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Deterministic Chaos Theory: Basic Concepts

Teoria do Caos Determinístico:Conceitos Básicos

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This article was written to students of mathematics, physics and engineering. In general, the word chaos may refer to any state of confusion or disorder and it may also refer to mythology or philosophy. In science and mathematics it is understood as irregular behavior sensitive to initial conditions. In this article we analyze the deterministic chaos theory, a branch of mathematics and physics that deals with dynamical systems (nonlinear differential equations or mappings) with very peculiar properties. Fundamental concepts of the deterministic chaos theory are briefly analyzed and some illustrative examples of conservative and dissipative chaotic motions are introduced. Complementarily, we studied in details the chaotic motion of some dynamical systems described by differential equations and mappings. Relations between chaotic, stochastic and turbulent phenomena are also commented.
Keywords: chaos theory, differential equations; Poincaré sections; mapping; Lyapunov exponent.

Este artigo foi escrito para estudantes de matemática, física e engenharia. Em geral, a palavra caos pode se referir a qualquer estado de confusão ou a desordem, mas também se referir a mitologia ou filosofia. Em ciência e matemática é entendido como um comportamento irregular sensível às condições iniciais. Neste artigo vamos analisar a teoria do caos determinístico, um ramo da matemática e da física que lida com sistemas dinâmicos (equações diferenciais não-lineares ou mapeamentos), com propriedades muito peculiares. Conceitos fundamentais da teoria do caos determinístico são brevemente analisados e alguns exemplos ilustrativos de movimentos caóticos conservativos e dissipativos são introduzidos. Complementarmente, estudamos em detalhes o movimento caótico de alguns sistemas dinâmicos descritos por equações diferenciais e mapeamentos. As relações entre fenômenos caóticos, estocásticos e turbulentos também são comentados.
Palavras-chave: teoria do caos, equações diferencias, seções de Poincaré; mapeamento; expoente de Lyapunov.

1. Introduction

This paper was written for students of mathematics, physics and engineering. Are briefly analyzed essential aspects of the growing field of mathematics and physics that has been applied to study a large number of phenomena generically named chaotic. These are present in many areas in science and engineering [1–4], including astronomy, plasma physics, statistical physics, hydrodynamics and biology. As in Greek the word chaos ($\chi\alpha o\varsigma$) means confusion, random, stochastic, and turbulent processes may be misleading associated with chaos. However, rigorously they are different in the framework of physics and mathematics, as will be shown. This article analyzes only the basic points of chaos theory, as exactly as possible from the mathematical point of view, avoiding sometimes a rigorous approach. In Section 2 we define chaos, in the context of the deterministic chaos theory, as a consequence of peculiar properties of deterministic nonlinear ordinary differential equations (NLODE) [5]. These equations that describe dynamic systems have a time evolution strongly dependent on initial conditions. Chaotic motion occurs depending of initial conditions and parameters values of the nonlinear equations.

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In Section 3 is seen the difference between chaotic and stochastic (or random) processes. In Section 4, to give a general idea about the chaos, we study in details the Duffing equation and the dissipative motion of a driven damped pendulum, introducing the Poincaré technique. In Section 5 we show that it is possible to get a good description of chaotic processes using an iterative algebraic model named mapping. As examples, we introduce the logistic and the Hénon maps. In Section 6 is presented the Lyapunov characteristic exponent, used to quantify the sensitive dependence on initial conditions for chaotic behavior. Finally, in Section 7 is briefly discussed an open problem: the relation between turbulence and chaos.

2. Definition of Deterministic Systems and Chaos

Usually in physics basics courses [1,3,4,6,7] we learn that all physical laws are described by differential equations. So, integrating, that is, solving analytically or numerically these equations, knowing the initial and boundary conditions (see Section 4), we would know the future of a physical system for all times. This is the deterministic view of nature. In other words, physics systems are deterministic because they obey deterministic differential equations. They can be conservative or dissipative. Remark that the deterministic development refers to the way as a system develops from one moment to the next, where the present system depends on the one just past in a well-determined way through physical laws [1,3,4,6,7]. If the initial states of deterministic systems were exactly known, future states would be precisely theoretically predicted.

This deterministic view survived to be questioned after the famous visionary works of Henri Poincaré on celestial mechanics [8] performed at the end of the 19^{th}. These works begin in 1880 when he found non-periodic orbits in the three-body problem.

According to Poincaré [8,9]: "If we knew exactly the law of nature and the situation of the universe at the initial moment, we could predict exactly the situation of that same universe at a succeeding moment. But even if it were the case that the natural laws had no longer any secret for us, we could still only know the initial situation approximately. If it enabled us to predict the succeeding situation with the same approximation, that is all we require, and we should say that the phenomenon had been predicted, that it is governed by laws. But it is not always so: it may happen that small differences in the initial conditions produce very great ones in the final phenomena. A small error in the former will produce an enormous error in the latter. Prediction becomes impossible, and we have the fortuitous phenomenon".

In practice, as observed for many systems, knowledge about the future state is limited by the precision with which the initial state can be measured. That is, knowing the laws of nature is not enough to predict the future. There are deterministic systems whose time evolution has a very strong dependence on initial conditions. That is, the differential equations that govern the evolution of the system are very sensitive to initial conditions. Usually we say that even a tiny effect, such as a butterfly flying nearby, may be enough to vary the conditions such that the future is entirely different than what it might have been, not just a tiny bit different [1–3, 10]. In this way, measurements made on the state of a system at a given time may not allow us to predict the future situation even moderately far ahead, despite the fact that the governing equations are exactly known. By definition, these equations are named chaotic and that they predict a deterministic chaos.

Only in recent years, with advent of computers that was allowed chaos to be studied because now it is possible to perform numerical calculations of the time evolution of the properties of systems sensitive to initial conditions. We begin to understand the existence of chaos when computers were readily available to calculate the long-time histories required to explain the discussed behavior. It did not happen until the 1970s. After almost one century of investigations we learned that chaotic systems can only be solved numerically, and there are no simple, general ways to predict when a system will exhibit chaos [1–3, 10]. We have also learned that deterministic chaos is always associated with nonlinear systems; nonlinearity is a necessary condition for chaos but not a sufficient one.

3. Random or Stochastic Process

According to Section 2 the deterministic model will always produce the same output from a given starting condition or initial state. On the other hand, a random process, sometimes called stochastic process, is a collection of random variables, representing the evolution of some system of random values over time [11]. Instead of describing a process which can only evolve in one way (as, for example, the solutions of an ordinary differential equation), in a stochastic process there is some indeterminacy: even if the initial condition is known, there are several (often infinitely many) directions in which the process may evolve. There is a probabilistic evolution of the initial states.

