta \phi \\
+ I_2 \frac{\partial^3 w}{\partial y^2 \partial t} \delta w - I_3 \frac{\partial^3 w}{\partial y \partial x \partial t} \delta \psi - I_4 \frac{\partial^2 \psi}{\partial t^2} \delta \psi
\end{array} \right] dx dy dz = 0
\]

where, stress resultants \( \left( M_x, M_y, M_{xy}, N_{xx}, N_{yy}, N_{xy}, N_{txx}, N_{txy} \right) \) are defined as:

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\[
(M_x, M_y, M_{xy}) = \int_{-h/2}^{h/2} \left( \sigma_x, \sigma_y, \tau_{xy} \right) z \, dz;
\]
\[
(N_{xx}, N_{xy}, N_{xy}) = \int_{-h/2}^{h/2} \left( \sigma_x, \sigma_y, \tau_{xy} \right) f(z) \, dz;
\]
\[
(N_{TCx}, N_{TCy}) = \int_{-h/2}^{h/2} \left( \tau_{cx}, \tau_{cy} \right) g(z) \, dz;
\]
\[
V_x = \frac{\partial M_x}{\partial x} + 2 \frac{\partial M_{xy}}{\partial y} - N_{xx}^0 \frac{\partial w}{\partial x} - N_{xy}^0 \frac{\partial w}{\partial y};
\]
\[
V_y = \frac{\partial M_y}{\partial y} + 2 \frac{\partial M_{xy}}{\partial x} - N_{xy}^0 \frac{\partial w}{\partial y} - N_{xy}^0 \frac{\partial w}{\partial x};
\]
\[
(I_1, I_2, I_3, I_4) = \rho \int_{-h/2}^{h/2} (1, z^2, z f(z), f^2(z)) \, dz.
\]

Substituting Eqs. (7) and (9) into Eq. (15) and integrating through the thickness of the plate, the stress resultants are related to the generalized displacements \((w, \phi \text{ and } \psi)\) by the relations
\[
\begin{bmatrix}
M_x \\
M_y \\
N_{xx} \\
N_{xy}
\end{bmatrix} =
\begin{bmatrix}
D_{11} & B_{s11} & D_{12} & B_{s12} \\
D_{12} & B_{s12} & D_{22} & B_{s22} \\
B_{s11} & A_{ss11} & B_{s12} & A_{ss12} \\
B_{s12} & A_{ss12} & B_{s22} & A_{ss22}
\end{bmatrix}
\begin{bmatrix}
k_x^e \\
k_y^e \\
k_x^c \\
k_y^c
\end{bmatrix},
\begin{bmatrix}
M_{xy} \\
N_{xy}
\end{bmatrix} =
\begin{bmatrix}
D_{66} & B_{s66} \\
B_{s66} & A_{ss66}
\end{bmatrix}
\begin{bmatrix}
k_x^{ec} \\
k_y^{ec}
\end{bmatrix},
\]
\[
\begin{bmatrix}
N_{TCx} \\
N_{TCy}
\end{bmatrix} =
\begin{bmatrix}
Acc_{55} & 0 \\
0 & Acc_{44}
\end{bmatrix}
\begin{bmatrix}
\phi \\
\psi
\end{bmatrix}
\]

Various stiffnesses used in Eq. (16) are expressed below
\[
(D_{11}, D_{12}, D_{22}, D_{66}) = \left( \overline{Q}_{11}, \overline{Q}_{12}, \overline{Q}_{22}, \overline{Q}_{66} \right) \int_{-h/2}^{+h/2} z^2 \, dz;
\]
\[
(B_{s11}, B_{s12}, B_{s22}, B_{s66}) = \left( \overline{Q}_{11}, \overline{Q}_{12}, \overline{Q}_{22}, \overline{Q}_{66} \right) \int_{-h/2}^{+h/2} z \, f(z) \, dz;
\]
\[
(A_{ss11}, A_{ss12}, A_{ss22}, A_{ss66}) = \left( \overline{Q}_{11}, \overline{Q}_{12}, \overline{Q}_{22}, \overline{Q}_{66} \right) \int_{-h/2}^{+h/2} f^2(z) \, dz;
\]
\[
(Acc_{44}, Acc_{55}) = \left( \overline{Q}_{44}, \overline{Q}_{55} \right) \int_{-h/2}^{+h/2} g^2(z) \, dz.
\]

Integrating the Eq. (14) by parts, collecting the coefficients of \(\delta w, \delta \phi, \text{ and } \delta \psi\) the equations of motion and boundary conditions for the orthotropic plate are obtained as follows:
\[
\frac{\partial^2 M_x}{\partial x^2} + 2 \frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} + q(x, y) + N_{xx}^0 \frac{\partial^2 W}{\partial x^2} + N_{yy}^0 \frac{\partial^2 W}{\partial y^2} + 2N_{xy}^0 \frac{\partial^2 W}{\partial x \partial y} = I_1 \frac{\partial^2 W}{\partial t^2} - I_2 \left( \frac{\partial^4 W}{\partial x^2 \partial t^2} + \frac{\partial^4 W}{\partial y^2 \partial t^2} \right) + I_3 \left( \frac{\partial^3 \phi}{\partial x \partial t^2} + \frac{\partial^3 \psi}{\partial y \partial t^2} \right)
\]

The associated boundary conditions of a plate are as follows:

<table>
<thead>
<tr>
<th></th>
<th>( x = 0 ) and ( x = a )</th>
<th>( y = 0 ) and ( y = b )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Either ( V_x = 0 )</td>
<td>( w ) is prescribed</td>
<td>Either ( V_y = 0 )</td>
</tr>
<tr>
<td>Either ( M_x = 0 )</td>
<td>( \frac{\partial w}{\partial x} )</td>
<td>( \frac{\partial w}{\partial y} )</td>
</tr>
<tr>
<td>Either ( N_{xx} = 0 )</td>
<td>( \phi ) is prescribed</td>
<td>Either ( N_{xy} = 0 )</td>
</tr>
<tr>
<td>Either ( N_{xy} = 0 )</td>
<td>( \psi ) is prescribed</td>
<td>Either ( N_{xy} = 0 )</td>
</tr>
</tbody>
</table>

