

On the development of the Navier–Stokes equation by Navier

Sylvio R. Bistafa*¹

¹Universidade de São Paulo, São Paulo, SP, Brazil

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After the work of Navier, the Navier-Stokes equation was re-obtained by different arguments by numerous investigators. We have chosen to revisit the approaches of Navier not only because they were pioneering, but also because, unexpectedly, by undergirding his theory on Laplace's new concept of molecular forces – thought to be capable of capturing the effects of viscosity – Navier managed to derive for the first time the ultimate equation for the laminar motion of viscous fluids. A fragile model was thus capable of generating a true prediction in comparison to other, more rigorous models of the Navier-Stokes equation. Navier's derivation appeared in two almost simultaneous publications. In the first one of them, he extended his theory for the motion of elastic solids to the motion of viscous fluids. In the second publication, Navier again derived his equation using Lagrange's method of moments, which could yield the boundary conditions. However, both derivations were not influential, and were neglected by his contemporaries and by specialized publications alike. The fact that his theory could only be applied to slow motion in capillaries may have later discouraged Navier, who abandoned his theory of fluid motion in favour of experiment-based formulations for ordinary applications.

Keywords: history of the Navier–Stokes equation, history of fluid mechanics, history of viscosity.

1. Introduction

The first derivations of the Navier–Stokes equation appeared in two memoirs by Claude-Louis Navier (1785–1836): *Sur les lois des mouvements des fluides, en ayant égard à l'adhésion des molécules* [1], here referred to as the 1st memoir, published in the *Annales de Chimie et de Physique*; and *Sur Les Lois du Mouvement des Fluides* [2], here referred to as the 2nd memoir, read at the Academy on 18th March, 1822, and which appeared in the *Mémoires de L'Académie Royale des Sciences de L'Institut de France* for the year of 1823.

These publications formally introduced, for the first time, friction in the equations of fluid motion. Up until then, all the theoretical developments related to the motion of fluids had considered them as media with perfect fluidity, devoid of viscosity. It is possible to say that the equations of motion of fluids had been limited to perfect fluids since 1755, following the publication of the well-

known Euler equation of motion for non-viscous fluids [3].

However, it had been recognized for a long time that the friction of fluids was the main cause for the deviation of experiments from theory. Nonetheless, despite its importance, only a few theoretical developments attempted to include the effects of viscosity in the equations of motion of fluids. One of these was proposed by Leonhard Euler (1707-1783) himself [4], in which he incorrectly assumed that, similar to the case of friction in solids, fluid friction was proportional to pressure [5].

After Euler, it appears that only Navier was motivated to formally tackle this problem and to succeed in solving it. His interest in the subject was not fortuitous, but motivated by his recognized competence in hydraulics, given that he had edited Belidor's popular treatise *Architecture hydraulique*¹.

Navier's inclination for scholarship and his background in higher analysis at the École Polytechnique and in prac-

*Endereço de correspondência: sbistafa@usp.br.

¹ Bernard Forest de Belidor (1697?–1761), a military engineer, taught mathematics at the artillery school at La Fère where he authored several textbooks. Seeking to introduce mathematics into practical engineering, he wrote *La science des ingénieurs* (1729) and *Architecture hydraulique* (1737-1739). In the 1810s Belidor's two works were issued in revised and expanded editions by Navier, who had been recruited by the École des Ponts et Chaussées to edit the works of his great-uncle, the great French engineer Émiland Gauthey. By 1813 Navier had completed this task, and had also issued a revised and expanded edition of Belidor's *La science des ingénieurs*. Navier's success as editor of Belidor's *Science des ingénieurs* and of Gauthey's works led their publisher, Firmin Didot, to invite him to prepare a revised edition of Belidor's *Architecture hydraulique*. Navier sought to correct the errors found in this work and to give it a mathematical sophistication that would make it useful to the graduates of the École Polytechnique. Navier's contributions to the *Architecture hydraulique* are confined to the first volume, which contains notes and commentary equal or surpassing the original text in length. The remaining volumes consist of reissues, with new titles dated 1810, of the edition published in 1780. (Text extracted and adapted from *Jeremy Norman's Historyofscience.com* <https://www.jnorman.com/cgi-bin/hss/38462.html> accessed on March 9, 2017.)

tical engineering at the *École des Ponts et Chaussées* put him in the ideal position to make significant contributions to engineering. From 1807 to 1820 he made mathematical analysis a fundamental tool for the civil engineer and codified the nascent concept of mechanical work for the science of machines. Therefore, Navier had the essential pre-requisites necessary to develop what later became known as the Navier–Stokes equation (referred hereafter simply as the N-S equation), namely, mathematical rigor associated with practicality, or the embodiment of an engineer in a scientist.

After Navier, the N-S equation was re-obtained by different arguments by a number of authors². Nonetheless, Navier was never acknowledged for his contribution, as his approaches were not influential and were ignored by his successors and by specialized publications on the subject alike. At the time, though each new discoverer of the N-S equation seemed to have been aware of the derivation by Navier, they chose to ignore it.

Despite the fact that several authors have obtained the N-S equation perhaps in a more rigorous fashion, we chose to revisit the approaches of Navier, not only because they were pioneering, but also because even after basing his developments on uncertain methods to deal with molecular forces deemed responsible for the friction, Navier unexpectedly managed to develop for the first time the ultimate equation for the motion of real viscous fluids.

The modern theory of elasticity may be considered to have its birth in 1821 when Navier first gave the equations for the equilibrium and motion of an (isotropic, one-constant) elastic solid in a memoir titled *Sur les Lois de l'Équilibre et du Mouvement des Corps Solides Élastiques* [6], read at the Academy on 14th May, 1821, and which appeared in the *Mémoires de L'Académie Royale des Sciences de L'Institut de France* in 1827. He soon perceived that these equations could be extended to other continuous media, and taking as a starting point the equations for elastic solids, he wrote in the 1st memoir the equation for the motion of viscous fluids, substituting fluid particle velocities for elastic solid displacements, and the fluid viscosity constant (called ‘adherence constant’ by Navier) for the elastic solid constant. Other investigators such as Cauchy, Poisson, and Saint-Venant, presumably encouraged by Navier’s publications, took the opportunity to offer the equation for viscous fluid motion from their equations of elasticity. From the case for a ‘non-elastic body’, and by assuming that the stress tensor is proportional to the rate of deformation tensor, Cauchy [7] obtained the equation of motion for viscous fluids given by Navier. By assuming that the stresses in a fluid are related to the fluid’s rate of deformation, in the same manner that the stresses in a solid are related to strain, Poisson [8] obtained the N–S equation, with some additions to the pressure gradient term. Saint-Venant [9], in turn, thinking in terms of transverse pressure act-

ing on the faces of the sliding fluid particles, obtained a stress tensor that yields the differential equations of Navier, Cauchy, and Poisson with one constant parameter. All of these 19th century investigators tried to fill the gap between the rational fluid mechanics of the perfect non viscous fluid developed in the 18th century by the Bernoullis (Daniel and Johann), d’Alembert, Euler, and Lagrange, and the actual behaviour of real viscous fluids in hydraulic systems.

As for the involvement of Stokes with the equation of motion for viscous flows, whose name is also associated with the N–S equation, it begins with his interest on pendulum experiments that seem to indicate that viscosity could play a role on the altered behaviour of the ideal pendulum motion. Stokes’ first strategy was to calculate cases of motion including oscillating spheres and cylinders that would represent the bob and the suspending thread of a pendulum, considering perfect fluidity, to investigate the departure of real fluids from perfect ones [10]. He pointed out that friction, of course, would be a cause of departure, but not the only one. He then indicates other causes such as the discontinuity of flow, and instabilities that lead to a turbulent wake. However, Stokes recognized that without accurate experiments it was not possible to advance in his initial proposal. He then changed his strategy, and decided to include internal fluid friction into the fundamental equations of hydrodynamics. By applying methods similar to Cauchy’s and Poisson’s, he arrived at the N–S equation by saying that this equation and the equation of continuity “[...] are applicable to the determination of the motion of water in pipes and canals, to the calculation of friction on the motions of tides and waves, and such questions” [11].