As an example, let us consider the Langevin [11, 12] stochastic process. He proposed in 1908 the following stochastic differential equation to describe the Brownian (random) motion of a particle immersed in a fluid [11, 12]:

$$m\frac{d^2x}{dt^2} = -\lambda\frac{dx}{dt} + \eta(t). \tag{1}$$

The degree of freedom of interest here is the position x of the particle, m denotes the particle's mass. The force acting on the particle is written as a sum of a viscous force proportional to the particle's velocity (Stokes' law), and a noise term η(t) (the name given in physical contexts to terms in stochastic differential equations which are stochastic processes) representing the effect of the collisions with the molecules of the fluid. The force η(t) has a Gaussian probability distribution with correlation function

$$< \eta_i(t)\eta_j(t') >= 2\lambda\kappa_B T\delta_{i,j}\delta(t-t'), \tag{2}$$

where κ_B is Boltzmann's constant and T is the temperature. The δ-function form of the correlations in time means that the force at a time t is assumed to be completely uncorrelated with it at any other time. This is an approximation; the actual random force has a nonzero correlation time corresponding to the collision time of the molecules. However, Langevin's equation is used to describe the motion of a macroscopic particle at a much longer time scale, and in this limit the δ-correlation and the Langevin equation become exact.

It can be difficult to tell from data whether a physical or other observed process is random or chaotic [11, 13]. In reference [3] one can see some procedures proposed to distinguish between deterministic chaos and stochastic behavior.

Finally, in quantum mechanics, the Schrödinger equation, which describes the continuous time evolution of a system's wave function, is deterministic [14], besides the well known relationship between the wave function and the observable properties of the system.

4. Deterministic Chaos

According to Section 2, after 130 years of investigations, it is known that chaotic phenomenon may be observed when dynamic systems obey nonlinear ordinary differential equations (NLODE)[1] or par-

[1] **Ordinary Differential Equations**: In mathematics, an ordinary differential equation (ODE) is an equation containing a function of one independent variable and its derivatives [15, 16]. The term ordinary is used in contrast with the term partial differential equation (PDE) which may be with respect to more than one independent variable. Let x be an independent variable and $y = y(x)$ a linear and continuous function of x. Indicating by $y(n) = d^n y/dx^n$ the derivative of order n of the function $y(x)$ an implicit ODE of order n can be generally written as

$$F(x, y, y', ..., y(n)) = 0, \tag{3}$$

where F is a continuous linear function of x and of the continuous $y(x)$ and of their derivatives $y^n(x)$. In this case the equation is defined as linear differential equation or simply ODE. When nonlinear terms are present, F is an ordinary nonlinear differential equation (NLODE).

Existence and Uniqueness of Solutions of ODE: It can be shown [15–18] that there is one and only one solution of (3) in an interval $(x_o - \Delta, x_o + \Delta)$, with $\Delta > 0$, given by a continuous function (or trajectory)

$$y = y(x, c_o, c_1, c_2, ..., c_n), \tag{4}$$

where $c_o = y(x_o)$ and $c_n = y^{(n)}(x_o)$ (n=1,. . . , n) are arbitrary constants (initial conditions). Note that general solutions of ODEs involve the knowledge of arbitrary constants. The solution (4) can be obtained analytically or by graphical and numerical methods. The existence and uniqueness of the ODE solutions are established by several theorems [15, 17, 19]. Now let us assume that at x_o there are two different initial conditions: one given $y(x_o, c_o, c_1, c_2, ..., c_n)$ and another $y(x_o, C_o, C_1, C_2, ..., C_n)$ when $C_n = c_n + \delta_n$, with $\delta_n << c_n$. At a point $x \neq x_o$ we have the difference Δy given by $\Delta y = y(x, c_o, c_1, c_2, ..., c_n) - y(x, C_o, C_1, C_2, ..., C_n)$. Since y is as a continuous function of the variables x, c_n and C_n, Δy can be expanded in a series in a first order approximation of the increments δ_n. In this way, for arbitrarily small increments δ_n the difference Δy becomes also arbitrarily small. Conclusion: "for arbitrarily small variations δ_n of the initial conditions the trajectories are practically the same". Consequently, chaotic systems cannot be governed by ODE. In absence of analytic solutions, graphical and numerical methods, applied by hand or by computer, may give approximate solutions of ODE and perhaps yield useful information.

Existence and Uniqueness of Solutions of NLODE: There are a few methods of solving NLODE analytically; those that are known typically depend on equation having particular symmetries. There are no general techniques that work for all such equations, and usually each individual equation has to be studied as a separate problem. In absence of analytic solutions, graphical and numerical methods applied tial differential equations [PDE][2] [22, 23]. In this article to avoid complex mathematical analysis we only consider chaos generated by NLODE.

In this way, let us recall the definitions of NLODE. An ordinary differential equation is an equation containing a function of one independent variable and its derivatives [15, 16, 19]. The term ordinary is used in contrast with the term PDE which may be with respect to more than one independent variable. Let x be an independent variable, $y = (x)$ a function of x, and $y^{(n)} = d^n y/dx^n$ the derivative of order n of the function $y(x)$. An ODE of order n can be generally written as $F(x, y, y', ..., y(n)) = 0$. If x, $y(x)$ and $y(n)$ are linear functions and F is a linear function of these functions we say that F is an ODE without any chaotic solutions (see Footnote 1). When nonlinear terms are present, F is a NLODE. In the N-dimensional case it is assumed that the time evolution of the dynamic of a system is described by continuous and continuous flux created by ordinary nonlinear differential equation

$$\frac{dx}{dt} = f_\alpha[x(t)], \tag{6}$$

with, $x(0) = x_o, x, f_\alpha$ (flow equation) are N-vectors $\epsilon\ R_m$, m is the number of degrees of freedom of the system, f_α is explicitly independent of time and α is a control parameter. Usually it is assumed that any NLODE can be integrated in the sense that they are resolved analytically or numerically and that the solutions obtained are unique. Note that rigorously in Mathematics, differential equations can be integrated [28, 29] when are manifested the following features: (a) existence of enough number of conserved quantities; (b) existence of an algebraic geometry and (c) ability to give explicit solutions.

by hand or by computer, may give approximate solutions of ODE. One extremely popular is the Runge-Kutta method [20]. NLODE can exhibit very complicated behavior over extended time intervals, characteristic of chaos. The questions of existence and uniqueness of solutions of NLODE and PDE are hard problems and their resolution are of fundamental importance to the mathematical theory [20]. However, if the differential equation is a correctly formulated representation of a meaningful physical process, then one expects it to have a unique solution [18]. Linear differential equations frequently appear as approximations to nonlinear equations. These approximations are only valid under restricted conditions. For example, the harmonic oscillator equation is an approximation to the nonlinear pendulum equation that is valid for small amplitude oscillations.

[2]**Partial Differential Equations and Chaos**: The formulation of a physical problems in mathematical terms often results in a partial differential equation (PDE) that contains unknown multivariable functions $u(x_1, x_2, ..., x_n)$ and their partial derivatives [22, 23] $\partial u/\partial x_1, ..., \partial u/\partial x_n, \partial^2 u/\partial x_1 \partial x_1, ..., \partial^2 u/\partial x_1 \partial x_n$ and so on. A PDE for the function $u(x_1, x_2, ..., x_n)$ can be written in an implicit form:

$$F\left(x_1, \ldots, x_n, u, \frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n}, \frac{\partial^2 u}{\partial x_1 \partial x_1}, \ldots, \frac{\partial^2 u}{\partial x_1 \partial x_n}\right) = 0, \tag{5}$$

which must generally satisfy additional conditions, which are dependent on the nature of the problem. This is the so called boundary value problem. F can be a linear (LPDE) or nonlinear (NLPDE) function of u and its derivatives [22, 23]. Common examples of PDE include sound and heat equations, fluid flow or Navier-Stokes equation, electrostatics, wave equation, electrodynamics, Laplace's equation, quantum mechanics, Klein-Gordon and Poisson's equations and gravitation. PDE as ODE often model multidimensional systems.

