APPENDIX A: THE POSITION OF THE CM OF A SYSTEM OF PARTICLES WITH VARIABLE MASSES

Consider a system of \( N \) particles with the vector position of their CM, \( \vec{R}_{CM}(t) \),

\[
\vec{R}_{CM}(t) = \frac{m_1 \vec{r}_1(t) + m_2 \vec{r}_2(t) + \ldots + m_N \vec{r}_N(t)}{m_1 + m_2 + \ldots + m_N} = \sum_{i=1}^{N} \frac{m_i \vec{r}_i(t)}{\sum_{i=1}^{N} m_i} \tag{A1}
\]

\[
\equiv X_{CM}(t) \hat{i} + Y_{CM}(t) \hat{j} + Z_{CM}(t) \hat{k}, \tag{A2}
\]

in which \( m_i, i \in \{1, 2, \ldots, N\} \) is the mass of the \( i \)-th particle and

\[
\vec{r}_i(t) = x_i(t) \hat{i} + y_i(t) \hat{j} + z_i(t) \hat{k}, \tag{A3}
\]

its position vector, expressed in some orthonormal basis \( \{\hat{i}, \hat{j}, \hat{k}\} \).

Correspondingly, we may write

\[
\vec{R}_i(t) = \sum_{i=1}^{N} \frac{m_i x_i(t)}{\sum_{i=1}^{N} m_i} \hat{i} + \sum_{i=1}^{N} \frac{m_i y_i(t)}{\sum_{i=1}^{N} m_i} \hat{j} + \sum_{i=1}^{N} \frac{m_i z_i(t)}{\sum_{i=1}^{N} m_i} \hat{k}, \tag{A4}
\]

\[
= X_{CM}(t) \hat{i} + Y_{CM}(t) \hat{j} + Z_{CM}(t) \hat{k}. \tag{A5}
\]

Let us consider at first one of the components of \( \vec{r}_i(t) \), say, \( x_i(t) \). Let \( \overline{x}(t) \) and \( \underline{x}(t) \) be the maximum and minimum values of this component at time \( t \), respectively,

\[
\overline{x}(t) \equiv \max_{j=1}^{N} \left( x_j(t) \right), \quad \underline{x}(t) \equiv \min_{j=1}^{N} \left( x_j(t) \right), \tag{A6}
\]

in which all particles in the set are considered.

If \( \overline{x}(t) \) is the maximum of all components \( x_i(t) \) at time \( t \), it means that, for all \( i \in \{1, 2, \ldots, N\} \),

\[
\overline{x}(t) - x_i(t) \geq 0. \tag{A7}
\]

Multiplying this inequality by any non-negative number, say \( m_i \) for convenience, yields

\[
m_i \left( \overline{x}(t) - x_i(t) \right) \geq 0. \tag{A8}
\]

Hence

\[
\sum_{i=1}^{N} m_i \left( \overline{x}(t) - x_i(t) \right) \geq 0. \tag{A9}
\]

Due to linearity,

\[
\overline{x}(t) \sum_{i=1}^{N} m_i - \sum_{i=1}^{N} m_i x_i(t) \geq 0, \tag{A10}
\]

which can be solved for \( \overline{x}(t) \)

\[
\overline{x}(t) \geq \frac{\sum_{i=1}^{N} m_i x_i(t)}{\sum_{i=1}^{N} m_i}. \tag{A11}
\]

On the other hand, if \( \underline{x}(t) \) is the minimum of all components \( x_i(t) \), it means that, for all \( i \in \{1, 2, \ldots, N\} \),

\[
x_i(t) - \underline{x}(t) \geq 0 \tag{A12}
\]
and, by similar reasoning, we obtain
\[ \frac{\sum_{i=1}^{N} m_i x_i(t)}{\sum_{i=1}^{N} m_i} \geq \bar{x}(t). \] (A13)

Hence,
\[ \bar{x}(t) \leq \frac{\sum_{i=1}^{N} m_i x_i(t)}{\sum_{i=1}^{N} m_i} \leq \bar{x}(t) \Rightarrow \bar{x}(t) \leq X_{CM}(t) \leq \bar{x}(t), \] (A14)

the \( x \)-component of \( \vec{R}_{CM}(t) \) lies between the minimum \( \bar{x}(t) \) and maximum \( \bar{x}(t) \) values of that component, all particles considered. Analogous results are obtained for the remaining components of \( \vec{R}_{CM}(t) \),
\[ y(t) \leq \frac{\sum_{i=1}^{N} m_i y_i(t)}{\sum_{i=1}^{N} m_i} \leq \bar{y}(t) \Rightarrow y(t) \leq Y_{CM}(t) \leq \bar{y}(t), \] (A15)
\[ z(t) \leq \frac{\sum_{i=1}^{N} m_i z_i(t)}{\sum_{i=1}^{N} m_i} \leq \bar{z}(t) \Rightarrow z(t) \leq Z_{CM}(t) \leq \bar{z}(t), \] (A16)

in which
\[ \bar{y}(t) \equiv \max_{j=1}^{N} \left( y_j(t) \right), \quad \bar{y}(t) \equiv \min_{j=1}^{N} \left( y_j(t) \right), \] (A17)
\[ \bar{z}(t) \equiv \max_{j=1}^{N} \left( z_j(t) \right) \quad \text{and} \quad \bar{z}(t) \equiv \min_{j=1}^{N} \left( z_j(t) \right). \] (A18)

Notice that no assumptions on \( m_i \) have been made other than \( m_i \geq 0 \). The \( m_i \) are not necessarily constant, for instance.

We conclude that \( \dot{R}_{CM}(t) \in \Sigma(t) \) in which \( \Sigma(t) \subset \mathbb{R}^3 \) is the (basis-dependent) parallelepiped at time \( t \), defined by the maximum and minimum values of all components,
\[ \Sigma(t) = \left\{ (x, y, z) \in \mathbb{R}^3 \mid x \in [\bar{x}(t), \bar{x}(t)] \wedge y \in [\bar{y}(t), \bar{y}(t)] \wedge z \in [\bar{z}(t), \bar{z}(t)] \right\}. \] (A19)