Substituting Eq. (16) into Eqs. (18)-(20), the governing equations of the plate in terms of generalized displacements are as follows:

\[
D_{11} \frac{\partial^4 w}{\partial x^4} + (2D_{12} + 4D_{66}) \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_{22} \frac{\partial^4 w}{\partial y^4} - \frac{\partial N_{xx}}{\partial x} (B_{s12} + 2B_{s66}) \frac{\partial^3 \phi}{\partial x \partial y \partial t^2} + \frac{\partial N_{xy}}{\partial y} (B_{s12} + 2B_{s66}) \frac{\partial^3 \psi}{\partial x \partial y \partial t^2}
\]

\[
-(B_{s22} \frac{\partial^3 \psi}{\partial y^3} - (B_{s12} + 2B_{s66}) \frac{\partial^3 \psi}{\partial x^2 \partial y} + I_1 \frac{\partial^2 w}{\partial t^2} - I_2 \left( \frac{\partial^4 w}{\partial x^2 \partial t^2} + \frac{\partial^4 w}{\partial y^2 \partial t^2} \right) + I_3 \left( \frac{\partial^3 \phi}{\partial x \partial t^2} + \frac{\partial^3 \psi}{\partial y \partial t^2} \right) = q(x, y) + N_{xx}^0 \frac{\partial^2 W}{\partial x^2} + N_{yy}^0 \frac{\partial^2 W}{\partial y^2} + 2N_{xy}^0 \frac{\partial^2 W}{\partial x \partial y}
\]

\[
B_{s11} \frac{\partial^3 w}{\partial x^3} + (B_{s12} + 2B_{s66}) \frac{\partial^3 w}{\partial x \partial y^2} - Ass_{s11} \frac{\partial^2 \phi}{\partial x^2} - Ass_{s66} \frac{\partial^2 \phi}{\partial y^2} + Acc_{ss} \phi
\]

\[
-(Ass_{s2} + Ass_{s66}) \frac{\partial^2 \psi}{\partial x \partial y} - I_3 \frac{\partial^3 w}{\partial x \partial t^2} + I_4 \frac{\partial^3 \phi}{\partial t^2} = 0
\]
\[ BS_{22} \frac{\partial^3 w}{\partial y^3} + (Bs_{12} + 2Bs_{66}) \frac{\partial^3 w}{\partial x^2 \partial y} - Ass_{66} \frac{\partial^2 \psi}{\partial x^2} - Ass_{22} \frac{\partial^2 \psi}{\partial y^2} + Acc_{44} \psi = 0 \] (23)

\[-(Ass_{12} + Ass_{66}) \frac{\partial^2 \phi}{\partial x \partial y} - I_3 \frac{\partial^2 \psi}{\partial y \partial t^2} + I_4 \frac{\partial^2 \psi}{\partial t^2} = 0 \]

4 Navier solution for simply supported square orthotropic plates

The Navier method is only applied for simply supported boundary conditions on all four edges of the rectangular plate. The following are the boundary conditions of the simply supported orthotropic plates.

\[ w = \psi = M_x = N_{sx} = 0 \quad \text{at} \quad x = a \quad \text{and} \quad x = a \] (24)

\[ w = \phi = M_y = N_{sy} = 0 \quad \text{at} \quad y = 0 \quad \text{and} \quad y = b \] (25)

In order to solve the governing equations with the prescribed boundary conditions, a generalized Navier’s approach is employed to obtain the closed-form solutions.

Example 1: Free vibration analysis of simply supported square orthotropic plates

The governing equations for free vibration analysis of plates can be obtained by setting the applied transverse load \(q\) and in-plane compressive forces \(N_{Ax}, N_{Ay}\) and \(N_{xy}\) equal to zero in Eqs. (21)-(23).

\[ D_{11} \frac{\partial^4 w}{\partial x^4} + (2D_{12} + 4D_{66}) \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_{22} \frac{\partial^4 w}{\partial y^4} - Bs_{11} \frac{\partial^3 \phi}{\partial x^3} - (Bs_{12} + 2Bs_{66}) \frac{\partial^3 \phi}{\partial x \partial y^2} - Bs_{22} \frac{\partial^3 \psi}{\partial y^3} = 0 \] (26)

\[-(Bs_{12} + 2Bs_{66}) \frac{\partial^3 \psi}{\partial x^2 \partial y} + I_4 \frac{\partial^2 w}{\partial t^2} - I_1 \left( \frac{\partial^4 w}{\partial x^4} + \frac{\partial^4 w}{\partial y^4} \right) + I_2 \left( \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^2 \partial x^2} \right) + I_3 \left( \frac{\partial^3 \phi}{\partial x \partial y^2} + \frac{\partial^3 \psi}{\partial y \partial t^2} \right) = 0 \]

\[ Bs_{11} \frac{\partial^3 w}{\partial x^3} + (Bs_{12} + 2Bs_{66}) \frac{\partial^3 w}{\partial x \partial y^2} - Ass_{11} \frac{\partial^2 \phi}{\partial x^2} - Ass_{66} \frac{\partial^2 \phi}{\partial y^2} + Acc_{55} \phi \] (27)

\[-(Ass_{12} + Ass_{66}) \frac{\partial^2 \psi}{\partial x \partial y} - I_3 \frac{\partial^3 w}{\partial x \partial y \partial t^2} + I_4 \frac{\partial^2 \phi}{\partial t^2} = 0 \]

\[ Bs_{22} \frac{\partial^3 w}{\partial y^3} + (Bs_{12} + 2Bs_{66}) \frac{\partial^3 w}{\partial x^2 \partial y} - Ass_{66} \frac{\partial^2 \psi}{\partial x^2} - Ass_{22} \frac{\partial^2 \psi}{\partial y^2} + Acc_{44} \psi \] (28)

\[-(Ass_{12} + Ass_{66}) \frac{\partial^2 \phi}{\partial x \partial y} - I_3 \frac{\partial^3 w}{\partial y \partial x \partial t^2} + I_4 \frac{\partial^2 \psi}{\partial t^2} = 0 \]

The following displacement functions \(w(x, y), \phi(x, y)\) and \(\psi(x, y)\) are chosen to automatically satisfy boundary conditions in Eqs. (24) and (25).