Existence and Uniqueness of Solutions: Although the issue of existence and uniqueness of solutions of ODE which has a very satisfactory answer, as seen in Section 2, that is not the case for PDE. General solutions of ODE involve arbitrary constants. Solutions of PDE are much more complicate because they involve arbitrary functions. A solution of a PDE is generally not unique: it depends on additional conditions that must be specified on the boundary of region where the solution is defined. The Cauchy-Kowalevski theorem states that the Cauchy problem for any LPDE whose coefficients are analytic in the unknown function and its derivatives, has a locally unique analytic solution. Although this result might appear to settle the existence and uniqueness of solutions, there are examples of LPDE which have no solutions at all.The NLPDE are more difficult to integrate analytically [22, 24]: there are almost no general techniques that work for all such equations, and usually each individual equation has to be studied as a separate problem. A fundamental problem for any PDE is the existence and uniqueness of a solution for given boundary conditions. For LPDE these questions are in general very hard. It is often possible to obtain analytic solutions as occurs, for instance, with solitons in hydrodynamics, electromagnetic waves and non-linear quantum mechanics. Numerical solution on a computer is almost the only method that can be used for getting information about arbitrary PDE. A list of NLPDE is given in reference [23]. As said in (see Footnote 1), if the differential equation is a correctly formulated representation of a meaningful physical process and if a solution can be found consistently with all the given boundary conditions, it is accepted without proof that this solution is unique [18].

Simplest Chaotic Partial Differential Equation: As commented before in spite of extensive investigations it was not possible to prove, in the general case, the existence of chaos in infinite-dimensional systems [11, 25–27]. However, it was shown that very simple NLPDE permit chaos [23]. These equations have the form $\partial u(x,t)/\partial t = F(u(x,t))$, where $F(u(x,t))$ can consist of derivatives in space but not in time, can contain a constant term, and must contain exactly one quadratic nonlinearity (e.g., u^2 or $u.\partial^n u(x,t)/\partial x^n, etc...$). For instance, $\partial u/\partial t = -u.(\partial u/\partial x) - A(\partial^2 u/\partial x^2) - (\partial^4 u/\partial x^4)$ [19–21].

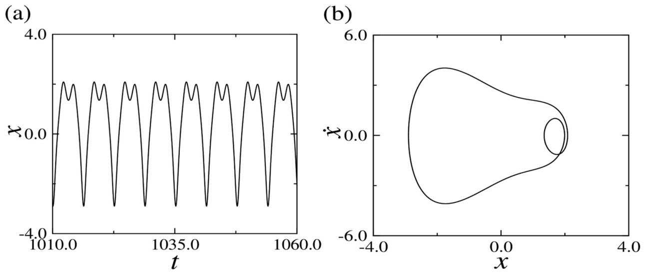

Figure 1: (a) Periodic time evolution of x, from a numerical solution of the Duffing equation for parameters β=0.05, γ=1.0, ω=1.0, F=4.0. (b). Orbit in phase space for the same solution.

To give a general idea about the chaos theory we study in details two examples of dissipative chaotic systems. Thus, in Section (5.1) a dissipative non linear oscillator and in (4.2) the dissipative motion of a damped and driven pendulum.

There are, however other illustrative examples of conservative chaotic systems. We suggest the lecture of two conservative processes [30], described by the Hamiltonian formalism, with chaotic solutions. One is the motion of a particle of mass m in a double quartic non-harmonic potential (Duffing potential) governed by the Duffing Hamiltonian:

$$H(p, x, t) = \frac{p^2}{2m} - kx^2 + x^4 + Fcos(\omega t), \qquad (7)$$

where the oscillating term $Fcos(\omega t)$ is a perturbative potential. A didactic approach of this case was done, for instance, in [30]. The second case is the conservative motion of a double pendulum seen, for instance, in reference [31] where are found animation pictures of the chaotic motion.

Another classical example is the chaos in the solar system (see, for instance, reference [3]).

4.1. Duffing Equation

A dissipative illustrative case is the motion of a particle with mass m submitted to a Duffing potential and to a dissipative force $\beta(dx/dt)$. That is, the motion is governed by the NLODE (Duffing equation) [2–4]:

$$\ddot{x} + \beta\dot{x} - x + \gamma x^3 = Fcos(\omega t), \qquad (8)$$

The NLODE (8) can only be solved for x using numerical methods, given the parameters β, k, and ω.

The motion in the phase space associated with Eq.(8) can be efficiently studied using the technique

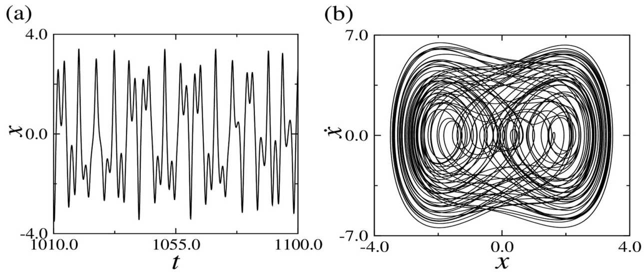

Figure 2: (a) Chaotic time evolution of x, from a numerical solution of the Duffing equation for parameters β=0.05, γ=1.0, ω=1.0, F=6.0. (b). Orbit in phase space for the same solution.

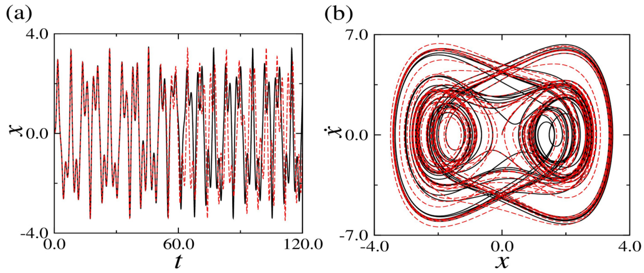

Figure 3: Sensitivity dependence on initial conditions. Chaotic time evolution of two solutions for the same parameters of Fig. 2, for initial conditions (x_0, x'_0, t_0) = (0.0, 0.0, 0.0) for the black solid line and (x_0, x'_0, t_0) = (0.001, 0.0, 0.0) for the red dashed line.

