COMPUTATIONAL COMPLEXITY OF CLASSICAL PROBLEMS FOR HEREDITARY CLIQUE-HELLY GRAPHS

Flavia Bonomo
Departamento de Computación
Facultad de Ciencias Exactas y Naturales
Universidad de Buenos Aires
Buenos Aires – Argentina
fbonomo@dc.uba.ar

Guillermo Durán *
Departamento de Ingeniería Industrial
Facultad de Ciencias Físicas y Matemáticas
Universidad de Chile
Santiago – Chile
gduran@dii.uchile.cl

* Corresponding author/autor para quem as correspondências devem ser encaminhadas

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Abstract

A graph is clique-Helly when its cliques satisfy the Helly property. A graph is hereditary clique-Helly when every induced subgraph of it is clique-Helly. The decision problems associated to the stability, chromatic, clique and clique-covering numbers are NP-complete for clique-Helly graphs. In this note, we analyze the complexity of these problems for hereditary clique-Helly graphs. Some of them can be deduced easily by known results. We prove that the clique-covering problem remains NP-complete for hereditary clique-Helly graphs. Furthermore, the decision problems associated to the clique-transversal and the clique-independence numbers are analyzed too. We prove that they remain NP-complete for a smaller class: hereditary clique-Helly split graphs.

Keywords: computational complexity; hereditary clique-Helly graphs; split graphs.
1. Introduction

All graphs in this paper are finite, without loops or multiple edges. For a graph $G$ we denote by $V(G)$ and $E(G)$ the vertex set and the edge set of $G$, respectively.

A graph is complete if every pair of vertices is connected by an edge. A complete in a graph $G$ is a subset of pairwise adjacent vertices of $G$. A clique in a graph is a complete maximal under inclusion. The clique number of a graph $G$ is the cardinality of a maximum clique of $G$ and is denoted by $\omega(G)$.

The chromatic number $\chi(G)$ of a graph $G$ is the smallest number of colours that can be assigned to the vertices of $G$ in such a way that no two adjacent vertices receive the same colour.

A clique cover of a graph $G$ is a subset of cliques covering all the vertices of $G$. A clique-transversal is a set of vertices intersecting all the cliques of $G$. The clique-covering number $k(G)$ and the clique-transversal number $\tau_c(G)$ are the cardinalities of a minimum clique cover and a minimum clique-transversal of $G$, respectively.

A stable set in a graph $G$ is a subset of pairwise non-adjacent vertices of $G$. A clique-independent set is a subset of pairwise disjoint cliques of $G$. The stability number $\alpha(G)$ and the clique-independence number $\alpha_c(G)$ are the cardinalities of a maximum stable set and a maximum clique-independent set of $G$, respectively.

Consider a finite family of non-empty sets. The intersection graph of this family is obtained by representing each set by a vertex, two vertices being connected by an edge if and only if the corresponding sets intersect. The clique graph $K(G)$ of $G$ is the intersection graph of the cliques of $G$.

A family $S$ of subsets satisfies the Helly property when every subfamily of $S$ consisting of pairwise intersecting subsets has a common element. A graph is clique-Helly ($CH$) when its cliques satisfy the Helly property. A graph $G$ is hereditary clique-Helly ($HCH$) when $H$ is clique-Helly for every induced subgraph $H$ of $G$. These graphs have been characterized in [Pr93] as the graphs which contains none of the four graphs in Figure 1 as an induced subgraph. This characterization leads to a polynomial time recognition algorithm for hereditary clique-Helly graphs.

An interesting survey on clique-Helly and hereditary clique-Helly graphs appears in [Fa02].

![Figure 1 – Hajós graphs](image)

A graph is split if its vertices can be partitioned into a clique and a stable set.

The neighborhood of a vertex $v$ in a graph $G$ is the set $N(v)$ consisting of all the vertices that are adjacent to $v$. The closed neighborhood of $v$ is $N[v] = N(v) \cup \{v\}$. A vertex $v$ of $G$ is called simplicial when $N[v]$ is a complete of $G$, and universal when $N[v] = V(G)$.
It is easy to see that the decision problems associated to the stability, chromatic, clique and clique-covering numbers are NP-complete for clique-Helly graphs. The reduction is trivial: we have to add a universal vertex to the general graph $G$ in order to generate a clique-Helly graph $G'$.

However, $\omega(G)$ can be obtained in polynomial time for $HCH$ graphs. The number of cliques is bounded by the number of edges [Pr93] and all the cliques can be generated in $O(nmk)$, where $m$ is the number of edges, $n$ the number of vertices and $k$ the number of cliques of the graph [TIAS77].