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\[ w(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} w_{mn} \sin \left( \frac{m\pi x}{a} \right) \sin \left( \frac{n\pi y}{b} \right) \sin \omega_{mn} t; \]  

\[ \phi(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \phi_{mn} \cos \left( \frac{m\pi x}{a} \right) \sin \left( \frac{n\pi y}{b} \right) \sin \omega_{mn} t; \]  

\[ \psi(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \psi_{mn} \sin \left( \frac{m\pi x}{a} \right) \cos \left( \frac{n\pi y}{b} \right) \sin \omega_{mn} t \]

where \( w_{mn} \) is the amplitude of translation and \( \phi_{mn}, \psi_{mn} \) are the amplitudes of rotation. \( \omega_{mn} \) is the natural frequency of \( m^{th} \) and \( n^{th} \) mode of vibration. Substitution of this solution form into the Eqs. (26)-(28), results in following standard Eigen value problem.

\[
\begin{bmatrix}
K_{11} & K_{12} & K_{13} \\
K_{21} & K_{22} & K_{23} \\
K_{31} & K_{32} & K_{33}
\end{bmatrix} - \omega_{mn}^2 
\begin{bmatrix}
M_{11} & M_{12} & M_{13} \\
M_{21} & M_{22} & M_{23} \\
M_{31} & M_{32} & M_{33}
\end{bmatrix}
\begin{bmatrix}
w_{mn} \\
\phi_{mn} \\
\psi_{mn}
\end{bmatrix} = 0
\]

(30)

where \( [K_{ij}] \) is the stiffness matrix, \( [M_{ij}] \) is the mass matrix. Elements of \( [K_{ij}] \) and \( [M_{ij}] \) are given as below:

\[
K_{11} = D_{11} \frac{m^4 \pi^4}{a^4} + (2D_{12} + 4D_{66}) \frac{m^2 n^2 \pi^4}{a^2b^2} + D_{22} \frac{n^4 \pi^4}{b^4};
\]

\[
K_{12} = -B_{s11} \frac{m^3 \pi^3}{a^3} - (B_{s12} + 2B_{s66}) \frac{mn^2 \pi^3}{ab^2};
\]

\[
K_{13} = -B_{s22} \frac{n^3 \pi^3}{b^3} - (B_{s12} + 2B_{s66}) \frac{m^2 n \pi^3}{a^2b};
\]

\[
K_{22} = A_{s11} \frac{m^2 \pi^2}{a^2} + A_{s66} \frac{n^2 \pi^2}{b^2} + Ac_{55}; \quad K_{23} = (A_{s12} + A_{s66}) \frac{mn \pi^2}{ab};
\]

\[
K_{33} = A_{s22} \frac{n^2 \pi^2}{b^2} + A_{s66} \frac{m^2 \pi^2}{a^2} + Ac_{44}; \quad K_{21} = K_{12}; \quad K_{31} = K_{13}; \quad K_{32} = K_{23};
\]

\[
M_{11} = \left( I_1 + I_2 \frac{m^2 \pi^2}{a^2} + I_2 \frac{n^2 \pi^2}{b^2} \right); \quad M_{12} = -I_3 \frac{m \pi}{a}; \quad M_{13} = -I_3 \frac{n \pi}{b}; M_{22} = I_4;
\]

\[
M_{23} = 0; \quad M_{33} = I_4; \quad M_{21} = M_{12}; \quad M_{31} = M_{13}; \quad M_{32} = M_{23};
\]
From the solution of Eq. (30), lowest natural frequency for all modes of vibration can be obtained. The orthotropic plate has following material properties.

\[ E_i / E_2 = 0.52500, \quad G_{12} / E_2 = 0.26293, \quad G_{13} / E_2 = 0.15991, \]  

\[ G_{23} / E_2 = 0.26681, \quad \mu_{12} = 0.44046, \quad \mu_{21} = 0.23124 \]

The bending mode and shear mode frequencies of orthotropic plate are presented in the following non-dimensional form.

\[ \bar{\omega}_{mn} = \omega_{mn} h \sqrt{\frac{\rho}{Q_{11}}} \quad \text{where} \quad Q_{11} = \frac{E_i}{1-\mu_{12}\mu_{21}} \]  

(33)

Example 2: Buckling Analysis of simply supported orthotropic plates subjected in-plane compressive forces

When a plate is subjected to in-plane compressive forces, and if the forces are small enough, the equilibrium of the plate is stable and the plate remains flat until ascertains load is reached. At that load, called the critical buckling load, the stable state of the plate is disturbed and plate seeks an alternative equilibrium configuration accompanied by a change in the load-deflection behavior. A simply supported rectangular plate subjected to the loading conditions, as shown in Fig. 1, is considered to illustrate the accuracy of the present theory in predicting the buckling behavior of the orthotropic plate

(a) Uniaxial compression along x-axis

(b) Uniaxial compression along y-axis

(c) Biaxial compression

Fig. 1 The loading conditions of rectangular plate: (a) uniaxial compression along x-axis, (b) uniaxial compression along y-axis (c) biaxial compression
The governing equations of plate in case of static buckling are obtained by setting \( q(x, y) = 0 \), \( N_{xx}^0 = k_1 N_0 \), \( N_{yy}^0 = k_2 N_{xx}^0 \) and \( N_{xy}^0 = 0 \) in Eqs. (21)-(23).

\[
D_{11} \frac{\partial^4 w}{\partial x^4} + (2D_{12} + 4D_{66}) \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_{22} \frac{\partial^4 w}{\partial y^4} - B_{s11} \frac{\partial^3 \phi}{\partial x^3} - (B_{s12} + 2B_{s66}) \frac{\partial^3 \phi}{\partial x \partial y^2} = \frac{m \pi x}{a} \sin \frac{m \pi x}{a} \sin \frac{n \pi y}{b} ;
\]

\[
B_{s11} \frac{\partial^3 w}{\partial x^3} + (B_{s12} + 2B_{s66}) \frac{\partial^3 \psi}{\partial x \partial y^2} - A_{s11} \frac{\partial^2 \phi}{\partial x^2} - A_{s66} \frac{\partial^2 \phi}{\partial y^2} + A_{c55} \phi = 0
\]

The following displacement functions \( w(x, y) \), \( \phi(x, y) \) and \( \psi(x, y) \) are chosen to automatically satisfy the boundary conditions in Eqs. (24) and (25).

\[
w(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} w_{mn} \sin \frac{m \pi x}{a} \sin \frac{n \pi y}{b} ;
\]

\[
\phi(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \phi_{mn} \cos \frac{m \pi x}{a} \sin \frac{n \pi y}{b} ;
\]

\[
\psi(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \psi_{mn} \sin \frac{m \pi x}{a} \cos \frac{n \pi y}{b} .
\]