invented by Poincaré, named Poincaré sections, illustrated in Fig. 4 and 5. First is constructed a 3-dim phase space with orthogonal axis (x, y, z), where $y = dx/dt'$ and $z = \omega t'$ and second, are taken parallel planes (y, x) orthogonal to the axis z distant one of the other by a given interval Δz (see Fig.4(a)). These planes, or Poincaré sections, are used to drawn a stroboscopic map of the flow. This name is given because such map consists in observe the system in discrete times $t_k = n/\omega$ ($n = 1, 2, \ldots, n$). Taking for $t = 0$ the initial values $x(0) = x_o$ and $y(0) = y_o$ we integrate numerically Eq.(8) up to the instant t_1 determining the point A_1 =[$x(t_1)$, $y(t_1)$] of the path. These values are now taken as new initial values to calculate the next point A_2 =[$x(t_2)$, $y(t_2)$] for t_2 and so on. Note that the calculated path is a continuous curve. The calculated path in the phase space (x, y, z) pierces the planes (stroboscopic sections) as a function of speed ($y = dx/dt$), time ($z = n/\omega$) and the coordinate x , according to Fig. 4 (a). The points on the intersections are labelled as A_1, A_2 and A_3, etc. This set of points A_i forms a pattern (stroboscopic map) when projected on the plane (y, x) (see Figs. 4(b) and (5). Poincaré realized that

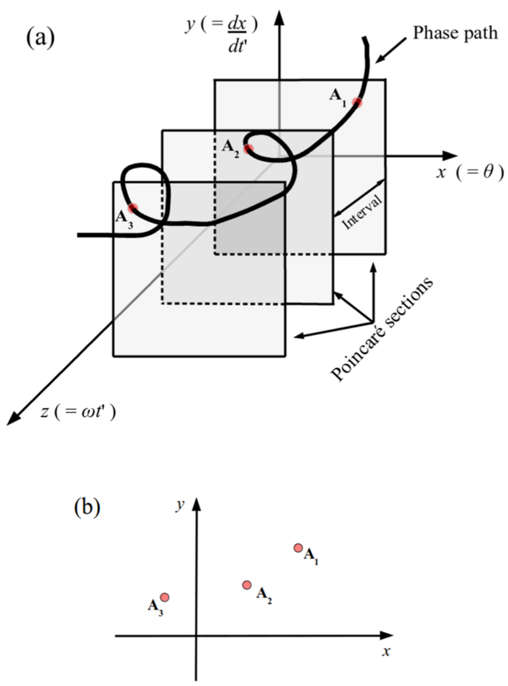

Figure 4: Technique invented by Poincaré to represent the phase space diagrams. The parallel planes are stroboscopic sections of the motion. The path pierces these planes at the points A_1, A_2, A_3. (b) Points A_1, A_2, A_3, projected on the plane (x, y).

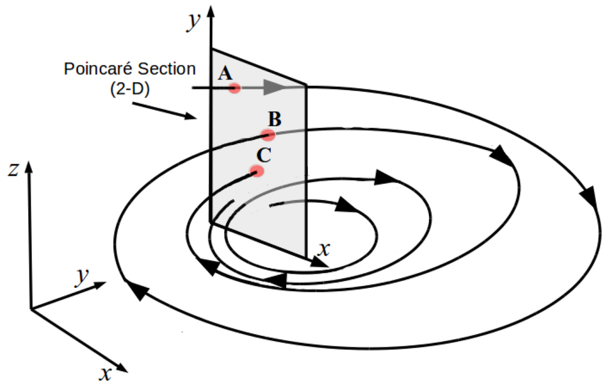

Figure 5: Illustration of the stroboscopic technique where are shown the intersections of the path with the Poincaré section.

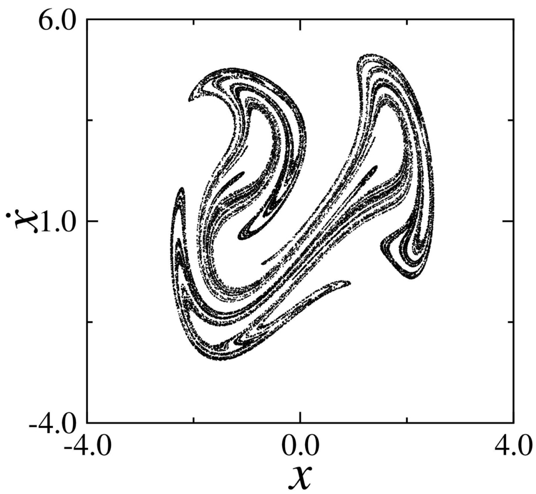

Figure 6: Chaotic attractor of Duffing equation obtained for parameters β=0.05, γ=1.0, ω=1.0, F=6.0.

the simple curves represent motion with possibly analytic solutions, but the many complicated, apparently irregular, curves represent chaos.

Now let us analyze results seen in Fig.1 (a). The displays x versus time t when transient effects have died out. The value ε=4.0 shows a simple periodic motion (only one vibrational frequency), but the results for ε=6.0 is not periodic Fig.2 (a). In Figs.1 (b) and 2 (b), we observe the phase-space plot, namely, $y = dx/dt$ versus x [6]. These results indicate the beautiful and surprising behavior obtained from nonlinear dynamics: the motion is periodic for ε=4.0, but chaotic for ε=6.0.

In Fig.6 is displayed the stroboscopic map, for F=6.0. This Poincaré section represents a chaotic motion: the system never comes back to the same position (x, y) after z goes through multiples of $z = n/\omega$. The illustrated motion presents a complicated variation of points expected for the chaotic motion (with a period T $\to \infty$). In these cases we have aperiodic motions which is a characteristic of the deterministic chaos [32].

Finally, we remark that only for dissipative systems there are set of points (attractors) or a point on which the motion converges. In chaotic motion, nearby trajectories in phase space are continually diverging from one another following the attractor. This effect is shown in Fig. 3, for two motions obtained for the same parameters but with two different neighbor initial conditions. Due to these attractors, named strange or chaotic attractors, the motions in the phase space are necessarily bounded.

The attractors create intricate patterns, folding and stretching the trajectories must occur because no trajectory intersects in the phase space, which is ruled out by deterministic dynamical motion [6]. The figures reveal a complex folded, layered structure of the attractors. Amplifying figure we would note that the lines are really composed of a set of sub lines. Amplifying a sub line we would see another set of sub lines and so on . . . verifying that the strange attractors usually are fractals [3, 31, 33].

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Table 1: Damped and driven pendulum dimensionless variables and parameters.

Variables and parameters	
Damping coefficient	$\left(c = \dfrac{b}{ml^2\omega_o}\right)$
Driving force strength	$\left(F = \dfrac{N_d}{ml^2\omega_o} = \dfrac{N_d}{mgl}\right)$
Dimensionless time	$\left(t' = \dfrac{t}{t_o} = t\dfrac{\sqrt{g}}{l}\right)$
Driving angular frequency	$\left(\omega = \dfrac{\omega_d}{\omega_o} = \omega_d\dfrac{\sqrt{l}}{g}\right)$

4.2. Chaos in damped and driven pendulum

Another example of one-dimensional nonlinear motion is the one described by the damped and driven pendulum around its pivot point shown in Fig.7 [1].