Since many investigators had corroborated the equation of motion for viscous flows as developed by Navier, one may wonder why Stokes became also associated with this equation. The answer may be that he made extensive comparisons of theory and experiments of different researchers with cylindrical rods, spheres, spheres at the end of long and short rods, oscillating disks, long and short pendula oscillating in air and water etc. [12]. Despite having wrongly assumed that viscosity is proportional to density, and by neglecting the convective term in the N–S equation, Stokes succeeded in obtaining good agreement with the N–S equation by comparing predictions with experimental data for oscillating pendula. Therefore, differently from the other authors of the N–S equation, and similarly to Navier, Stokes had a very clear intention on the practicality of his efforts by confronting theory with experiments, and this may be a reason why he and Navier became associated with the equation of motion for viscous flows.

A matter of such importance as the development of the N–S equation has been, of course, the subject of many historians of mechanics. René Dugas’ *A History of Mechanics* [13] offers an exposition of Navier’s 2nd mem-

² We refer the reader to the book by Darrigol [15, pp.101-144] for a detailed and thorough analysis of the history of the N-S equation.

air, and a brief account of Navier’s memoir on elasticity. Grattan-Guinness’ *Convolutions in French Mathematics* [14] reviewed both Navier’s memoirs on viscous fluids, in association with his works on elastic rods, surfaces and bodies. Darrigol’s *Worlds of Flow* [15] gives an excellent brief summary and contextualization of all of Navier’s works on fluids; it also fully describes their influence on (and neglect by) Cauchy, Saint-Venant, Poisson and Stokes. Therefore, one might wonder what is new in the present work. The answer to that question can be found at the end of this article, in the Conclusions. The present work uncovers current concepts on the topic of viscous fluid flow that have Navier’s imprint, but whose authorship has often been overlooked, taken for granted and other results that customarily have not been directly attributed to him. These new discoveries show that the impact of Navier’s works on viscous flow was more profound than Navier scholars have so far proposed.

2. What is the Navier–Stokes equation?

The Navier–Stokes equation is a nonlinear partial differential equation, which governs the motion of *real viscous fluids* and can be seen as Newton’s second law of motion for fluids. In the case of an incompressible Newtonian fluid, this yields

Navier–Stokes equation written in vector notation:

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = \mathbf{g} - \frac{1}{\rho} \nabla p + \nu \nabla^2 \mathbf{u} ,$$

where \mathbf{u} is the velocity vector, \mathbf{g} is the acceleration vector due to a body force, p is pressure, $\nu = \frac{\mu}{\rho}$ is the kinematic viscosity, μ is the dynamic viscosity, and ρ is the density.

If we call $\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = \mathbf{a}$ the acceleration of the fluid particle, and $\left(\mathbf{g} - \frac{1}{\rho} \nabla p + \nu \nabla^2 \mathbf{u}\right) = \sum \mathbf{f}_{\text{ext.}}$, the summation of the external forces that act in the fluid particle per unit of mass, we have that $\mathbf{a} = \sum \mathbf{f}_{\text{ext.}}$, which is Newton’s second law of motion per unit of mass.

The acceleration \mathbf{a} is composed of two parts, the first part $\frac{\partial \mathbf{u}}{\partial t}$ is the local acceleration, and reflects the change in velocity with time observed in a fixed point in the flow as the particles pass by; the second part $\mathbf{u} \cdot \nabla \mathbf{u}$ is the convective acceleration, and reflects the change in velocity in space, as the particle moves from one point to another in the flow field in infinitesimal time.

Here \mathbf{g} , represents the body force term per unit of mass, and indicates the force (such as the gravitational force or the electromagnetic force) that acts in the fluid particle; $\frac{1}{\rho} \nabla p$ is the pressure term, and indicates that the fluid flows in the direction of the largest change in pressure; $\nu \nabla^2 \mathbf{u}$ is the viscous term, and indicates the friction force due to the viscosity acting on the fluid particle as it flows with velocity \mathbf{u} . Both the pressure force and the viscous force are forces that act at the surface of the fluid particle, and as such are classified as external forces.

The Navier–Stokes equation is an evolution of the Euler’s equation. This equation governs the motion of the *perfect non viscous fluid* and as such can be seen as the Navier–Stokes equation without the viscous term $\nu \nabla^2 \mathbf{u}$

Euler’s equation written in vector notation:

$$\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u}\right) = \mathbf{g} - \frac{1}{\rho} \nabla p.$$

It is possible to write Euler’s equation along a streamline in the so-called intrinsic coordinate system (s, n) (see Figure 1) as

$$\left(\frac{\partial V_s}{\partial t} + V_s \frac{\partial V_s}{\partial s}\right) = -g \frac{\partial z}{\partial s} - \frac{1}{\rho} \frac{\partial p}{\partial s}.$$

For steady incompressible flow

$$\frac{d}{ds} \left(\frac{V_s^2}{2}\right) + g \frac{d}{ds} (z) + \frac{1}{\rho} \frac{d}{ds} (p) = 0,$$

$$\frac{d}{ds} \left(\frac{V_s^2}{2} + gz + \frac{p}{\rho}\right) = 0,$$

which upon integration yields

$$\frac{V_s^2}{2} + gz + \frac{p}{\rho} = cte., \text{ along a streamline.}$$

which is recognized as Bernoulli’s equation, the most popular equation of fluid dynamics.

2.1. A word about notation

When Navier wrote his memoirs on viscous flows, the vector notation had not been invented yet. We have already written the Navier–Stokes equation in the more modern vector notation at the beginning of this section. We can write it in another form, in the so-called index notation, also known as tensor notation, in which case the Navier–Stokes equation is written as

Navier–Stokes equation written in index notation:

$$\frac{\partial u_i}{\partial t} + u_k \frac{\partial u_j}{\partial x_k} = g_j - \frac{1}{\rho} \frac{\partial p}{\partial x_j} + \nu \frac{\partial^2 u_j}{\partial x_i \partial x_i},$$

where repeated index means summation.

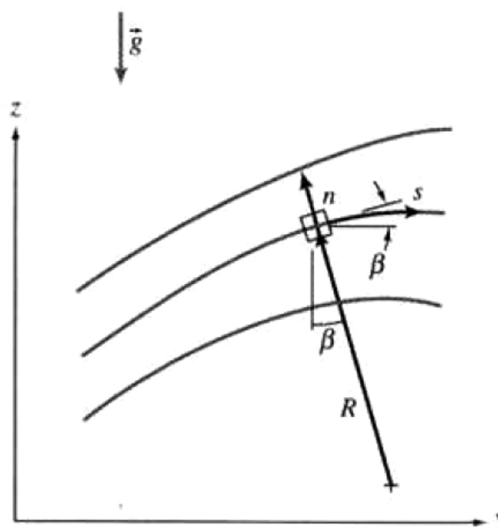


Figure 1: Streamlines with the indication of the intrinsic coordinate system.