Substituting Eq. (36) into Eqs. (33)-(35), the following system is obtained:

\[
\begin{bmatrix}
K_{11} & K_{12} & K_{13} \\
K_{21} & K_{22} & K_{23} \\
K_{31} & K_{32} & K_{33}
\end{bmatrix}
- N_0
\begin{bmatrix}
N_{11} & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
w_{mn} \\
\phi_{mn} \\
\psi_{mn}
\end{bmatrix} = \begin{bmatrix} 0 \\
0 \\
0
\end{bmatrix}
\]

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where, \([K_0]\) are given in Eq. (31) and \(N_{11}\) as given below:

\[
N_{11} = -\left( k_1 \frac{m^2 \pi^2}{a^2} + k_2 \frac{n^2 \pi^2}{b^2} \right)
\]  

(38)

For nontrivial solution, the determinant of the coefficient matrix in Eq. (37) must be zero. This gives the following equation for buckling load:

\[
N_0 = \begin{vmatrix}
K_{11} & K_{22} & K_{23} & -K_{12} & K_{23} & +K_{13} & K_{21} & K_{22} \\
K_{32} & K_{33} & & & & & & \\
K_{32} & K_{33} & & & & & & \\
\end{vmatrix}
\]

(39)

\[
N_{11} = \begin{vmatrix}
K_{22} & K_{23} & -K_{12} & K_{23} & +K_{13} & K_{21} & K_{22} \\
K_{32} & K_{33} & & & & & & \\
K_{32} & K_{33} & & & & & & \\
\end{vmatrix}
\]

For each choice of \(m\) and \(n\), there is a corresponding unique value of \(N_0\). The critical buckling load is the smallest value of \(N_0 (m, n)\). For verification purposes, a simply supported rectangular plate subjected to the uniaxial and biaxial loading conditions, as shown in Fig. 1, is considered to illustrate the accuracy of the present theory in predicting the buckling behavior of the orthotropic plate. The following material properties are used.

\[
E_1 / E_2 = \text{open}, \ G_{12} / E_2 = G_{13} / E_2 = 0.5, \ G_{23} / E_2 = 0.2, \ \mu_{12} = 0.25
\]

(40)

Critical buckling loads are presented in the following non-dimensional form:

\[
N_{cr} = \frac{N_0 a^2}{E_2 h^3}
\]

(41)

**Table 1:** Comparison of non-dimensional natural predominantly bending mode frequencies \(\bar{\omega}_n\) of simply-supported orthotropic square plate (\(b / a = 1, h / a = 0.1\))

<table>
<thead>
<tr>
<th>((m, n))</th>
<th>Exact (Srinivas et al., 1970)</th>
<th>Present (Reddy, 1984)</th>
<th>HSDT (Ghugul and Sayyad, 2011b)</th>
<th>TSDT (Shimpi and Patel, 2006)</th>
<th>RPT (Mindlin, 1951)</th>
<th>FSPT (Mindlin, 1951)</th>
<th>CPT</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1, 1)</td>
<td>0.0474</td>
<td>0.0474</td>
<td>0.0474</td>
<td>0.0477</td>
<td>0.0474</td>
<td>0.0474</td>
<td></td>
</tr>
<tr>
<td>(1, 2)</td>
<td>0.1033</td>
<td>0.1033</td>
<td>0.1033</td>
<td>0.1040</td>
<td>0.1032</td>
<td>0.1120</td>
<td></td>
</tr>
<tr>
<td>(1, 3)</td>
<td>0.1888</td>
<td>0.1888</td>
<td>0.1888</td>
<td>0.1793</td>
<td>0.1898</td>
<td>0.2154</td>
<td></td>
</tr>
<tr>
<td>(1, 4)</td>
<td>0.2969</td>
<td>0.2969</td>
<td>0.2969</td>
<td>0.2932</td>
<td>0.2980</td>
<td>0.3599</td>
<td></td>
</tr>
<tr>
<td>(2, 1)</td>
<td>0.1188</td>
<td>0.1190</td>
<td>0.1189</td>
<td>0.1196</td>
<td>0.1198</td>
<td>0.1187</td>
<td>0.1354</td>
</tr>
<tr>
<td>(2, 2)</td>
<td>0.1694</td>
<td>0.1697</td>
<td>0.1695</td>
<td>0.1696</td>
<td>0.1722</td>
<td>0.1692</td>
<td>0.1987</td>
</tr>
<tr>
<td>(2, 3)</td>
<td>0.2475</td>
<td>0.2480</td>
<td>0.2477</td>
<td>0.2478</td>
<td>0.2520</td>
<td>0.2469</td>
<td>0.3029</td>
</tr>
<tr>
<td>(2, 4)</td>
<td>0.3476</td>
<td>0.3482</td>
<td>0.3479</td>
<td>0.3468</td>
<td>0.3534</td>
<td>0.3463</td>
<td>0.4480</td>
</tr>
<tr>
<td>(3, 1)</td>
<td>0.2180</td>
<td>0.2191</td>
<td>0.2184</td>
<td>0.2199</td>
<td>0.2197</td>
<td>0.2178</td>
<td>0.2779</td>
</tr>
<tr>
<td>(3, 2)</td>
<td>0.2624</td>
<td>0.2637</td>
<td>0.2629</td>
<td>0.2671</td>
<td>0.2675</td>
<td>0.2619</td>
<td>0.3418</td>
</tr>
<tr>
<td>(3, 3)</td>
<td>0.3320</td>
<td>0.3337</td>
<td>0.3326</td>
<td>0.3326</td>
<td>0.3407</td>
<td>0.3310</td>
<td>0.4470</td>
</tr>
<tr>
<td>(4, 1)</td>
<td>0.3319</td>
<td>0.3351</td>
<td>0.3330</td>
<td>0.3346</td>
<td>0.3344</td>
<td>0.3311</td>
<td>0.4773</td>
</tr>
<tr>
<td>(4, 2)</td>
<td>0.3707</td>
<td>0.3743</td>
<td>0.3720</td>
<td>0.3727</td>
<td>0.3774</td>
<td>0.3696</td>
<td>0.5415</td>
</tr>
</tbody>
</table>
4.2 Discussion of Results

Example 1: In the present study, free vibration analysis of an orthotropic plate with all edges simply supported is considered. Natural predominantly bending mode ($\tilde{\omega}_b$) and thickness shear modes ($\tilde{\omega}_\phi$ and $\tilde{\omega}_\psi$) frequencies of square plate are obtained for aspect ratio 10. Non-dimensional frequencies of simply supported square plate are presented in Tables 1-3 and compared with higher order shear deformation theory (HSDT) of Reddy (1984), trigonometric shear deformation theory (TSDT) of Ghugal and Sayyad (2011b), refined plate theory (RPT) of Shimpi and Patel (2006), first order shear deformation theory (FSDT) of Mindlin (1951) and classical plate theory (CPT).