The torque τ around the pivot point can be written as

$$\tau = I\frac{d^2\theta}{dt^2} = -b\frac{d\theta}{dt} - mglsin(\theta) + N_d cos(\omega_d t), \quad (9)$$

where I is the moment of inertia, b the damping coefficient and N_d is the driving force of angular frequency ω_d. Dividing Eq. 9 by $I = ml^2$ results the nonlinear equation

$$\frac{d^2\theta}{dt^2} = -\left(\frac{b}{ml^2}\right)\left(\frac{d\theta}{dt}\right) - \frac{g}{l}sin(\theta) + \frac{N_d}{ml^2}cos(\omega_d t). \quad (10)$$

If we want to deal with Eq.(10) with a computer it is more convenient to use dimensionless parameters. So, let us divide Eq.(10) by ω_o^2=g/l and define the dimensions less parameters: time $t' = t/t_o$ with $t_o = 1/\omega_o$ and driving frequency $\omega' = \omega_d/\omega_o$. The new dimensionless variables and parameters are presented in Table 1:

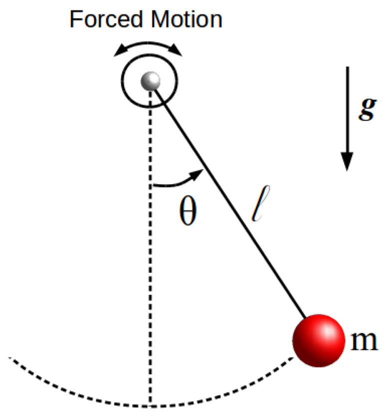

Figure 7: Damped and Driven pendulum with length l.

Using the variables and parameters defined in Table1, we verify that Eq.11 becomes,

$$\frac{d^2\theta}{dt'^2} = -c\left(\frac{d\theta}{dt'}\right) - sin(\theta) + Fcos(\omega t). \quad (11)$$

Defining $y = dx/dt$ and $z = \omega t$, the second-order non-linear differential equation (11) is substituted by a system of two first order-differential equations:

$$y = \frac{dx}{dt'} \quad (12)$$

$$\frac{dy}{dt'} = -cy - sin(x) + Fcos(z) \quad (13)$$

Integrating numerically Eq. (12) and (13), we find periodic and chaotic attractors which depend on the chosen parameters and initial conditions. As an example, in Fig. 8 we present the only three periodic oscillations (represented by blue, red, and green lines) that are obtained by a specific choice of parameters, for all possible initial conditions. These solutions correspond to three different periodic attractors.

Furthermore, to show the attractor dependence on initial conditions, we present in Fig.9 the parameter space obtained by a grid of initial conditions. For each initial condition we obtain the numerical solution and identify the corresponding atractor, associated with one of the three lines shown in Fig.8, and represent it in Fig.9 as a point with the same color used in Fig.8. Figure 9 (a) is denominated basin of attraction of teh solutions of Eq. (7) [32]. The successive amplifications of the basin of attraction, shown in Fig.9 (b), (c) and (d), indicate the basin of attraction fractality.

5. Mapping

In some cases it is very difficult to study the evolution of a nonlinear system integrating their differential equations. Sometimes it is also difficult to construct an exact nonlinear mathematical model to study physical system. In these cases it is possible to get a good description of the chaotic process using an iterative algebraic model named mapping. To understand the origin of this model let us assume that the motion of a system is described by nonlinear first-order differential equations of the form [8]

$$\frac{dx}{dt} = f(x), \quad (14)$$

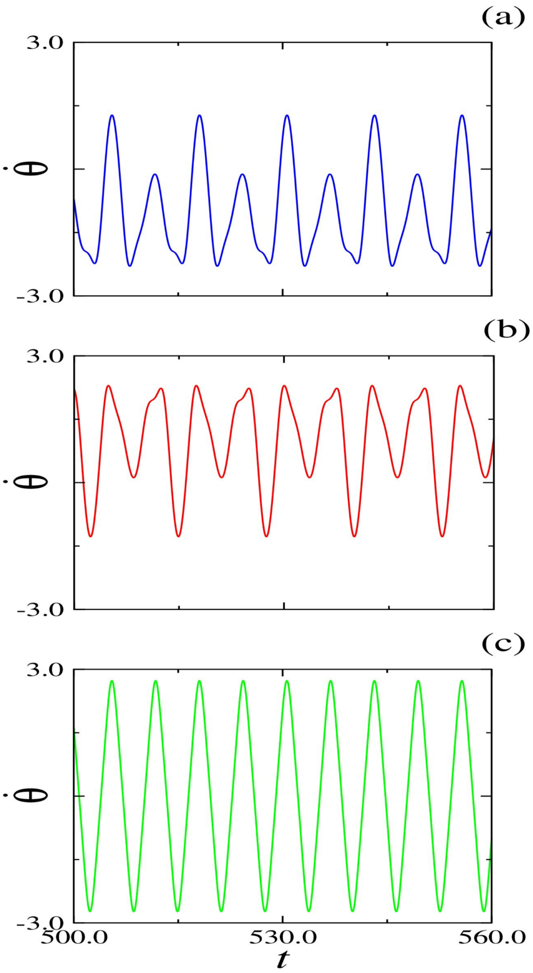

Figure 8: Three different numerical periodic solutions obtained for the damped driven pendulum with distinct initial conditions and the same parameters $c = 0.2$, $F = 1.67$, $\omega = 1.0$.

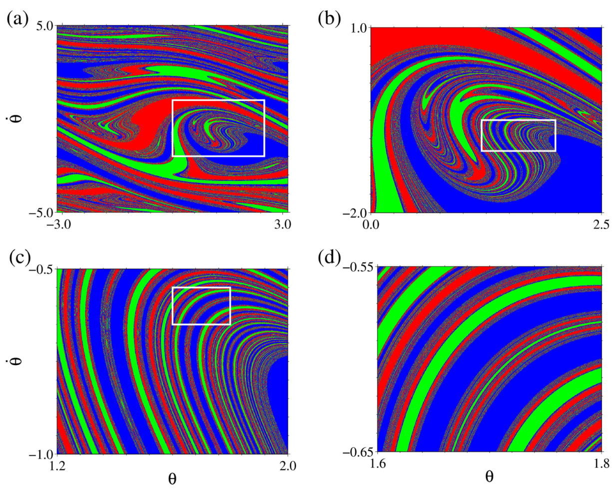

Figure 9: (a) Basin of attraction of the damped driven pendulum solutions for the same parameters of Fig. 8 (b), (c) and (d) Successive amplifications of the previous figure revealing the basin of attraction fractality.

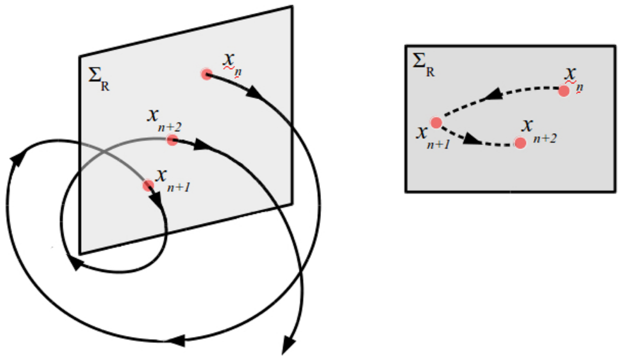

Figure 10: Trajectory of the motion piercing Poincaré section $\sum_R$. The right figure shows only the points x_n, x_{n+1} e x_{n+2} on $\sum_R$.

where x and $f(x)$ are explicitly independent of time and that the motion is represented in Poincaré section $\sum_R$ in Fig. 5.