In Cartesian coordinates this equation reads

$$\begin{aligned}\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} &= g_x - \frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right), \\ \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} &= g_y - \frac{\partial p}{\partial y} + \nu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right), \\ \frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} &= g_z - \frac{\partial p}{\partial z} + \nu \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right).\end{aligned}$$

Vector and index notations are not only elegant; they are also useful because of their compactness. They make it possible to handle vectors as single objects rather than collections of components. However, vector notation has some difficulties, a major one of them being that there is a whole heap of vector algebraic and differential identities that are hard to remember and hard to derive. In index notation, there are certain rules to be followed for operations such as dot and cross products, which can represent a challenge in the developments. A major hindrance, however, is that both notations tend to overshadow the physics of the problem, particularly for the reader unfamiliar with compact notations. This can be readily seen by comparing the Navier–Stokes equation written in the compact notations with the equation written in Cartesian coordinates – the latter is seen to be more readily connected with the physical meanings of the various terms.

As we shall see, Navier writes his equations in the Cartesian coordinate system. As for his developments and equations, the only concession asked of the modern reader is that he notices that Navier writes partial derivatives as $\frac{du}{dt}$, $\frac{du}{dx}$, $\frac{d^2u}{dx^2}$, etc., instead of $\frac{\partial u}{\partial t}$, $\frac{\partial u}{\partial x}$, $\frac{\partial^2 u}{\partial x^2}$, etc. As far as notation is concerned, with this in mind, the reader will encounter no obstacles or additional difficulties in following Navier developments.

3. The nature of the molecular forces according to Navier

Shortly before the 2nd memoir [2], Navier presented a first attempt to include viscous effects in the equations of fluid motion of the 1st memoir [1]. Navier begins the latter memoir by saying that the illustrious geometers d’Alembert and Euler were the first to express the laws of equilibrium and motion of fluids by means of differential equations with partial derivatives, by considering these media as an ensemble of molecules, susceptible to have free motion without opposing any resistance between each other, and that the scholars who followed them had also adopted the same hypothesis. He then points out that Laplace³ was the only one who had investigated the laws of equilibrium of incompressible fluids, with regard to the molecular actions. Navier then proposes to discuss the effects of molecular actions in the state of fluid motion.

³ Imitating Newton’s gravitation theory, Laplace sought to explain the properties of matter in terms of central forces acting between molecules. His first successful attempt in this direction was a theory of capillarity published in 1805/1806. He also indicated in his *Système du monde*, published in 1808, how physical processes including elasticity and viscosity could all be reduced to short-range forces between molecules. Navier adopted Laplace’s ideas about molecular forces, and emphasized that the conditions of equilibrium of fluids and fluid motion could not be established without the molecular viewpoint. [15].

⁴ This terminology was first used by D’Alembert in the *Traité de Dynamique* associated with a principle on how to deduce the solution of problems of dynamics from conditions of equilibrium: “If we impress to several bodies motions that are forced to change as a result of their mutual action, it is clear that we can regard these motions as composed of those they will really take and of motions that are *destroyed*; from what it follows that the latter must be such that the animated bodies be in equilibrium under their own motions” [20].

According to Navier [1]:

[...] the positions of the molecules in the state of fluid equilibrium are such that the mutual distances are only determined by the actual temperature, which is assumed uniform throughout the fluid. This state of affair is the same when these molecules are animated with a common motion. In both cases, the actions exercised on each molecule by all others that surround it are reciprocally destroyed⁴, such that these actions have no influence in the state of the fluid. However, when a fluid is animated with a motion such that the molecules are continuously displaced with respect to each other, actions are exercised among them, which must necessarily be investigated if we wish to know the true laws of fluid motion [1].

Navier [1] continues by saying that:

[...] from a great number of experiences it appears that the pressure does not sensibly influences the resistances coming from the molecular actions that are exercised among the parts of a fluid in motion. The causes of these resistances must be sought, therefore, in the differences, either in the magnitude or in the direction, of the velocities of the neighbouring molecules [1].

When considering the equilibrium of an incompressible fluid whose surface is submitted to a pressure which tends to reduce its volume, Navier [1] conceives that:

[...] any two neighbouring molecules are in such state as if there existed between them a spring that has contracted by an insensible quantity, and which opposes the approach of these molecules. If the fluid moves, and all the molecules, being carried away by a common motion, preserve their respective situations, the state of these springs does not change, and no new action is established in the interior of the fluid. In the two cases just mentioned, a molecule is equally pressed by all those which surround it. But if, by the effect of the diversity of motions of the neighbouring particles, the two molecules in question tend to approach or to move away from each other, the spring established between them will be contracted more in the first case, and less in the second [1].

According to this circumstance, Navier [1] admits that:

[...] the effect of the diversity of these motions is to modify the actions which existed between the molecules in a state of equilibrium, or of a motion common to all, actions which had no influence on this motion. In the general case of any motion, this molecule is pressed more closely by those which approach it; it is less pressed by those which withdraw from it. The increase in pressure on the part of the former and the decrease in pressure on the part of the latter constitute forces which tend to modify the motion of the molecule, and which are combined with the accelerating forces acting on the fluid. The modification of motion does not depend on the intensity of the pressure or on the intensity of the forces existing between the molecules. It depends on the increase or decrease which these forces undergo when the molecules tend to approach or to depart from each other. This increase or reduction itself depends entirely or almost entirely, as experience shows, on the relative velocity of the molecule, and not on the magnitude of the pressure [1].

Navier [1] then adopts the following principle:

[...] when two molecules of the fluid, in consequence of the diversity of their motions, approach or depart from one another, there

exists between them a repulsion or attraction, the intensity of which depends on the speed with which these molecules approach or depart from each other [1].

Navier [1] then concludes by saying that:

[...] this attraction or repulsion is in the nature of molecular forces, which take place only between very neighbouring molecules, and may be considered as null for molecules whose distance from each other is of sensible magnitude. The molecules, of which the mutual action will be considered, will never have but very different velocities, that is to say, they will approach or depart from each other with an extremely small speed. Consequently, whatever may be the general function of the relative velocity of the molecules, which must exhibit their mutual action, not having to consider very small values of this (relative) velocity, we may suppose that the action of the molecules is proportional to it⁵ [1].

Navier's hypothesis that no viscous forces are manifested when the molecules are being carried away by a common motion, and that these only arise when there is a relative motion of the molecules, is in perfect agreement with the modern view of considering viscous effects to be confined in the layer in the immediate vicinity of a bounding surface where the particles present relative motion and the effects of viscosity are significant (the boundary-layer); as opposed to the flow outside the boundary-layer (the outer potential flow), where viscous effects are small because the particles are animated by a common motion in this region.

Therefore, the equationing of the molecular forces in the state of fluid motion by Navier is essentially based on the difference in velocities between two molecules, and is established by him as follows: two molecules with masses m and m' , animated with velocities v and v' , and angles λ and λ' which are formed between the directions of the velocities and the line joining the molecules, the repulsive force between these two molecules would be proportional to

$$m m' (v \cos \lambda - v' \cos \lambda'). \quad (1)$$

This quantity, when multiplied by a function of the distance of these two molecules (which decreases very fast as the distance increases), and by a constant relative to the 'adherence of the fluid molecules' (viscosity), gives the repulsive force between the two molecules (force that will be of attraction if this quantity is negative).

⁵ In 1687 Newton had already hypothesized viscous forces in fluid motion proportional to the relative velocities of the fluid particles: "The resistance arising from the want of lubricity in the parts of a fluid is *caeteris paribus*, proportional to the velocity with which the parts of the fluid are separated from each other" (Newton, *Principia* Section IX, Book II).