Table 1 shows comparison of predominantly bending mode frequencies of orthotropic square plate for various modes of vibration and aspect ratio 10. It is observed that the present theory yields excellent values of frequencies for all modes of vibration. The present theory and HSDT of Reddy predicts exact result of bending frequency for $(m = 1, n = 1)$, $(m = 1, n = 2)$, $(m = 1, n = 3)$ and $(m = 1, n = 4)$ modes of vibration whereas RPT and CPT overestimates the same. From Tables 2-3 it is observed that, thickness shear modes frequencies of orthotropic square plate obtained by the present theory and HSDT of Reddy are in close agreement with the exact values. The FSDT marginally overestimates the value of frequencies of this mode compared to the exact one.

Table 2: Comparison of non-dimensional predominantly thickness shear mode frequencies $\tilde{\omega}_\phi$ of simply-supported orthotropic square plate ($b / a = 1, h / a = 0.1$)

<table>
<thead>
<tr>
<th>$(m, n)$</th>
<th>Exact (Srinivas et al., 1970)</th>
<th>Present</th>
<th>HSDT (Reddy, 1984)</th>
<th>TSDT (Ghugal and Sayyad, 2011b)</th>
<th>FSDT (Mindlin, 1951)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1, 1)</td>
<td>1.3077</td>
<td>1.2999</td>
<td>1.3086</td>
<td>1.3077</td>
<td>1.3159</td>
</tr>
<tr>
<td>(1, 2)</td>
<td>1.3331</td>
<td>1.3290</td>
<td>1.3339</td>
<td>1.3332</td>
<td>1.3410</td>
</tr>
<tr>
<td>(1, 3)</td>
<td>1.3665</td>
<td>1.3638</td>
<td>1.3772</td>
<td>1.3766</td>
<td>1.3841</td>
</tr>
<tr>
<td>(1, 4)</td>
<td>1.4372</td>
<td>1.4281</td>
<td>1.4379</td>
<td>1.4371</td>
<td>1.4445</td>
</tr>
<tr>
<td>(2, 1)</td>
<td>1.4205</td>
<td>1.4168</td>
<td>1.4216</td>
<td>1.4203</td>
<td>1.4285</td>
</tr>
<tr>
<td>(2, 2)</td>
<td>1.4516</td>
<td>1.4277</td>
<td>1.4323</td>
<td>1.4316</td>
<td>1.4383</td>
</tr>
<tr>
<td>(2, 3)</td>
<td>1.4596</td>
<td>1.4562</td>
<td>1.4603</td>
<td>1.4598</td>
<td>1.4671</td>
</tr>
<tr>
<td>(2, 4)</td>
<td>1.5068</td>
<td>1.5039</td>
<td>1.5076</td>
<td>1.5063</td>
<td>1.5142</td>
</tr>
<tr>
<td>(3, 1)</td>
<td>1.5777</td>
<td>1.5744</td>
<td>1.5789</td>
<td>1.5766</td>
<td>1.5857</td>
</tr>
<tr>
<td>(3, 2)</td>
<td>1.5651</td>
<td>1.5612</td>
<td>1.5658</td>
<td>1.5644</td>
<td>1.5727</td>
</tr>
<tr>
<td>(3, 3)</td>
<td>1.5737</td>
<td>1.5701</td>
<td>1.5744</td>
<td>1.5737</td>
<td>1.5812</td>
</tr>
<tr>
<td>(4, 1)</td>
<td>1.7179</td>
<td>1.7119</td>
<td>1.7189</td>
<td>1.7168</td>
<td>1.7265</td>
</tr>
<tr>
<td>(4, 2)</td>
<td>1.6940</td>
<td>1.6890</td>
<td>1.6947</td>
<td>1.6942</td>
<td>1.7022</td>
</tr>
</tbody>
</table>

Table 3: Comparison of non-dimensional predominantly thickness shear mode frequencies $\tilde{\omega}_\psi$ of simply-supported orthotropic square plate ($b / a = 1, h / a = 0.1$)

<table>
<thead>
<tr>
<th>$(m, n)$</th>
<th>Exact (Srinivas et al., 1970)</th>
<th>Present</th>
<th>HSDT (Reddy, 1984)</th>
<th>TSDT (Ghugal and Sayyad, 2011b)</th>
<th>FSDT (Mindlin, 1951)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1, 1)</td>
<td>1.6530</td>
<td>1.6448</td>
<td>1.6550</td>
<td>1.6530</td>
<td>1.6647</td>
</tr>
<tr>
<td>(1, 2)</td>
<td>1.7160</td>
<td>1.7105</td>
<td>1.7209</td>
<td>1.7145</td>
<td>1.7307</td>
</tr>
<tr>
<td>(1, 3)</td>
<td>1.8115</td>
<td>1.8052</td>
<td>1.8210</td>
<td>1.8044</td>
<td>1.8307</td>
</tr>
<tr>
<td>(1, 4)</td>
<td>1.9306</td>
<td>1.9249</td>
<td>1.9466</td>
<td>1.9121</td>
<td>1.9562</td>
</tr>
<tr>
<td>(2, 1)</td>
<td>1.6805</td>
<td>1.6728</td>
<td>1.6827</td>
<td>1.6817</td>
<td>1.6922</td>
</tr>
<tr>
<td>(2, 2)</td>
<td>1.7509</td>
<td>1.7462</td>
<td>1.7562</td>
<td>1.7513</td>
<td>1.7657</td>
</tr>
<tr>
<td>(2, 3)</td>
<td>1.8523</td>
<td>1.8418</td>
<td>1.8622</td>
<td>1.8458</td>
<td>1.8717</td>
</tr>
<tr>
<td>(2, 4)</td>
<td>1.9749</td>
<td>1.9701</td>
<td>1.9912</td>
<td>1.9524</td>
<td>2.0004</td>
</tr>
<tr>
<td>(3, 1)</td>
<td>1.7334</td>
<td>1.7274</td>
<td>1.7361</td>
<td>1.7373</td>
<td>1.7452</td>
</tr>
<tr>
<td>(3, 2)</td>
<td>1.8195</td>
<td>1.8068</td>
<td>1.8255</td>
<td>1.8255</td>
<td>1.8343</td>
</tr>
<tr>
<td>(3, 3)</td>
<td>1.9289</td>
<td>1.9203</td>
<td>1.9395</td>
<td>1.9301</td>
<td>1.9418</td>
</tr>
<tr>
<td>(4, 1)</td>
<td>1.8458</td>
<td>1.8437</td>
<td>1.8583</td>
<td>1.7163</td>
<td>1.7267</td>
</tr>
<tr>
<td>(4, 2)</td>
<td>1.9447</td>
<td>1.9351</td>
<td>1.9514</td>
<td>1.9568</td>
<td>1.9588</td>
</tr>
</tbody>
</table>
Example 2: Present study also deals with the buckling analysis of an orthotropic square and rectangular plate with all edges simply supported. Three different in-plane loading conditions are used in this numerical study: (1) uniaxial compression along the $x$-axis; (2) uniaxial compression along the $y$-axis; and (3) biaxial compression. For the comparison studies, results are also generated using higher order shear deformation theory (HSDT) of Reddy (1984), trigonometric shear deformation theory (TSDT) of Ghugal and Sayyad (2011b), first order shear deformation theory (FSDT) of Mindlin (1951) and classical plate theory.