The Poincaré map is found by choosing a point x_n on $\sum_R$ and integrating Eq. (14) to find the next intersection x_{n+1} of the orbit with $\sum_R$. In this way we construct the map $x_{n+1} = f(x_n)$.

In a few words, denoting by n the time sequence of a system and by x the physical observable of this system we can describe the progression of a nonlinear system at a particular moment by investigating how the $(n+1)^{th}$ state depends on the n^{th} state. The evolution $n \to n+1$ can be written as a difference equation using a function $f(\alpha, x_n)$ as follows

$$x_{n+1} = f_\alpha(x_n), \qquad (15)$$

where α is a model dependent control parameter, α and x are real numbers. The function $f_{alpha}(x_n)$ generates the value x_{n+1} from x_n and the collection of points generated is said to be a map of the function itself. The difference equation (14), which is an evolution equation in the Poincaré section is considered a milestone in the field of nonlinear phenomena. Note that n must be iterated from $n = 1$ up to $N >> 1$.

5.1. Logistic Equation and Logistic Map

There are innumerous chaotic systems studied with the mapping approach. Famous examples are the map models for ecological and economic interactions: symbiosis, predator prey and competition [34, 35]. Malthus, for instance, claimed that the human population p grows obeying the law [34]

$$\frac{dp}{dt} = kp. \qquad (16)$$

Verhulst [35] argued that the population grow has inhibitory term ap^2 so that Eq. (16) is actually given by a nonlinear equation, called logistic function

$$\frac{dp}{dt} = kp - ap^2, \tag{17}$$

which shows that the population tends asymptotically to the constant k/a.

One century later, indicating the population by x the differential equation (17) was substituted by the logistic equation [34, 35]

$$x_{n+1} = \alpha x_n(1 - x_n), \tag{18}$$

where $0 < \alpha < 4$ in order to assure that $0 < x_n < 1$.

Note that the Eq. (18) must be calculated (iterated) from $n = 1$ up to the cycle $n >> 1$. An n cycle is an orbit that returns to its original position after n iterations. In reference [1] are presented logistic maps of x_{n+1} as a function of x_n showing that x assume one stable value and only two discrete values for α values in the interval 2.8-3.1, characterizing a periodic motion.

A more general view of the evolution can be obtained plotting a bifurcation diagram [1, 34, 35] (see Fig. 11) where the x_n is calculated numerically after many 1nteractions to avoid initial effects is plotted as a function of the parameter α [1].

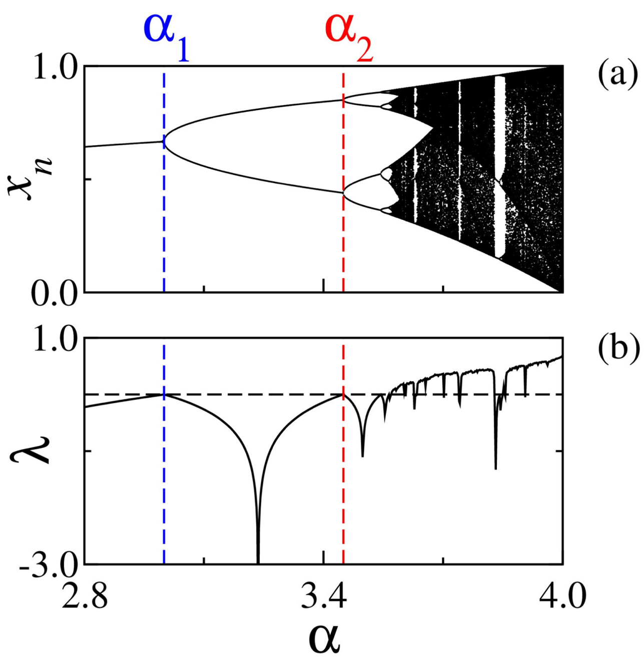

Figure 11: (a) Bifurcation diagram x_n as function of α for logistic equation map ($2.8 < \alpha < 4.0$). (b) Lyapunov exponentes λ as a function of α.

Analyzing this figure we verify that for $2.80 < \alpha < 3.00$ there is a stable population (the period is one cycle; $x_{n+1} = x_n$). At $\alpha = 3.1$ we see a bifurcation (because of obvious shape of the diagram) where there is a period doubling effect ($x_{n+2} = x_n$): x begins to oscillate periodically between 0.558 and 0.765. At $\alpha = 3.45$ there are two different points of bifurcation: now there appear four possible periodic oscillations. The bifurcation and period doubling continues up to an infinite number of cycles near 3.57. Chaos (black regions) occurs for many of α values between 3.57 and 4.0, but there are still windows of periodic motions (white region). Detailed description of these regions can be seen, for instance, in references [34, 36], where is also shown a cobweb diagram of the logistic map showing chaotic behavior for most values of $\alpha > 3.57$. The special case of $r = 4$ can in fact be solved exactly [10], as can the case with $\alpha = 2$; however the general case can only be calculated numerically. For $\alpha = 4$ is $x_n = sin^2(2^n\theta\phi)$ where the initial condition parameter θ is given by $\theta = (1/\phi)arcsin(x_o^{1/2})$. For rational θ after a finite number of iterations x_n maps into a periodic sequence. But almost all θ are irrational, and, for irrational θ, x_n never repeats itself-it is non-periodic. This solution equation clearly demonstrates the two key features of chaos stretching and folding: the factor 2^n shows the exponential growth of stretching, which results in sensitive dependence on initial conditions, while the squared sine function x_n keeps folded within the range $\{0, 1\}$.

5.2. Hénon Map

Another example is the bidimensional dissipative Hénon map given by the equations

$$x_{n+1} = a + by_n - x_n^2 \tag{19}$$

$$y_{n+1} = x_n, \tag{20}$$

where the parameters a and b are the control parameters [26].

Examples of numerical solutions of Eq. (19) and (20) are in Fig. 12, which shows a periodic and a chaotic attractors, obtained, respectively, for (a) $a = 1.45$, $b = 0.2$ and (b) $a = 0.2$, $b = 1.48$.

To show how the numerical solutions depend on the control parameters, we present in Fig. 13 (a) the bifurcation diagram of Eq. (19) solutions for a fixed a and $1.42 < b < 1.48$. An interval with a

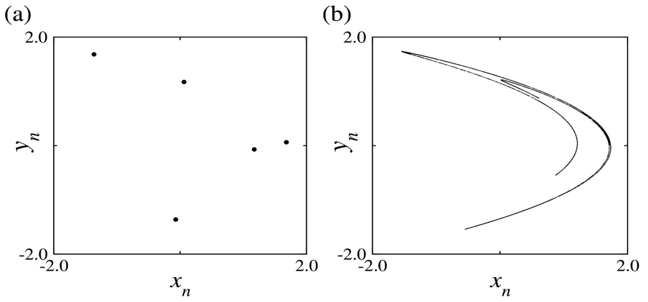

Figure 12: Examples of periodic and chaotic attractors of the Hénon map for the parameters the attractors depend on the control parameters.