4. The derivation of the N–S equation in the 1st memoir

In the 1st memoir [1], after having established the nature of the molecular forces for fluids in motion, Navier makes an analogy between the motion of viscous fluids and the motion of elastic solids as developed in the memoir *Sur les Lois de l'Équilibre et du Mouvement des Corps Solides Élastiques* [6] in order to write the expressions for the components of the force that acts on the molecule m due to the motion of the neighbouring molecules as

$$\epsilon \left(3 \frac{d^2 u}{dx^2} + \frac{d^2 u}{dy^2} + \frac{d^2 u}{dz^2} + 2 \frac{d^2 v}{dxdy} + 2 \frac{d^2 w}{dxdy} \right),$$

Equating this force to the accelerative forces X , Y , and Z in the x , y and z directions that act on the molecule m , with density ρ , subjected to the pressure p gives

$$\begin{aligned} \frac{1}{\rho} \frac{dp}{dx} &= X + \epsilon \left(\frac{d^2 u}{dx^2} + \frac{d^2 u}{dy^2} + \frac{d^2 u}{dz^2} \right) - \frac{du}{dt} - u \frac{du}{dx} - v \frac{du}{dy} - w \frac{du}{dz}, \\ \frac{1}{\rho} \frac{dp}{dy} &= Y + \epsilon \left(\frac{d^2 v}{dx^2} + \frac{d^2 v}{dy^2} + \frac{d^2 v}{dz^2} \right) - \frac{dv}{dt} - u \frac{dv}{dx} - v \frac{dv}{dy} - w \frac{dv}{dz}, \\ \frac{1}{\rho} \frac{dp}{dz} &= Z + \epsilon \left(\frac{d^2 w}{dx^2} + \frac{d^2 w}{dy^2} + \frac{d^2 w}{dz^2} \right) - \frac{dw}{dt} - u \frac{dw}{dx} - v \frac{dw}{dy} - w \frac{dw}{dz}, \end{aligned} \quad (3)$$

which are recognized as the three components of the Navier–Stokes equation for an incompressible fluid.

Astonishingly Navier did not indicate in the 1st memoir that once the viscous terms given by Eq. (2) are rewritten by invoking the equation of continuity for an incompressible fluid $\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0$, the result would be the standard form of the N–S equation as given by Eq. (3).

Next, Navier gives an application for a flow through a conduit with rectangular cross section, by considering a non-slip boundary condition at the walls of the conduit⁶. He then found the average fluid velocity in a tube with square cross section to be proportional to the square of its perimeter. This result was in contrast with Girard's formulation which experimentally found an average fluid velocity proportional to the diameter of capillaries with circular cross sections.

In fact, Girard⁷ [16] also adopted a non-slip boundary condition, by assuming that a layer of fluid remained at rest at the tube walls, and gave the formula $U = g \frac{Dh}{4a}$ for the average velocity through a capillary with diameter D , length l , subjected to the gravity g , and head h , where a represents a coefficient that is constant for the same capillary, but which significantly diminishes as the

$$\epsilon \left(2 \frac{d^2 u}{dxdy} + \frac{d^2 v}{dx^2} + 3 \frac{d^2 v}{dy^2} + \frac{d^2 v}{dz^2} + 2 \frac{d^2 w}{dydz} \right), \quad (2)$$

$$\epsilon \left(2 \frac{d^2 u}{dxdz} + 2 \frac{d^2 v}{dydz} + \frac{d^2 w}{dx^2} + \frac{d^2 w}{dy^2} + 3 \frac{d^2 w}{dz^2} \right),$$

where ϵ is what Navier called the 'adherence of the fluid molecules constant' (later called viscosity), which had been substituted for the elastic solid constant, called ϵ as well by Navier.

In these expressions, the components of a fluid particle velocity u , v and w , at a point (in the Euler sense) whose rectangular coordinates are x , y and z , had been substituted for the elastic solid displacements x , y and z at a point whose rectangular coordinates are a , b , and c .

temperature rises, and that varied with the diameter of the capillary.

It is well known, that the average velocity in a laminar flow through a tube ('linear motion in a capillary' according to the Navier/Girard terminology), known as the Hagen-Poiseuille flow, is given by $U = g \frac{R^2 h_L}{8(\frac{\mu}{\rho})l}$, where h_L is the head loss due to viscous effects in the tube, μ is the dynamic viscosity, and ρ is the density.

The application formulated by Navier in the 1st memoir, where he considers a tube with square cross section of side R , which is very small (a capillary), yields for the average velocity approximately $U \cong g \frac{R^2 h_L}{8(\frac{2\epsilon}{\rho})l}$, where it can be seen that Navier's adherence of molecules constant ϵ is roughly equal to half the dynamic viscosity μ .

It is also surprising that already in the 1st memoir, Navier had anticipated, on purely theoretical grounds, what became later known as the Hagen-Poiseuille law. Both Hagen and Poiseuille confirmed the same result experimentally [17, 18].

Unfortunately, because of the contrasting results related to the dependence of the average velocity on the tube diameter as experimentally found by Girard, Navier

⁶ This is what we call a 'strict non-slip boundary condition'. More details on the different types of boundary conditions considered by the 19th century investigators can be found in the last section (Supplementary material).

⁷ Pierre-Simon Girard (1765-1836) worked on fluid mechanics at the École des Ponts et Chaussées. He was in charge of planning and constructing the Amiens canal and the Ourcq canal. He collaborated with Gaspard de Prony on the *Dictionnaire des Ponts et Chaussées*. He authored works on fluids and on the strength of materials

was led to consider changing the non-slip boundary condition. Moreover, at the end of the 1st memoir, by comparing Girard's experimental results of water flowing through capillaries made of glass [16] and capillaries made of copper [19], Navier verified in these experimental results that in capillaries of about the same length and diameter, the flow is three to four times slower in tubes made of glass than in tubes made of copper.

Intrigued by this discrepancy, Navier concludes the 1st memoir with the following conjecture:

[...] For a reconciliation of this result it appears that an action on the liquid in motion is exerted on the part of the solid wall which greatly influences the effects, and that it would not be allowed in this kind of phenomena to suppose that a very thin layer of fluid adhering to the surface of the bodies which it wets acts alone on the fluid in motion. The solution of our application is well suited to the hypothesis on which it is based: it gives the general law of the motion of the fluid, a law which is the same, whatever the nature of the wall is; and moreover, it would also give the absolute value of the velocity in the particular case where the action of the molecules of the wall upon those of the fluid would be equal to the action of these molecules upon themselves. But, in any other case, the velocity has other absolute values; and the theory of this kind of phenomena will be entirely formed only so far as the particular forces emanating from the molecules of the fixed walls have been taken into consideration [1].

Therefore, if Navier had wanted to explore more deeply the interaction between the molecules of the fluid layer in contact with the wall and the molecules of the wall itself, he would have to rely on a method that could yield the boundary conditions. This method is Lagrange's method of moments, which Navier had already applied in the memoir *Sur les Lois de l'Équilibre et du Mouvement des Corps Solides Élastiques* [6]. In fact, in this memoir, Navier gave two different derivations of the equations of elasticity for an isotropic body with one elastic constant, the first by direct summation of forces acting on a molecule, and the second by Lagrange's method of moments, with both methods yielding the same "undefined equations", as Navier called the equations to be satisfied in all interior points of the body. However, in order to obtain the particular conditions that exist on points situated at the boundaries of the bodies, Navier proposed the use of Lagrange's method of moments. By applying this method in the 2nd memoir, Navier reconfirmed the same so-called undefined equations as obtained in the 1st memoir, and further obtained the equations to be satisfied for the forces applied at the boundaries of the bodies in contact with the fluid.

Since Navier's main concern now was to know what would be the appropriate boundary conditions to be satisfied at the tube boundaries, for cases where the molecules of the walls exert a particular action upon those of the fluid, he then, in the 2nd memoir, applied the method of moments, obtaining again the N-S equation, but with new boundary conditions.