Tables 4-5 shows the comparison of non-dimensional critical buckling load for simply supported square plate subjected to uniaxial and biaxial compression with the variation of modular and aspect ratios. It can be seen that the present theory gives excellent results of critical buckling load for all aspect ratios and modular ratio 3, whereas TSDT overestimate and FSDT underestimate the same. The differences between present theory and HSDT of Reddy will slightly increases with respect to increase in modular ratios. Classical plate theory overestimates the value of critical buckling load for all aspect ratios and modular ratios. Tables 6-8 shows the comparison of non-dimensional critical buckling load for simply supported rectangular plate. Examination of these Tables reveals that, non-dimensional critical buckling load decreases with increase in $b/a$ ratios when subjected to uniaxial compression along $x$-axis whereas increases when subjected to uniaxial compression along $y$-axis and biaxial buckling. Critical buckling load for rectangular plate increases with increase.

**Table 4:** Comparison of non-dimensional buckling load factors ($N_o$) for simply supported orthotropic square plate under uniaxial compression ($b/a = 1, k_1 = -1, k_2 = 0, m = n = 1$)

<table>
<thead>
<tr>
<th>$a/h$</th>
<th>Model</th>
<th>Modular Ratio ($E_y/E_x$)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>3</td>
</tr>
<tr>
<td>5</td>
<td>Present</td>
<td>3.9650</td>
</tr>
<tr>
<td>TSDT (Ghugal and Sayyad, 2011b)</td>
<td>4.0572</td>
<td>6.3212</td>
</tr>
<tr>
<td>RPT (Kim et al, 2009)</td>
<td>---</td>
<td>6.3478</td>
</tr>
<tr>
<td>FSDT (Mindlin, 1951)</td>
<td>3.9386</td>
<td>6.1804</td>
</tr>
<tr>
<td>CPT</td>
<td>5.4248</td>
<td>11.163</td>
</tr>
<tr>
<td>10</td>
<td>Present</td>
<td>4.9612</td>
</tr>
<tr>
<td>TSDT (Ghugal and Sayyad, 2011b)</td>
<td>5.0128</td>
<td>9.3646</td>
</tr>
<tr>
<td>RPT (Kim et al, 2009)</td>
<td>---</td>
<td>9.3732</td>
</tr>
<tr>
<td>FSDT (Mindlin, 1951)</td>
<td>4.9562</td>
<td>9.2734</td>
</tr>
<tr>
<td>CPT</td>
<td>5.4248</td>
<td>11.163</td>
</tr>
<tr>
<td>20</td>
<td>Present</td>
<td>5.3004</td>
</tr>
<tr>
<td>TSDT (Ghugal and Sayyad, 2011b)</td>
<td>5.3194</td>
<td>10.653</td>
</tr>
<tr>
<td>RPT (Kim et al, 2009)</td>
<td>---</td>
<td>10.653</td>
</tr>
<tr>
<td>FSDT (Mindlin, 1951)</td>
<td>5.2994</td>
<td>10.620</td>
</tr>
<tr>
<td>CPT</td>
<td>5.4248</td>
<td>11.163</td>
</tr>
<tr>
<td>50</td>
<td>Present</td>
<td>5.4044</td>
</tr>
<tr>
<td>HSDT (Reddy, 1984)</td>
<td>5.4040</td>
<td>11.072</td>
</tr>
<tr>
<td>TSDT (Ghugal and Sayyad, 2011b)</td>
<td>5.4116</td>
<td>11.081</td>
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<tr>
<td>RPT (Kim et al, 2009)</td>
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<td>11.078</td>
</tr>
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<td>FSDT (Mindlin, 1951)</td>
<td>5.4046</td>
<td>11.072</td>
</tr>
<tr>
<td>CPT</td>
<td>5.4248</td>
<td>11.163</td>
</tr>
<tr>
<td>100</td>
<td>Present</td>
<td>5.4196</td>
</tr>
<tr>
<td>HSDT (Reddy, 1984)</td>
<td>5.4192</td>
<td>11.130</td>
</tr>
<tr>
<td>TSDT (Ghugal and Sayyad, 2011b)</td>
<td>5.4250</td>
<td>11.145</td>
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<tr>
<td>RPT (Kim et al, 2009)</td>
<td>---</td>
<td>11.142</td>
</tr>
<tr>
<td>FSDT (Mindlin, 1951)</td>
<td>5.4206</td>
<td>11.142</td>
</tr>
<tr>
<td>CPT</td>
<td>5.4248</td>
<td>11.163</td>
</tr>
</tbody>
</table>
Table 5: Comparison of non-dimensional buckling load ($N_{cr}$) for simply supported orthotropic square plate under biaxial compression ($b/a=1$, $k_1=-1$, $k_2=-1$, $m=n=1$)