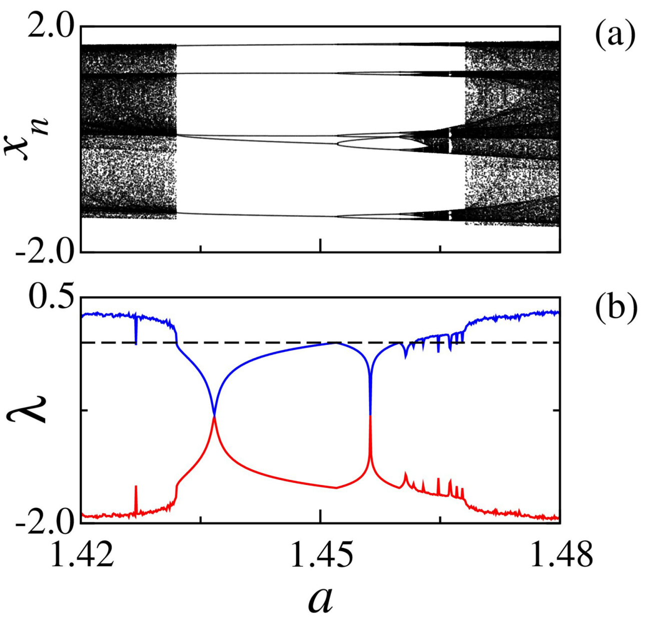

Figure 13: (a) Bifurcation diagram x_n as a function of a for the Hénon map. (b) Lyapunov exponents λ as a function of α and b=0.2.

period 5 attractor can be observed in Fig. 13. In the parameter space of Fig. 14 (a) we indicate the period 5 window in the parameter space. The amplification in Fig. 13 (b) shows better the same periodic window. Such windows are also called shrimps [37] and have been observed in several dynamical systems [38,39].

6. Lyapunov Exponents

The nonlinear terms of the differential equations amplify exponentially small differences in the initial conditions. In this way the deterministic evolution laws can create chaotic behavior, even in the absence of noise or external fluctuations. In the chaotic regime it is not possible to predict exactly the evolution of the system state during a time arbitrarily

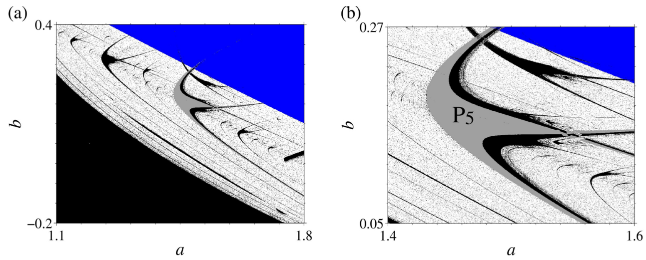

Figure 14: (a) Parameter space for the Hénon Map. Periodic windows are in black. White points represent parameters with chaotic attractor. In gray is a periodic-5 window. (b) Amplification of a.

long. This is the unpredictability characteristic of the chaos. The temporal evolution is governed by a continuous spectrum of frequencies responsible for an aperiodic behavior (see, for instance, 4). The motions present stationary patterns, that is, patterns that are repeated only non-periodically [2,3]

Lyapunov created a method [1–3,34] known as Lyapunov characteristic exponent to quantify the sensitive dependence on initial conditions for chaotic behavior. It gives valuable information about the stability of dynamic systems. With this method it is possible to determine the minimum requirements of differential equations that are necessary to create chaos (see footnote 2). To each variable of the system is a Lyapunov exponent. Let us study the case of systems with only one variable [1] that assume two initial states x_o and $x_o+\varepsilon$, differing by a small amount ε. We want to investigate the possible values of x_n after n iterations from the two initial values. The difference d_n between the two x_n values after n iterations (omitting for simplicity the subscript α) is given approximately by

$$d_n = f(x_n+\varepsilon) - f(x_n) = \varepsilon \exp(\mathrm{n}\lambda), \qquad (21)$$

where λ is the Lyapunov exponent that represents the coefficient of the average exponential growth per unit of time between the two states. From Eq.(21) we see that if λ is negative, the two orbits will eventually converge, but if positive, the nearby trajectories diverge resulting chaos. The difference d_1 between the two initial states is written as

$$d_1 = f(x_o+\varepsilon) - f(x_o) \approx \varepsilon \left(\frac{df}{dx}\right)_{x0}. \qquad (22)$$

Now, in order to avoid confusion that sometimes is found in the chaotic literature, we remember that

$$\begin{aligned} x_{n+1} &= f(x_n) = f(f(x_{n-1})) = f(f(f(x_{n-2}))) = \\ \ldots &= f(f(f(\ldots(f(x_o))))), \qquad (23) \end{aligned}$$

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that also is written as

$$x_{n+1} = f(x_n) = f^n(x_o), \tag{24}$$

where the superscript n indicates the n^{th} iterate of the map.

After a large number n of iterations the difference between the nearby states, using Eq. (21) and Eq. (23), will be given by

$$\begin{aligned} d_n &= f(x_n + \varepsilon) - f(x_n) = f^n(x_o + \varepsilon) - f^n(x_o) \\ &= \varepsilon \exp(\mathrm{n}\lambda). \end{aligned} \tag{25}$$

Dividing Eq. (23) by ε and taking the logarithm of both sides, results

$$\ln\left\{\left[f^n(x_o+\varepsilon) - f^n\frac{x_o}{\varepsilon}\right]\right\} = \ln[\exp(\mathrm{n}\lambda)] = \mathrm{n}\lambda. \tag{26}$$

Taking into account that ε is small we obtain from Eq. (24),

$$\lambda(x_o) = \frac{1}{n}\ln\left|df^n\frac{x_o}{dx_o}\right|. \tag{27}$$

Since $f^n(x_o)$ is obtained iterating $f(x_o)$ n times we have $f^n(x_o) = f(f((f(x_o)))$, that is, $f^n(x_o) = f(f^{n-1}(x_o)) = f(f^{n-1}(f^{n-2}(x_o))) =$, where $x_i = f_i(x_o)$ is the result of the i^{th} iteration of the map $f(x)$ from the initial condition x_o. So, using the derivative chain rule we get

$$\begin{aligned} df^n\frac{x_o}{dx_o} &= \left\{\frac{df^n(x_{n-1})}{dx_o}\right\}\left\{\frac{df^n(x_{n-2})}{dx_o}\right\} \\ &\quad \ldots \left\{\frac{df^n(x_o)}{dx_o}\right\}. \end{aligned} \tag{28}$$

Thus, for $\varepsilon \to \infty$ we get, using Eq. (25) and Eq. (26),

$$\begin{aligned} \lambda(x_o) &= \lim \to \infty \left(\frac{1}{n}\right)\ln\left|\prod df\frac{x_i}{dx_o}\right| \\ &= \lim \to \infty \frac{1}{n}\ln\left|\frac{df(x_i)}{dx_o}\right|, \end{aligned} \tag{29}$$

where $x_i = f_i(x_o)$. In the lim n$\to \infty$ the Lyapunov exponent becomes independent of the initial condition x_o. This occurs because when is done an infinite numbers of iterations. the attractor is entirely covered by $x(t)$, and it does not matter the initial point x_o. As in practice n are large, but finite numbers, we calculate λ for different initial conditions and take an average of these values.