5. Expressions derived by Navier for forces arising from molecular-actions occurring in a fluid in a state of equilibrium by the method of moments

In *Mécanique Analytique* [20], Lagrange considers a body to be composed of infinitely small particles, each of which subject to forces P , Q , R , etc, arising from the action of gravity. These forces tend to drive the particles along the directions p , q , r , etc with infinitely small variations of ${}^8\delta p$, δq , δr , etc. The *moment* of each one of these forces was defined as the product of the force and the corresponding variation, e.g., $P\delta p$, $Q\delta q$, $R\delta r$, etc. The sum of the moments that acts on each particle was given by ${}^9(P\delta p + Q\delta q + R\delta r + \dots)$. Then, the integral $\int (P\delta p + Q\delta q + R\delta r + \dots)$ yielded the sum of the moments of all the forces of the system, which in general was expected to be equal to zero in the state of equilibrium¹⁰. This condition then led to the particular equations for the equilibrium of the system.

As for the application of the method of moments, we now present the derivation of the equations of equilibrium for an incompressible fluid as developed by Navier in the first part of the 2nd memoir [2]. In this derivation, Navier begins by considering two molecules (fluid particles) M and M' , the first of which is located at point (x, y, z) and the second placed in the vicinity of M at $(x + \alpha, y + \beta, z + \gamma)$. The Euclidean distance r between these two points, then, is $r = \sqrt{\alpha^2 + \beta^2 + \gamma^2}$. The repulsive force established between these two molecules depends on the location of point M as it must balance the pressure, which can vary in various parts of the fluid. This repulsive force depends on the distance r , and similar to all molecular actions, it decreases rapidly as the distance increases. The repulsive force, denoted as $f(r)$, depends on the coordinates (x, y, z) . Consequently, each fluid molecule M is subjected to similar forces that arise from each molecule M' that surrounds it. It is supposed that this molecule is also subjected to the accelerative forces (per unit volume) P , Q , and R in the x , y , and z directions, respectively.

When a minor motion (*virtual displacement*) is imprinted on the molecule M such that its components cor-

⁸ These are called *virtual displacements*, and are assumed to be infinitesimal along with no passage of time and constant applied forces. Strictly, they should also not violate the constraints of the system.

⁹ This sum is called *virtual work*

¹⁰ This is the so-called *principle of the virtual work*

responding to each orthogonal axis are δx , δy , and δz , the molecule M' will be displaced along the same axis, and the corresponding distances between these molecules will change according to

$$\delta\alpha = \frac{d\delta x}{dx}\alpha + \frac{d\delta x}{dy}\beta + \frac{d\delta x}{dz}\gamma, \tag{4}$$

$$\delta\beta = \frac{d\delta y}{dx}\alpha + \frac{d\delta y}{dy}\beta + \frac{d\delta y}{dz}\gamma, \tag{5}$$

$$\delta\gamma = \frac{d\delta z}{dx}\alpha + \frac{d\delta z}{dy}\beta + \frac{d\delta z}{dz}\gamma, \tag{6}$$

$$\delta r = \frac{\alpha\delta\alpha + \beta\delta\beta + \gamma\delta\gamma}{r}. \tag{7}$$

Therefore,

$$\delta r = \frac{1}{r} \left(\frac{d\delta x}{dx}\alpha^2 + \frac{d\delta x}{dy}\alpha\beta + \frac{d\delta x}{dz}\alpha\gamma + \frac{d\delta y}{dx}\alpha\beta + \frac{d\delta y}{dy}\beta^2 + \frac{d\delta y}{dz}\beta\gamma + \frac{d\delta z}{dx}\alpha\gamma + \frac{d\delta z}{dy}\beta\gamma + \frac{d\delta z}{dz}\gamma^2 \right). \tag{8}$$

The product $f(r) \cdot \delta r$ represents the sum of the moment of the force $f(r)$, acting between the two molecules M and M' , considered as acting on the point M and the moment of the same force acting on the point M' .

Next, Navier considers eight M' molecules around the molecule M , situated at the same distance r such that the relative coordinates (α, β, γ) have values that differ two-by-two only by the sign of one of the coordinates. Therefore, the sum of the eight values of the product $f(r) \cdot \delta r$ that correspond to these eight points is given by

$$\frac{8 \cdot f(r)}{r} \left(\frac{d\delta x}{dx}\alpha^2 + \frac{d\delta y}{dy}\beta^2 + \frac{d\delta z}{dz}\gamma^2 \right). \tag{9}$$

This latter expression is then integrated with respect to α , β , and γ in the $1/8^{\text{th}}$ part of the sphere, where these quantities have only positive values. To this end, the orthogonal coordinates α , β , and γ are changed to polar coordinates, where ψ represents the elevation angle with respect to the plane $\alpha\beta$ and φ is the azimuth angle measure from α such that

$$\alpha = r \cdot \cos \psi \cdot \cos \varphi, \tag{10}$$

$$\beta = r \cdot \cos \psi \cdot \sin \varphi, \tag{11}$$

$$\gamma = r \cdot \sin \psi. \tag{12}$$

Considering that the volume element is given by $dr \cdot d\psi \cdot d\varphi \cdot r^2 \cdot \cos \psi$, integrating¹¹ between the limits r from zero to infinity, ψ from zero to $\pi/2$, and φ from zero to $\pi/2$, yields

$$\frac{4\pi}{3} \int_0^\infty f(r) \cdot r^3 \cdot dr \cdot \left(\frac{d\delta x}{dx} + \frac{d\delta y}{dy} + \frac{d\delta z}{dz} \right), \tag{13}$$

¹¹ Navier's method of summing molecular virtual moments, replacing discrete summations by integrals, essentially presupposes the continuity of the fluid. This is the reproach addressed to Navier by Poisson, a purist of the molecular physics.

¹² Here Navier notes that the moments of the interior forces were considered twice, because the sum of the moments of the two forces that act along the line r , represented by $f(r) \cdot \delta r$, was counted for both the molecules M and M' . However, this in fact makes no difference in the final result, because the factor $1/2$ that should have been applied in the first term of this equation may be considered as being part of the quantity p , the absolute value of which always depends on the magnitude of the forces applied to the fluid.

which, on defining $\frac{4\pi}{3} \int_0^\infty f(r) \cdot r^3 \cdot dr = p$, gives

$$p \left(\frac{d\delta x}{dx} + \frac{d\delta y}{dy} + \frac{d\delta z}{dz} \right), \tag{14}$$

where p is a measure of the opposing resistance to the pressures that tends to approximate the fluid parts.

Above is the sum of the moments of the molecular forces that act between the molecule M and all the molecules that surround it.

Considering that geometrically, the molecule M defines a rectangular volume element with dimensions dx , dy , and dz , and the sum of the moments of all the forces acting in this volume element is given by $p \left(\frac{d\delta x}{dx} + \frac{d\delta y}{dy} + \frac{d\delta z}{dz} \right) + P\delta x + Q\delta y + R\delta z$. Integrating this last expression with respect to x , y , and z throughout the extent of the fluid and equating it to zero results in the equation that expresses the conditions of equilibrium of the system as¹²

$$\iiint \left[p \left(\frac{d\delta x}{dx} + \frac{d\delta y}{dy} + \frac{d\delta z}{dz} \right) + P\delta x + Q\delta y + R\delta z \right] dx dy dz = 0. \tag{15}$$

Integrating the first term of this equation by parts results in

$$\begin{aligned} & \iiint \left[\left(P - \frac{dp}{dx} \right) \delta x + \left(Q - \frac{dp}{dy} \right) \delta y + \left(R - \frac{dp}{dz} \right) \delta z \right] dx dy dz - \iint (p' \delta x' - p'' \delta x'') dy dz \\ & - \iint (p' \delta y' - p'' \delta y'') dx dz \\ & - \iint (p' \delta z' - p'' \delta z'') dx dy = 0, \end{aligned} \tag{16}$$

in which the prime and double primes in the surface integrals represent the quantities belonging to the limits of integration.