<table>
<thead>
<tr>
<th>$a/h$</th>
<th>Model</th>
<th>$N_{cr}$ (Non-dimensional Critical Buckling Load)</th>
<th>$E_1/E_2$ (Modular Ratio)</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>Present</td>
<td>1.9825</td>
<td>3.1507 4.0473 4.6083 5.0246</td>
</tr>
<tr>
<td></td>
<td>HSĐT (Reddy, 1984)</td>
<td>1.9717</td>
<td>3.1036 3.9146 4.3711 4.6726</td>
</tr>
<tr>
<td></td>
<td>TSĐT (Ghugal and Sayyad, 2011b)</td>
<td>2.0281</td>
<td>3.1606 3.9662 4.4209 4.7251</td>
</tr>
<tr>
<td></td>
<td>RPT (Kim et al, 2009)</td>
<td>---</td>
<td>3.1739 --- --- ---</td>
</tr>
<tr>
<td></td>
<td>FSĐT (Mindlin, 1951)</td>
<td>1.9693</td>
<td>3.0902 3.8725 4.2924 4.5542</td>
</tr>
<tr>
<td></td>
<td>CPT</td>
<td>2.7124</td>
<td>5.5814 9.6917 13.8034 17.9154</td>
</tr>
<tr>
<td>10</td>
<td>Present</td>
<td>2.4806</td>
<td>4.6499 7.0402 8.8741 10.3380</td>
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<tr>
<td></td>
<td>HSĐT (Reddy, 1984)</td>
<td>2.4784</td>
<td>4.6386 7.0002 8.7885 10.1929</td>
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<td>TSĐT (Ghugal and Sayyad, 2011b)</td>
<td>2.5064</td>
<td>4.6823 7.0582 8.8558 10.2674</td>
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<td>---</td>
<td>4.6866 --- --- 11.1290</td>
</tr>
<tr>
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<td>FSĐT (Mindlin, 1951)</td>
<td>2.4781</td>
<td>4.6367 6.9910 8.7662 10.1522</td>
</tr>
<tr>
<td></td>
<td>CPT</td>
<td>2.7124</td>
<td>5.5814 9.6917 13.8034 17.9154</td>
</tr>
<tr>
<td>20</td>
<td>Present</td>
<td>2.6502</td>
<td>5.3125 8.8405 12.0731 15.0470</td>
</tr>
<tr>
<td></td>
<td>HSĐT (Reddy, 1984)</td>
<td>2.6497</td>
<td>5.3101 8.8320 12.0540 15.0127</td>
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<tr>
<td></td>
<td>TSĐT (Ghugal and Sayyad, 2011b)</td>
<td>2.6597</td>
<td>5.3266 8.8574 12.0875 15.0537</td>
</tr>
<tr>
<td></td>
<td>RPT (Kim et al, 2009)</td>
<td>---</td>
<td>5.2265 --- --- 15.5345</td>
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<td>FSĐT (Mindlin, 1951)</td>
<td>2.6497</td>
<td>5.3100 8.8311 12.0513 15.0070</td>
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<tr>
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<td>CPT</td>
<td>2.7124</td>
<td>5.5814 9.6917 13.8034 17.9154</td>
</tr>
<tr>
<td>50</td>
<td>Present</td>
<td>2.7022</td>
<td>5.5364 9.5437 13.4911 17.3971</td>
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<tr>
<td></td>
<td>HSĐT (Reddy, 1984)</td>
<td>2.7020</td>
<td>5.5360 9.5424 13.4884 17.3744</td>
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<tr>
<td></td>
<td>TSĐT (Ghugal and Sayyad, 2011b)</td>
<td>2.7058</td>
<td>5.5407 9.5490 13.4969 17.3849</td>
</tr>
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<td>RPT (Kim et al, 2009)</td>
<td>---</td>
<td>5.5390 --- --- 17.4860</td>
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<tr>
<td></td>
<td>FSĐT (Mindlin, 1951)</td>
<td>2.7023</td>
<td>5.5362 9.5425 13.4885 17.3745</td>
</tr>
<tr>
<td></td>
<td>CPT</td>
<td>2.7124</td>
<td>5.5814 9.6917 13.8034 17.9154</td>
</tr>
<tr>
<td>100</td>
<td>Present</td>
<td>2.7098</td>
<td>5.5700 9.6542 13.7238 17.7779</td>
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<tr>
<td></td>
<td>HSĐT (Reddy, 1984)</td>
<td>2.7096</td>
<td>5.5697 9.6533 13.7230 17.7767</td>
</tr>
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<td>TSĐT (Ghugal and Sayyad, 2011b)</td>
<td>2.7124</td>
<td>5.5727 9.6571 13.7269 17.7811</td>
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<td>RPT (Kim et al, 2009)</td>
<td>---</td>
<td>5.5710 --- --- 17.8060</td>
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<tr>
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<td>FSĐT (Mindlin, 1951)</td>
<td>2.7103</td>
<td>5.5710 9.6544 13.7241 17.7772</td>
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<tr>
<td></td>
<td>CPT</td>
<td>2.7124</td>
<td>5.5814 9.6917 13.8034 17.9154</td>
</tr>
</tbody>
</table>

Table 6: Comparison of non-dimensional critical buckling load ($N_{cr}$) of simply supported orthotropic rectangular plates subjected to uniaxial compression along x-axis ($a/h=5$, $k_1=-1$, $k_2=0$, $m=n=1$)

<table>
<thead>
<tr>
<th>$E_1/E_2$</th>
<th>Model</th>
<th>Non-dimensional Critical Buckling Load ($N_{cr}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>Present</td>
<td>6.3015</td>
</tr>
<tr>
<td></td>
<td>TSĐT (Ghugal and Sayyad, 2011b)</td>
<td>6.3212</td>
</tr>
<tr>
<td></td>
<td>RPT (Kim et al, 2009)</td>
<td>6.3478</td>
</tr>
<tr>
<td></td>
<td>FSĐT (Mindlin, 1951)</td>
<td>6.1804</td>
</tr>
<tr>
<td></td>
<td>TSĐT (Ghugal and Sayyad, 2011b)</td>
<td>8.4398</td>
</tr>
<tr>
<td></td>
<td>RPT (Kim et al, 2009)</td>
<td>9.1039</td>
</tr>
<tr>
<td></td>
<td>FSĐT (Mindlin, 1951)</td>
<td>8.2199</td>
</tr>
<tr>
<td></td>
<td>TSĐT (Ghugal and Sayyad, 2011b)</td>
<td>9.4502</td>
</tr>
<tr>
<td></td>
<td>RPT (Kim et al, 2009)</td>
<td>10.579</td>
</tr>
<tr>
<td></td>
<td>FSĐT (Mindlin, 1951)</td>
<td>9.1084</td>
</tr>
<tr>
<td></td>
<td>CPT</td>
<td>35.830</td>
</tr>
</tbody>
</table>
Table 7: Comparison of non-dimensional critical buckling load \( N_c \) of simply supported orthotropic rectangular plates subjected to uniaxial compression along \( y \)-axis \((a/h = 5, k_1 = 0, k_2 = -1, m = n = 1)\)