From Eq. (21) we verify that if λ is negative, the two orbits will eventually converge; but if λ is positive, the nearby trajectories diverge resulting chaos. From Eq. (23) we see that at the bifurcation $\lambda = 0$ because $|df/dx| = 1$ (the solution becomes unstable). When $df/dx = 0$ we have $\lambda = -\infty$ (the solution becomes super stable).

The λ estimation using simply the flow equations (6), (8) and (3), that is, without maps, are in general difficult because one has to deal with solutions of NLDE and analytic calculations. This kind of calculation for the damped and driven pendulum is seen, for instance, in reference [1]. Using maps these calculations become easier. This is shown in what follows for logistic map and triangular map.

6.1. Lyapunov exponents for logistic map

According to Eq. (25) or Eq. (19) to obtain λ are used the iterated functions $f_n(x_o)$. For the logistic map we have the logistic equation (18).

As an example, the second order iterated function $f^2(x)$ is given by $f^2(x) = f(f(x)) = f(\alpha x(1-x)) = \alpha(f(x)(1-f(x))) = \alpha 2x(1-x[1-\alpha x(1-x)]$.

So, to get $\lambda(x_o)$ we can continue to iterate $f(x)$ up to $n >> 1$ and use Eq. (25) or use Eq. (26) taking into account $f(x_i)$, with $i = 1, 2, \ldots, n$, remembering that $f(x_i) = f_i(x)$.

In reference [36, 40] are seen cobweb plots (web diagrams) or Verhulst diagrams that are graphs that can be used to visualize successive iterations of the function $f(x)$. In particular , the segments of the diagram connect the points $(x, f(x)), (f(x), f(f(x))), (f(f(x)), f(f(f(x))))$. The diagram is so-named because its straight lines segments anchored to the functions x and $f(x)$ resemble a spider web. The cobweb plot is a visual tool used to investigate the qualitative behavior of one-dimensional iterated functions such as the logistic map. With this plot it is possible to infer the long term status of an initial condition under repeated application of a map.

In Fig. 11 are shown the Lyapunov exponents λ calculated numerically as a function of the parameter α for the logistic map x seen in Fig. 6.

6.2. Lyapunov exponents for triangular map

In the particular case of a triangular map [8, 31] λ can be calculated analytically. This map, represented in Fig. 15, obey the following equations:

$$\begin{aligned} x_{n+1} &= 2\beta X_n, 0 < x \geq 0.5 \\ x_{n+1} &= \alpha(1-x_n), 0.5 < x < 1, 0 < \beta \geq 1 \end{aligned} \quad (30)$$

Equations (30) can be rewritten as $x_{n+1} = f(x_n)$, where the function $f(x)$ is given by

$$f(x) = \beta[1 - 2|0.5 - x|]. \quad (31)$$

The n^{th} application on $2\beta x$ of the first region $0 < x < 0.5$ give $f_n(x) = (2\beta)^n x^n$.

The maximum value of $f_n(x)$ is βn at the point $x = 2^{-n}$. By symmetry the next point of minimum must be 2.2^{-n} and of maximum at 3.2^{-n} and so on.

This implies that $|f^n(x)/dx| = (2\beta)^n$ for the two regions. Taking into account Eq. (25) we get

$$\lambda(x_o) = \frac{1}{n}\left|\frac{df(x_o)}{dx_o}\right| = ln(2\beta). \quad (32)$$

Consequently, there is chaos only for $\beta > 0.5$, since $\lambda > 0$.

7. Turbulent Processes

As usually seen in basic physic courses [4,41], turbulence is originated from studies of fluid motion in classical mechanics. The general equation of motion for a viscous fluid is given by the Navier-Stokes nonlinear partial differential equation (NLPDE)

$$\frac{\partial v}{\partial t} + (v.\text{grad})v = -\text{grad}\frac{(P)}{\rho}\text{grad}(\phi) + \frac{\eta}{\rho}\text{lapl}(v), \quad (33)$$

where $v(r,t)$ is the velocity of the fluid at point r, P is the pressure, ρ the density of fluid, $\phi(r)$ the gravitational potential and η the viscosity. This equation is a miracle of brevity, relating a fluid's velocity, pressure, density and viscosity [20]. Since Eq. (33) is a NLPDE, it is not submitted to any general method of solution (see Footnote 2).

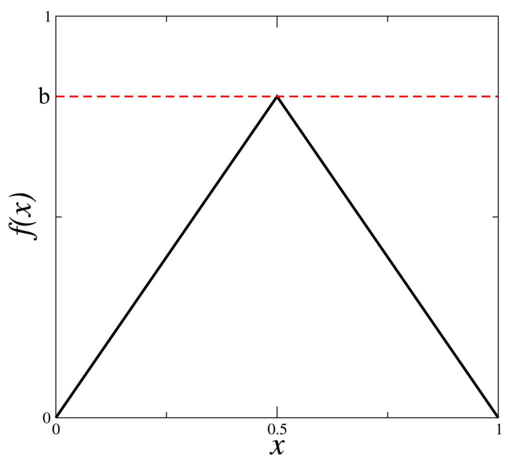

Figure 15: Triangular map.

Laminar flow occurs for very small Reynolds number $Re = \nu L\rho/\eta << 1$ [17,20], where ν is a typical fluid velocity and L is some characteristic length in the flow. In these conditions Eq. (33) can be approximated by a linear partial differential equation (LPDE) and all elements of volume of the fluid describe well defined trajectories $r = r(t)$. Since there are an infinite number of elements of volume δV the resulting LPDE has an infinite number of degrees of freedom which is a characteristic of the PDE (see Footnote 2). For $Re >> 1$ the nonlinear effects become dominant being responsible for the phenomenon called turbulence. In these conditions the flux becomes disordered: the trajectories of the fluid elements δV are irregular and develop eddies, ripples and whorls. In spite of this yet there is some sort of order found within the disorder or turbulence which could be described as self-similar or fractal [25]. An open problem is to find a mathematical formalism able to describe this disordered state [25–27].

Turbulence in fluid dynamics is being understood in infinite dimensional phase space under the flow defined by the Navier-Stokes equation. We have seen that in the finite dimensional phase space physical systems can be described with very good precision by LODE and NLODE that can solved exactly or numerically. They can in principle reveal all detailed structures of the dynamical systems. Turbulence in fluid mechanics is generated by a NLPDE anchored in an infinite dimensional phase space. Is turbulence a chaotic process? Up to nowadays it is well-known that the theory of chaos in finite-dimensional dynamical systems has been well-developed. Such theory has produced important mathematical theorems and led to important applications in physics, chemistry, biology, engineering, etc [17].

Note that, in the contrary, theory of chaos in PDE has not been well-developed. In terms of applications, most of important natural phenomena are described by linear and nonlinear partial differential equations (wave equations, Yang−Mills equations, Navier−Stokes ,General Relativity, Schrȯdinger equations, etc) (see Footnote 2). In spite of extensive investigations it was not possible to prove, in the general case, the existence of chaos in infinite-dimensional systems [10, 17, 18, 20].

Among the NLPDE there is a class of equations called soliton equations that are integrable Hamiltonian PDE and natural counterparts of finite-dimensional integrable Hamiltonian systems [10]. Many works have also been developed investigating the existence of chaos in perturbed soliton equations [20, 27].

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