The stable set and the colorability problems remain NP-complete for $HCH$ graphs. These results are direct corollaries of the NP-completeness of these problems for triangle-free graphs [Pol74], [MP96]. For triangle-free graphs, a subclass of $HCH$ graphs, the clique-covering number can be obtained in polynomial time [GJ79].

So, the following question arises naturally: what happens with the complexity of the clique-cover problem for hereditary clique-Helly graphs?

The decision problems associated to the problems of finding the clique-independence number and the clique-transversal number are NP-complete [CFT93] and NP-hard [EGT92], respectively. This last problem is not known to be in NP, in fact the problem of determining if a subset of vertices is a clique-transversal is NP-hard [DLS02].

The clique-transversal problem is NP-complete for $HCH$ graphs. Again, this result is a consequence of the NP-completeness of this problem for triangle-free graphs. In this class of graphs, the clique-transversal problem is equivalent to vertex cover, and vertex cover is NP-complete for triangle-free graphs [Pol74]. Remember that in this case the problem is in NP for the property of $HCH$ graphs above mentioned. This problem remains NP-complete for split graphs [GP00].

However, the clique-independence number can be obtained in polynomial time for triangle-free graphs, because it is equivalent in this case to maximum matching. This problem is NP-complete for split graphs [GP00] but, to our knowledge, it was not known its complexity for clique-Helly graphs.

Again, the following question appears naturally: what happens with the complexity of the clique-independence problem for hereditary clique-Helly graphs?

In this note, we prove that clique-cover and clique-independence problems remain NP-complete for $HCH$ graphs. Additionally, it is proved that clique-transversal and clique-independence problems remain NP-complete for a smaller class: the intersection between $HCH$ and split graphs.

2. Preliminaries

There are some relations between the parameters defined in the introduction in a graph $G$ and its clique graph $K(G)$.

**Theorem 2.1** Let $G$ be a graph. Then:

1. $\alpha_c(G) = \alpha(K(G))$.
2. If $G$ is a clique-Helly graph then $\tau_c(G) = k(K(G))$. 

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Proof: (i) It follows from the fact that independent cliques of \( G \) correspond to non adjacent vertices in \( K(G) \), and conversely, non adjacent vertices in \( K(G) \) correspond to independent cliques in \( G \).

(ii) Let \( v_1, \ldots, v_{|\tau_c(G)|} \) be a clique-transversal set of \( G \). For each \( i \), analyze the vertices in \( K(G) \) corresponding to the cliques in \( G \) that contain the vertex \( v_i \). They form a complete of \( K(G) \). This complete must be included in some clique \( L_i \) of \( K(G) \). Observe that these cliques \( L_i (i = 1, \ldots, |\tau_c(G)|) \) are not all necessarily different. Let us see that these at most \( \tau_c(G) \) cliques are a clique cover of \( K(G) \). Let \( w \) be a vertex of \( K(G) \). Then \( w \) corresponds to some clique \( M_w \) of \( G \). As the set \( v_1, \ldots, v_{|\tau_c(G)|} \) intersects all the cliques of \( G \), there is some vertex \( v_j \) that belongs to \( M_w \). This means that the corresponding vertex of \( M_w \) in \( K(G) \) belongs to the clique \( L_j \), i.e., \( w \in L_j \). Then, the size of the minimum clique cover of \( K(G) \) is at most the size of this clique cover which is at most \( \tau_c(G) \).

All we need to prove is that if \( G \) is clique-Helly, then \( \tau_c(G) \leq k(K(G)) \). By the Helly property, each clique \( L \) of \( K(G) \) has an associated vertex \( v_L \) in \( G \) such that \( v_L \) belongs to all the cliques of \( G \) corresponding to the vertices of \( L \) in \( K(G) \).

Let \( L_1, \ldots, L_{|K(G)|} \) be a clique cover of \( K(G) \). Let \( v_{L_1}, \ldots, v_{L_{|K(G)|}} \) be the vertices in \( G \) associated to those \( k(K(G)) \) cliques. Let us see that they form a clique-transversal set of \( G \). Let \( M \) be a clique of \( G \) and \( w_M \) its corresponding vertex in \( K(G) \). Then there is an index \( j \) such that \( w_M \) belongs to the clique \( L_j \) in \( K(G) \). It follows that \( v_{L_j} \) belongs to \( M \) in \( G \). □

Let \( M_1, \ldots, M_k \) and \( v_1, \ldots, v_n \) be the cliques and vertices of a graph \( G \), respectively. A clique matrix \( A_G \in \mathbb{R}^{k \times n} \) of \( G \) is a 0-1 matrix whose entry \( a_{ij} \) is 1 if \( v_j \in M_i \), and 0, otherwise.