Therefore, we firstly have for the equilibrium conditions at any point in the interior of the fluid, the undefined equations

$$\frac{dp}{dx} = P, \quad \frac{dp}{dy} = Q, \quad \frac{dp}{dz} = R, \tag{17}$$

which imply that the expressions for the forces P , Q and R given as functions of x , y and z , should be partial differentials with respect to x , y , and z , respectively, of

the same function p of these coordinates. Therefore, the total differential of this function is

$$dp = Pdx + Qdy + Rdz, \tag{18}$$

and consequently,

$$p = \int (Pdx + Qdy + Rdz) + const. \tag{19}$$

In the case that no force is applied to the interior points of the fluid, the value of p must be constant in the entire fluid.

Secondly, with respect to the boundary conditions, we begin with points belonging to the free surface. The variation of the coordinates of each point is entirely arbitrary, so, each surface integral vanishes independently, and the only way for this to be possible is $p = 0$ on the free surface. Therefore, on the free surface

$$0 = \int (Pdx + Qdy + Rdz) + const. \tag{20}$$

and the resulting differential equation is

$$(Pdx + Qdy + Rdz) = 0, \tag{21}$$

implying that the net force resulting from the combination of the forces P , Q , and R that acts on each molecule belonging to the free surface should be directed towards the normal of this surface.

With respect to the points belonging to the fluid in contact with a solid surface, the molecules cannot move, and the relation between the variations δx , δy and δz is given by the equation

$$\delta x \cdot \cos l + \delta y \cdot \cos m + \delta z \cdot \cos n = 0, \tag{22}$$

where l , m , and n are the angles formed by a tangent planar to the solid surface. Consequently, these terms add to zero in the above surface integrals, and there is no particular condition for the pressure at the solid surfaces.

6. The derivation of the N–S equation by the method of moments in the 2nd memoir

In the second part of the 2nd memoir [2], the steps to determine the expressions for the forces arising from the molecular actions that take place in a state of fluid motion follow quite closely the basic structure presented in the previous section for the case of a fluid in equilibrium. Therefore, and also because of the forbidding appearance

of the expressions thus developed by Navier, we will only highlight the new assumptions and the main results, to omit highly voluminous calculation passages for the sake of economy of space and readers' time.

For the case of a moving fluid, Navier assumes that the repulsive actions of the molecules are increased or diminished by a quantity proportional to the velocity with which the distance of the molecules decreases or increases.

In contrast, for the case of a fluid in equilibrium, the repulsive force between two molecules is $f(r)$, and for the case of a moving fluid, $f(r) \cdot \mathbf{v}$ represents the force that exists between any two fluid molecules, where \mathbf{v} is the difference in velocity of the molecules at points M and M' , calculated along the line MM' , and is given by

$$\begin{aligned} \mathbf{v} = & \frac{\alpha}{r} \left(\frac{du}{dx} \alpha + \frac{du}{dy} \beta + \frac{du}{dz} \gamma \right) \\ & + \frac{\beta}{r} \left(\frac{dv}{dx} \alpha + \frac{dv}{dy} \beta + \frac{dv}{dz} \gamma \right) \\ & + \frac{\gamma}{r} \left(\frac{dw}{dx} \alpha + \frac{dw}{dy} \beta + \frac{dw}{dz} \gamma \right), \end{aligned} \tag{23}$$

where u , v , and w are the components of the velocity of the molecule M and as before, x , y , and z are the coordinates of point M , while $(x + \alpha, y + \beta, z + \gamma)$ are the coordinates of point M' .

The moment of the molecular force is given by $f(r) \cdot \mathbf{v} \delta \mathbf{v}$, where $\delta \mathbf{v}$ represents the difference in velocities between M and M' , calculated along the line joining these two molecules, when a virtual displacement (here a *virtual velocity*) is applied to the moving fluid. $\delta \mathbf{v}$ is

$$\begin{aligned} \delta \mathbf{v} = & \frac{\alpha}{r} \left(\frac{\delta du}{dx} \alpha + \frac{\delta du}{dy} \beta + \frac{\delta du}{dz} \gamma \right) \\ & + \frac{\beta}{r} \left(\frac{d\delta v}{dx} \alpha + \frac{d\delta v}{dy} \beta + \frac{d\delta v}{dz} \gamma \right) \\ & + \frac{\gamma}{r} \left(\frac{\delta dw}{dx} \alpha + \frac{\delta dw}{dy} \beta + \frac{\delta dw}{dz} \gamma \right). \end{aligned} \tag{24}$$

Let's remember that in the previous section, for the case of a fluid in a state of equilibrium, eight M' molecules were situated at the same distance r around the molecule M such that the relative coordinates α , β , and γ had values that differed two-by-two only by the sign of one of the coordinates, and an expression was obtained for the sum of the eight values of the product $f(r) \cdot \mathbf{v} \delta \mathbf{v}$. Changing the orthogonal coordinates α , β , and γ to polar coordinates, and integrating this expression in the $1/8^{th}$ part of the sphere results in

$$\epsilon \left\{ \begin{aligned} & 3 \frac{du}{dx} \frac{\delta du}{dx} + \frac{du}{dy} \frac{\delta du}{dy} + \frac{du}{dz} \frac{\delta du}{dz} + \frac{dv}{dy} \frac{\delta du}{dx} + \frac{dv}{dx} \frac{\delta du}{dy} + \frac{dw}{dz} \frac{\delta du}{dx} + \frac{dw}{dx} \frac{\delta du}{dz} \\ & \dots\dots\dots + 3 \frac{dv}{dy} \frac{\delta dv}{dy} + \dots\dots\dots + 3 \frac{dw}{dz} \frac{\delta dw}{dz} \end{aligned} \right\}, \tag{25}$$

where $\epsilon = \frac{8\pi}{30} \int_0^\infty f(r) \cdot r^4 \cdot dr$.

Integrating this last expression by parts with respect to x , y , and z throughout the extent of the fluid gives¹³

$$\begin{aligned}
 & -\epsilon \iiint dx dy dz \left[\left(3 \frac{d^2 u}{dx^2} + \frac{d^2 u}{dy^2} + \frac{d^2 u}{dz^2} + 2 \frac{d^2 v}{dx dy} + 2 \frac{d^2 w}{dx dy} \right) \delta u + \left(2 \frac{d^2 u}{dx dy} + \frac{d^2 v}{dx^2} + 3 \frac{d^2 v}{dy^2} + \frac{d^2 v}{dz^2} + 2 \frac{d^2 w}{dy dz} \right) \delta v + \right. \\
 & \left. \left(2 \frac{d^2 u}{dx dz} + 2 \frac{d^2 v}{dy dz} + \frac{d^2 w}{dx^2} + \frac{d^2 w}{dy^2} + 3 \frac{d^2 w}{dz^2} \right) \delta w \right] \\
 & + \text{surface integral terms evaluated at the limits of integration.} \tag{26}
 \end{aligned}$$

By invoking the continuity equation $\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0$, the last integral transforms into

$$\begin{aligned}
 & -\epsilon \iiint dx dy dz \left[\left(\frac{d^2 u}{dx^2} + \frac{d^2 u}{dy^2} + \frac{d^2 u}{dz^2} \right) \delta u + \left(\frac{d^2 v}{dx^2} + \frac{d^2 v}{dy^2} + \frac{d^2 v}{dz^2} \right) \delta v + \left(\frac{d^2 w}{dx^2} + \frac{d^2 w}{dy^2} + \frac{d^2 w}{dz^2} \right) \delta w \right] \\
 & + \text{surface integral terms evaluated at the limits of integration.} \tag{27}
 \end{aligned}$$

This is the expression for the moments of forces arising from the molecular actions that occur in the state of fluid motion.