<table>
<thead>
<tr>
<th>( E_1 / E_2 )</th>
<th>Model</th>
<th>Non-dimensional Critical Buckling Load ( (N_c) )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>( (b / a) ) 1.0 1.5 2 2.5 3.0 3.5 4.0</td>
</tr>
<tr>
<td>10</td>
<td>Present</td>
<td>6.3014 11.930 20.059 30.587 43.485 58.743 76.356</td>
</tr>
<tr>
<td></td>
<td>HSDDT (Reddy, 1984)</td>
<td>6.2072 11.755 19.765 30.139 42.849 58.885 75.242</td>
</tr>
<tr>
<td></td>
<td>TSDDT (Ghugul and Sayyad, 2011b)</td>
<td>6.3212 11.907 19.975 30.426 43.229 58.375 75.859</td>
</tr>
<tr>
<td></td>
<td>RPT (Kim et al., 2009)</td>
<td>6.3478 --- 20.044 --- --- --- ---</td>
</tr>
<tr>
<td></td>
<td>FSDDT (Mindlin, 1951)</td>
<td>6.1804 11.705 19.682 30.013 42.670 57.644 74.929</td>
</tr>
<tr>
<td></td>
<td>CPT</td>
<td>11.163 21.048 35.371 53.918 76.638 103.514 134.53</td>
</tr>
<tr>
<td>25</td>
<td>Present</td>
<td>8.7062 17.634 30.403 46.904 67.107 90.999 118.57</td>
</tr>
<tr>
<td></td>
<td>HSDDT (Reddy, 1984)</td>
<td>8.3938 16.859 29.652 44.813 64.110 86.931 113.27</td>
</tr>
<tr>
<td></td>
<td>TSDDT (Ghugul and Sayyad, 2011b)</td>
<td>8.4398 16.968 29.171 44.943 64.253 87.089 113.44</td>
</tr>
<tr>
<td></td>
<td>RPT (Kim et al., 2009)</td>
<td>9.1039 --- 30.164 --- --- --- ---</td>
</tr>
<tr>
<td></td>
<td>FSDDT (Mindlin, 1951)</td>
<td>8.2199 16.606 28.611 44.131 63.132 85.604 111.54</td>
</tr>
<tr>
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<td>CPT</td>
<td>23.495 48.803 84.716 131.02 187.66 254.63 331.92</td>
</tr>
<tr>
<td>40</td>
<td>Present</td>
<td>10.049 20.769 36.058 55.801 79.968 108.55 141.42</td>
</tr>
<tr>
<td></td>
<td>HSDDT (Reddy, 1984)</td>
<td>9.3472 19.246 33.382 51.642 73.995 100.42 130.93</td>
</tr>
<tr>
<td></td>
<td>TSDDT (Ghugul and Sayyad, 2011b)</td>
<td>9.4502 19.353 33.487 51.744 74.092 100.52 131.02</td>
</tr>
<tr>
<td></td>
<td>RPT (Kim et al., 2009)</td>
<td>10.579 --- 35.034 --- --- --- ---</td>
</tr>
<tr>
<td></td>
<td>FSDDT (Mindlin, 1951)</td>
<td>9.1084 18.728 32.471 50.226 71.962 97.667 127.33</td>
</tr>
<tr>
<td></td>
<td>CPT</td>
<td>36.630 76.560 134.06 208.12 298.69 405.76 529.31</td>
</tr>
</tbody>
</table>

Table 8: Comparison of non-dimensional critical buckling load \( N_c \) of simply supported orthotropic rectangular plates subjected to biaxial compression \((a/h = 5, k_1 = -1, k_2 = -1, m = n = 1)\)

<table>
<thead>
<tr>
<th>( E_1 / E_2 )</th>
<th>Model</th>
<th>Non-dimensional Critical Buckling Load ( (N_c) )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>( (b / a) ) 1.0 1.5 2 2.5 3.0 3.5 4.0</td>
</tr>
<tr>
<td></td>
<td>RPT (Kim et al., 2009)</td>
<td>3.1739 --- 4.0087 --- --- --- ---</td>
</tr>
<tr>
<td></td>
<td>FSDDT (Mindlin, 1951)</td>
<td>3.0992 3.6017 3.9364 4.1398 4.2671 4.3505 4.4076</td>
</tr>
<tr>
<td></td>
<td>CPT</td>
<td>5.8914 6.4765 7.0743 7.4371 7.6638 7.8122 7.9137</td>
</tr>
<tr>
<td></td>
<td>RPT (Kim et al., 2009)</td>
<td>4.5519 --- 6.0527 --- --- --- ---</td>
</tr>
<tr>
<td>40</td>
<td>Present</td>
<td>5.0246 6.3907 7.2116 7.6967 7.9968 8.1920 8.3251</td>
</tr>
<tr>
<td></td>
<td>TSDDT (Ghugul and Sayyad, 2011b)</td>
<td>4.7251 5.9549 6.6975 7.1372 7.4092 7.5863 7.7071</td>
</tr>
<tr>
<td></td>
<td>RPT (Kim et al., 2009)</td>
<td>5.2895 --- 7.0069 --- --- --- ---</td>
</tr>
</tbody>
</table>

5 CONCLUSION
An exponential shear deformation theory (Sayyad and Ghugul, 2012a and 2012b; Sayyad, 2013) has been extended in this paper for buckling and free vibration analysis of orthotropic plates. The theory takes into account of transverse shear effects and parabolic distribution of the transverse shear strains through the thickness of the plate. From the numerical results and discussion it can be concluded that, the frequencies obtained by the present theory are accurate as seen from the comparison with exact results specially in case of natural predominately bending mode. Also, an
exponential shear deformation theory can accurately predict the critical buckling loads of the orthotropic plates with various plate aspect ratios and modular ratios.

References