Another characterization of \( HCH \) graphs is the following [Pr93]: a graph \( G \) is \( HCH \) if and only if \( A_G \) does not contain a vertex-edge incidence matrix of a triangle as a submatrix.

Let \( M_1, \ldots, M_k \) and \( v_1, \ldots, v_n \) be the cliques and vertices of a graph \( G \), respectively. Define the graph \( H(G) \) where \( V(H(G)) = \{q_1, \ldots, q_k, w_1, \ldots, w_n\} \), each \( q_i \) corresponds to the clique \( M_i \) of \( G \), and each \( w_j \) corresponds to the vertex \( v_j \) of \( G \). The edges of \( H(G) \) are the following: the vertices \( q_1, \ldots, q_k \) induce the graph \( K(G) \), the vertices \( w_1, \ldots, w_n \) are a stable set and \( w_j \) is adjacent to \( q_i \) if and only if \( v_j \) belongs to the clique \( M_i \) in \( G \).

Let \( A \in \mathbb{R}^{m \times m} \) and \( B \in \mathbb{R}^{n \times k} \) be two matrices. We define the matrix \( A|B \in \mathbb{R}^{n \times (m+k)} \) as \((A|B)(i, j) = A(i, j) \) for \( i = 1, \ldots, n, \ j = 1, \ldots, m \) and \((A|B)(i, m + j) = B(i, j) \) for \( i = 1, \ldots, n, \ j = 1, \ldots, k \). Let \( I_n \) be the \( n \times n \) identity matrix.

Theorem 2.2 [Ham68] Let \( G \) be a clique-Helly graph and \( H(G) \) as it is defined above. Then the cliques of \( H(G) \) are \( N[w_i] \) for each \( i \), \( w_i \) is a simplicial vertex of \( H(G) \) for every \( i \), and \( K(H(G)) = G \).
Corollary 2.1  Let $G$ be a clique-Helly graph, $|V(G)| = n$. Then $A_{H(G)} = A_{H(G)}^I | I_n$.

Proof: It follows directly from the fact that $N[w_i] \ (i=1,\ldots,n)$ are the cliques of $H(G)$ and each clique contains the vertex $w_i$ and the vertices $q_j$ whose corresponding cliques $M_j$ contain the vertex $v_j$ in $G$. □

This corollary leads us to prove the following result:

Theorem 2.3  Let $G$ be an HCH graph. Then $H(G)$ is HCH.

Proof: Let $G$ be an HCH graph, $|V(G)| = n$. By Corollary 2.1, $A_{H(G)} = A_{H(G)}^I | I_n$. Let us suppose that $A_{H(G)}$ contains a vertex-edge incidence matrix of a triangle as a submatrix. Since it has two 1’s in each column, it must be a submatrix of $A_{H(G)}^I$, but then $A_{H(G)}$ contains a vertex-edge incidence matrix of a triangle as a submatrix, which is a contradiction. □

3. Clique Cover

The decision problem associated to the problem of finding the clique-covering number of a graph is the following:

**CLIQUE COVER**

INSTANCE: A graph $G = (V,E)$, a positive integer $K \leq |V|$.

QUESTION: Are there $k \leq K$ cliques of $G$ covering all the vertices of $G$?

To prove that CLIQUE COVER is NP-complete for HCH graphs, we will use that the following problem is NP-complete [GJ79]:

**EXACT COVER BY 3-SETS (X3C)**

INSTANCE: A set $X$ such that $|X|=3q$ and a collection $C$ of 3-element subsets of $X$.

QUESTION: Does $C$ contain an exact cover (by $q$ sets) of $X$?

Theorem 3.1  The problem CLIQUE COVER is NP-complete for HCH graphs.

Proof: The transformation from X3C to CLIQUE COVER on HCH graphs is based on the transformation given in [GJ79] from X3C to PARTITION INTO TRIANGLES and is the following: let the set $X$ with $|X|=3q$ and the collection $C$ of 3-element subsets of $X$ be an arbitrary instance of X3C. We will construct an HCH graph $G=(V,E)$, with $|V|=3q'$, such that $G$ has a clique cover of size at most $q'$ if and only if $C$ contains an exact cover of $X$.