Equating to zero, the sum of the moments of the forces applied to all the molecules in the interior (no surface integral terms) of an incompressible fluid in the state of motion gives

$$\iiint dx dy dz \left\{ \begin{aligned} & \left[P - \frac{dp}{dx} - \rho \left(\frac{du}{dt} + u \frac{du}{dx} + v \frac{du}{dy} + w \frac{du}{dz} \right) + \epsilon \left(\frac{d^2 u}{dx^2} + \frac{d^2 u}{dy^2} + \frac{d^2 u}{dz^2} \right) \right] \delta u + \\ & \left[Q - \frac{dp}{dy} - \rho \left(\frac{dv}{dt} + u \frac{dv}{dx} + v \frac{dv}{dy} + w \frac{dv}{dz} \right) + \epsilon \left(\frac{d^2 v}{dx^2} + \frac{d^2 v}{dy^2} + \frac{d^2 v}{dz^2} \right) \right] \delta v + \\ & \left[R - \frac{dp}{dz} - \rho \left(\frac{dw}{dt} + u \frac{dw}{dx} + v \frac{dw}{dy} + w \frac{dw}{dz} \right) + \epsilon \left(\frac{d^2 w}{dx^2} + \frac{d^2 w}{dy^2} + \frac{d^2 w}{dz^2} \right) \right] \delta w \end{aligned} \right\} = 0. \tag{28}$$

Finally, the three components of the equation of motion, in the x , y , and z directions, for an incompressible fluid subjected to external forces P , Q , and R , and the repulsive molecular forces proportional to ϵ are

$$\begin{aligned}
 \rho \left(\frac{du}{dt} + u \frac{du}{dx} + v \frac{du}{dy} + w \frac{du}{dz} \right) &= P - \frac{dp}{dx} + \epsilon \left(\frac{d^2 u}{dx^2} + \frac{d^2 u}{dy^2} + \frac{d^2 u}{dz^2} \right), \\
 \rho \left(\frac{dv}{dt} + u \frac{dv}{dx} + v \frac{dv}{dy} + w \frac{dv}{dz} \right) &= Q - \frac{dp}{dy} + \epsilon \left(\frac{d^2 v}{dx^2} + \frac{d^2 v}{dy^2} + \frac{d^2 v}{dz^2} \right), \\
 \rho \left(\frac{dw}{dt} + u \frac{dw}{dx} + v \frac{dw}{dy} + w \frac{dw}{dz} \right) &= R - \frac{dp}{dz} + \epsilon \left(\frac{d^2 w}{dx^2} + \frac{d^2 w}{dy^2} + \frac{d^2 w}{dz^2} \right),
 \end{aligned} \tag{29}$$

which are seen to be the three components of the Navier–Stokes equation for an incompressible fluid.

The surface integrals in equation (27), and the surface integral given by ¹⁴ $\iint E(u\delta u + v\delta v + w\delta w)$ will give the conditions to be satisfied at the boundaries, whether they are for an empty space or a solid surface, where $\frac{4\pi}{6} \int_0^\infty f(r) \cdot r^2 \cdot dr = E$, which is a quantity that depends on the nature of the surface in contact with the fluid. It is understood that for an empty space, $E = 0$.

7. Navier’s new boundary conditions in the 2nd memoir

In the 2nd memoir [2], after a lengthy evaluation of the surfaces integrals, Navier summarizes the applicable con-

ditions for the cases where the solid surfaces are perpendicular to a particular axis. For these cases, the equations to be satisfied at these boundaries are given by

$$Eu + \epsilon \frac{du}{dz} = 0, \quad Ev + \epsilon \frac{dv}{dz} = 0, \tag{30}$$

when the surface is perpendicular to the z -axis.

$$Eu + \epsilon \frac{du}{dy} = 0, \quad Ew + \epsilon \frac{dw}{dy} = 0, \tag{31}$$

when the surface is perpendicular to the y -axis.

$$Ev + \epsilon \frac{dv}{dx} = 0, \quad Ew + \epsilon \frac{dw}{dx} = 0, \tag{32}$$

when the surface is perpendicular to the x -axis.

¹³ The following transformations should be made before integration by parts: $\frac{du}{dx} \frac{\delta du}{dx} = \frac{du}{dx} \frac{d\delta u}{dx}$, $\frac{dv}{dy} \frac{\delta dv}{dy} = \frac{dv}{dy} \frac{d\delta v}{dy}$, $\frac{dw}{dz} \frac{\delta dw}{dz} = \frac{dw}{dz} \frac{d\delta w}{dz}$, etc., that is, the derivative of an infinitesimal variation is equal to the infinitesimal variation of the derivative. This is an important step because it reveals that after the due integrations, the viscous forces become proportional to the second derivative of the velocity field.

¹⁴ Although this is a rather simple result, the derivation of this expression, which gives the condition to be satisfied at a solid surface, is quite lengthy and will not be presented here.

From these expressions, Navier provides a physical interpretation of the nature of the quantities ϵ and E . For this purpose, he considers a fluid flow where all molecules move along layers parallel to a given plane. In each layer, parallel to this plane, the molecules have the same velocity, but the velocities increase uniformly and progressively from layer to layer as the distance from the plane increases, such that two layers are apart by a linear unit and the difference in velocities is also equal to a linear unit. Under such a hypothesis, the quantity ϵ represents, in units of force, the resistance that arises from the sliding of two layers of fluid one over the other, per unit surface area. This is indeed what is currently known as the viscous stress, where ϵ is recognized as the dynamic viscosity. Similarly, E represents the resistance in units of force per unit of surface area that arises from the sliding of the layer of fluid in contact with a solid surface, which according to Navier, would depend on the nature of the surface in contact with the fluid.

The term $E u_i$ in these expressions implies a fluid–solid slip, which was later proved to be incorrect. Moreover, there is no such thing as a coefficient that depends on the nature of the surface in contact with the fluid, as E would represent. Nonetheless, the term $\epsilon \frac{\partial u_i}{\partial x_j}$ is the only term that should subsist at a solid boundary. This is in fact the viscous stress τ_{ij} that arises not only at a fluid–solid interface, but also from the sliding of two parallel layers of molecules in the inner parts of the fluid, although, in the former case, the non-slip condition (zero velocity) at a stationary solid boundary should still hold. This was the first time in the history of fluid mechanics that Newton’s law of viscosity was formally established¹⁵.

In the last part of the 2nd memoir, Navier presents two applications of his theory, one for a tube with square cross section, with side R , and another for a tube with circular cross section, with radius R . For both cases, in the limit of narrow tubes, he found the average velocity to be given by $= \rho g \frac{R h_f}{2 E l}$, that is, proportional to the tube radius R , and finally, reconciling with Girard’s formula. He goes on to comment that the average velocity is seen to be independent of the viscosity ϵ , and to depend only on the adherence between the fluid and the tube wall, characterized by the constant E . As the diameter of the tube increases, the effect of the viscosity ϵ should increase accordingly, and for sufficiently large diameters, the viscosity ϵ would solely determine the fluid average velocity.

An estimate for $E = 0.234$ units is then given and interpreted by Navier based on Girard’s experimental

results, for the flow of water inside a capillary of copper. This value would represent in units of weight, the resistance necessary to overcome the friction of a fluid layer, flowing over a wall with a velocity equal to a linear unit, for a layer with surface area equal to a linear unit. Therefore, the resistance from the friction of a layer of water of one square meter, flowing over copper with a velocity of one meter per second, at a temperature of 12°C is approximately $1/4$ kg.

As for the value of ϵ , according to Navier it would be difficult to conduct experiments in order to find its value with sufficient exactitude because capillary tubes are not proper¹⁶ and because with larger pipes, it would be hard to be certain that the motion is exactly linear¹⁷.