We will replace each subset $c_i = \{x_i, y_i, z_i\}$ in $C$ by the graph of Figure 2. Let $E_i$ be the set of 18 edges of the graph corresponding to $\{x_i, y_i, z_i\}$.
Thus $G=(V,E)$ is defined by

$$V = X \cup \bigcup_{i=1}^{q'} \{a_i[j] : 1 \leq j \leq 9\}, \quad E = \bigcup_{i=1}^{q'} E_i$$

It is easy to see that $G$ does not contain any graph of Figure 1 as an induced subgraph, thus $G$ is an $HCH$ graph, $|V| = |X| + 9|C| (q' = q + 3|C|)$ and the transformation can be made in polynomial time. Figure 3 shows an example of this transformation from an instance of X3C to an instance of CLIQUE COVER.
Let us suppose that $C$ contains an exact cover of $X$, then we construct a clique cover of $G$ of size $q'$, by taking for each $1 \leq i \leq |C|$

$$\{a_i[1], a_i[2], x_i \}, \{a_i[4], a_i[5], y_i \}, \{a_i[7], a_i[8], z_i \}, \{a_i[3], a_i[6], a_i[9] \},$$

whenever $c_i = \{x_i, y_i, z_i \}$ is in the exact cover and

$$\{a_i[1], a_i[2], a_i[3] \}, \{a_i[4], a_i[5], a_i[6] \}, \{a_i[7], a_i[8], a_i[9] \}.$$

otherwise.

Let us now suppose that $G$ has a clique cover of size at most $q'$. Since the cliques of $G$ are triangles, the number of cliques in the clique cover must be $q'$ and each vertex of $G$ must be covered exactly once.

In the graph of Figure 2, the only two ways of covering by triangles each vertex $a_i[j]$ $(j=1,\ldots,9)$ exactly once are the above mentioned, covering or not $x_i$, $y_i$ and $z_i$, respectively. Then the exact cover of $X$ is given by choosing those $c_i \in C$ such that $\{a_i[3], a_i[6], a_i[9] \}$ belongs to the clique cover of $G$.

Finally, the membership in NP for the restricted problem follows from that for the general problem. □

4. Clique Transversal and Clique-Independent Set

The decision problems associated to the problems of finding the clique-independence number and the clique-transversal number of a graph, respectively, are the following:

**CLIQUE-INDEPENDENT SET**

INSTANCE: A graph $G = (V,E)$, a positive integer $K \leq |V|$.

QUESTION: Is there a set of $K$ or more pairwise disjoint cliques of $G$?

**CLIQUE-TRANSVERSAL**

INSTANCE: $G = (V,E)$, a positive integer $K \leq |V|$.

QUESTION: Is there a set of $K$ or fewer vertices of $G$ intersecting all the cliques of $G$?

**Theorem 4.1** The problems CLIQUE-TRANSVERSAL and CLIQUE-INDEPENDENT SET are NP-complete for HCH split graphs.

*Proof:* We will show a polynomial time transformation from CLIQUE COVER on HCH graphs (by Theorem 3.1 it is NP-complete) to CLIQUE-TRANSVERSAL on HCH split graphs.

Define the graph $G^+$ where $V(G^+) = V(G) \cup \{u\}$, $V(G)$ induces the graph $G$ and $u$ is a universal vertex. Since for any graph $G$ all the cliques of $G^+$ share the vertex $u$, the graph $K'(G^+)$ is complete and thus the graph $H(G^+)$ is a split graph.
Let $G$ be an HCH graph. As the set of cliques of an HCH graph has polynomial size and can be computed in polynomial time, $H(G^+)$ can be built in polynomial time. By Theorem 2.3, since $G^+$ is an HCH graph, $H(G^+)$ is an HCH graph. By Theorem 2.2 $K(H(G^+)) = G^+$, and by Theorem 2.1 $k(G) = k(G^+) = \tau_c(H(G^+))$. Finally, the problem of determining if a subset of vertices is a clique-transversal is solvable in polynomial time for HCH graphs, and therefore CLIQUE-TRANSVERSAL is NP-complete for HCH split graphs.

In a similar way, using the equality $\alpha(G) = \alpha(G^+) = \alpha_c(H(G^+))$ instead of $k(G) = k(G^+) = \tau_c(H(G^+))$, and the NP-completeness of the STABLE SET problem for HCH graphs, CLIQUE-INDEPENDENT SET is NP-complete for HCH split graphs. □

**Corollary 4.1** The problem CLIQUE-INDEPENDENT SET is NP-complete for HCH graphs.

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**References**