8. The aftermath

Probably discouraged by the fact that his theory could only be applied to slow motion in capillaries (laminar flows), Navier never returned to his theory of viscous flow. In the hydraulic session of his course at the École des Ponts et Chaussées [21], he presents the formula for the velocity in narrow tubes proportional to the tube diameter, for which an estimate for the fluid–surface coefficient E was obtained from “Girard’s very curious experiments”. However, for the cases of ordinary applications he deemed the result of experiments as the only guide. He then turned to the experiment-based formulation of M. Prony¹⁸ for the calculation of the average velocities in viscous flows in tubes of lengths at least 400 times the diameter, in which the velocities do not surpass 2.5 m/s.

The correct dependence of the average velocity proportional to the square of the tube diameter, anticipated by Navier in the 1st memoir on purely theoretical grounds and by the adoption of a non-slip boundary condition, had to wait the experimental confirmations by Hagen, published in 1839 [17], and by Poiseuille, published in 1844 [18]¹⁹. Unfortunately, Navier, who died in 1836, did not live long enough to know these results.

9. Conclusions

The development of the Navier–Stokes equation by Navier shows his great ability as a scientist because, by undergirding his theory on Laplace’s new concept of molecular forces, thought to be also capable of capturing the effects of viscosity, Navier managed to reach for the first time the ultimate equation for the laminar motion of

¹⁵ $\tau_{ij} = \epsilon \frac{\partial u_i}{\partial x_j}$ is the constitutive relation for a Newtonian fluid, popularly known as Newton’s law of viscosity. Nonetheless, Newton never formulated this law in such terms, and never proposed a mathematical expression for it.

¹⁶ Because in capillaries, the coefficient E is dominant over ϵ

¹⁷ Navier here refers to the flow not being laminar

¹⁸ Gaspard Clair François Marie Riche de Prony (1755-1839) was a French mathematician and engineer who worked on hydraulics.

¹⁹ Hagen discovered his law without knowledge of Girard’s conflicting results. Poiseuille did not consider the capillaries used by Girard narrow enough, and used instead capillaries one hundred times narrower than Girard’s. Both Hagen and Poiseuille used very meticulous experimental protocols, which eliminated important sources of errors that had escaped Girard’s attention. All these factors seem to justify Girard’s wrong dependence of average velocity on tube diameter.

real viscous fluids. This seems to be the case of a fragile model that was capable of generating a true prediction, in comparison to the more rigorous models of the other developers of the Navier–Stokes equation.

Navier also reveals strong analytical skills in the use of Lagrange’s method of moments. His concern of confronting the theory with available experimental data also shows an engineer interested in the practical use of his contributions. However, it has been recognized that Navier’s derivation of the Navier–Stokes equation was not influential and has since been neglected in specialized publications on the subject. That fact that his theory of fluid motion could only be applied to slow motion in capillaries (laminar flows) may have later discouraged Navier, who abandoned it in favour of experiment-based formulations for ordinary applications.

Revisiting the two memoirs authored by Navier allowed the identification of current concepts on the topic of viscous fluid flow that have his imprint, but whose authorship has often been either overlooked by Navier scholars, taken for granted, or even customarily not directly attributed to him.

- By distinguishing between the relative motion and the common motion of molecules, he may be considered the precursor of the concepts of boundary layer and outer potential flow, which according to the established view, was only introduced much later, in the beginning of the 20th century.
- His ideas about molecular forces – the consideration of the independence of pressure from viscous stresses in incompressible flows, which translates into the equality between mechanical and thermodynamic pressure – are remarkable, considering his fragile molecular model.
- His principle for the viscous forces for fluids in motion being proportional to the relative velocities of the fluid particles redeemed the old Newtonian principle for imperfect fluidity, and provided the main undergird for his theory.
- Navier anticipated on purely theoretical grounds what later became known as the Hagen–Poiseuille law. Both Hagen and Poiseuille were able to later confirm experimentally Navier’s theoretical findings.
- Navier should also be acknowledged for establishing the constitutive relation for a Newtonian fluid for the first time, in the form of the so-called Newton’s law of viscosity.
- His expression for the fluid-solid slip boundary condition, characterized by a two constant model, opened up new perspectives for the quest for the conditions at the interface of a fluid in contact with a solid wall (more on that in the next section).

Supplementary material

The following online material is available for this article:

On the conditions at the interface of a fluid in contact with a solid wall

References

- [1] C.L. Navier, *Annales de Chimie et de Physique*, **XIX**, 244 (1821).
- [2] C.L. Navier, *Mémoires de L’Académie Royale des Sciences de L’Institut de France*, **VI**, 389 (1827).
- [3] L. Euler, *Mémoires de l’Académie des Sciences*, Berlin, **11**, 274 (1757).
- [4] L. Euler, *Novi Commentarii Academiae Scientiarum Petropolitanae*, Berlin, **6**, 338 (1761).
- [5] S.R. Bistafa, *Eur. Phys. J. H* **40**, 375 (2015).
- [6] C.L. Navier, *Mémoires de L’Académie Royale des Sciences de L’Institut de France*, **VII**, 376 (1827).
- [7] A.L. Cauchy, *Exercices de Mathématiques* (Chez de Bure Frères, Paris, 1828).
- [8] S.D. Poisson, *Journal de L’École Royale Polytechnique* **XX**, read at the Academy of Sciences on 12th October (1829).
- [9] A.B. Saint-Venant, *Comptes Rendus Hebdomadaires des Séances de l’Académie des Sciences*, 1240 (1843).
- [10] G.G. Stokes, *On Some Cases of Fluid Motion. Mathematical and Physical Papers by George Gabriel Stokes* (Cambridge University Press, Cambridge, 1880), pp. 17-68.
- [11] G.G. Stokes, *On the Theories of the Internal Friction of Fluids in Motion, and of the Equilibrium and Motion of Elastic Solids. Mathematical and Physical Papers by George Gabriel Stokes* (Cambridge University Press, Cambridge, 1880), pp. 75-129.
- [12] G.G. Stokes, *On the Effect of the Internal Friction of Fluids on the Motion of Pendulums. Mathematical and Physical Papers by George Gabriel Stokes* (Cambridge University Press, Cambridge, 1901), pp. 1-15.
- [13] R. Dugas, *A History of Mechanics* (Central Book Company, New York, 1955).
- [14] I. Grattan-Guinness, *Convolution in French Mathematics, 1800-1840. The Turns*, . (Birkhäuser, Basileia, 1990), v. 2.
- [15] O. Darrigol, *Worlds of Flow: A History of Hydrodynamics From the Bernoullis to Prandtl* (Oxford University Press, Oxford, 2005).
- [16] P.S. Girard, *Mémoires de la Classe des Sciences Mathématiques et Physiques de L’Institut de France*, p. 249 (1813, 1814, 1815).
- [17] G. Hagen, *Annalen der Physik* **46**, 423 (1839).
- [18] J.L. Poiseuille, *Extrait du Tome IX des Savants Étrangers de la Académie des Sciences, Paris* (1844).
- [19] P.S. Girard, *Mémoires de L’Académie Royale des Sciences de L’Institut de France, Paris* (1816), p. 187-274.
- [20] J.L. Lagrange, *Mécanique Analytique* (Éditions Jacques Gabay, Sceaux, 1989).
- [21] C.L. Navier, *Résumé des Leçons Données à l’École des Ponts et Chaussées sur l’Application de la Mécanique à l’Établissement des Constructions et des Machines. Deuxième Partie, Leçons sur le Mouvement et la Résistance des Fluides, la Conduite et la Distribution des Eaux* (Chez Carilian-Goeury, Paris, 1838).