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On the Foucault pendulum

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The Foucault pendulum is a physical apparatus that shows the Earth's rotation through the variation of the plane of oscillation over time. The main physical agent responsible for this phenomenon is the non-inertial Coriolis force, which couples the coordinates in the equation of motion. Although the Foucault pendulum is more than 170 years old, its theoretical description still lacks completeness, as there is no exact vector solution to the problem or even a decoupled equation of motion. Furthermore, the main prediction about the Foucault pendulum, the dependence of the pendulum's plane of oscillation rotation on the latitude, needs to be revised as such dependence is more complex than is shown in the literature, once it depends essentially on the initial conditions. Here all these points raised are successfully answered.

Keywords: Foucault pendulum, high-order differential equations, exact solution.

1. Introduction

The French physicist Jean Bernard Léon Foucault installed, in 1851 in the city of Paris, a simple pendulum whose string length was very large compared to the height reached by its bob [1]. The suspension point of this pendulum was fixed relative to a reference system located on the Earth's surface. As the Earth rotated, the suspension point followed such motion and, as a result, the plane of oscillation of the pendulum changed over time. This is a historical experiment that directly demonstrated, without requiring astronomical observations, that the succession of days and nights was due to the Earth's rotational motion. Such a device was then called Foucault pendulum. Another interesting consequence of this experiment is the possibility of estimating the latitude of a location, by measuring the angular displacement of the pendulum's plane of oscillation over the course of a day.

The physical-mathematical treatment of Foucault pendulum is generally done within the subject of motion in non-inertial frames, specifically to the motion relative to the Earth [2–5]. However, this treatment is done without finding a decoupled equation of motion and generally appeals to a scalar approach using vector components to solve the equation of motion in an approximative way.

Here I present a fully vector approach of decoupling the equation of motion and obtaining the exact solution of the Foucault pendulum problem for generic initial conditions within Coriolis approximation. The rotation of the pendulum's plane of oscillation is also investigated and I show that the angular displacement of this plane depends essentially on the initial conditions, predicting that even if a Foucault pendulum is exactly over the Equator, there may be a rotation of its oscillation plane.

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2. Equation of Motion and General Solution

For motions in a non-inertial frame, Coriolis acceleration $2\dot{\boldsymbol{r}} \times \boldsymbol{\omega}$ is more relevant compared to centrifugal terms and Euler acceleration, leading to the following equation of motion [2]:

$$\ddot{\boldsymbol{r}} = \frac{\boldsymbol{T}}{m} + \boldsymbol{g} + 2\dot{\boldsymbol{r}} \times \boldsymbol{\omega} \tag{1}$$

where $\boldsymbol{T}$ is the tension force of the string on the bob, $\boldsymbol{g}$ is the Earth's gravitational field, and $\boldsymbol{\omega}$ is the (constant) Earth's rotation angular vector velocity.

The system is represented by Fig. 1, from which the relation $\boldsymbol{L} = \boldsymbol{r} - L\boldsymbol{e}_z$ is taken. The orientation of the Cartesian axes in Fig. 1 is such that the z-axis points vertically, so that $\boldsymbol{g} = -g\boldsymbol{e}_z$. The orientations of the x and y axes in geographical terms are properly presented in the next section.

The tensile force $\boldsymbol{T}$ on the pendulum bob points along the direction of the string in the opposite direction to the vector $\boldsymbol{L}$, and can then be written as

$$\boldsymbol{T} = -T\frac{\boldsymbol{L}}{L} = -T\frac{\boldsymbol{r} - L\boldsymbol{e}_z}{L} \tag{2}$$

In general, the modulus of the tensile force $T = T(\boldsymbol{r}, \dot{\boldsymbol{r}})$ is a nonlinear function of $\boldsymbol{r}$ and $\dot{\boldsymbol{r}}$. Under the condition of a very large wire length compared to vertical displacements of the pendulum bob, the nonlinear terms of $T(\boldsymbol{r}, \dot{\boldsymbol{r}})$ can be neglected and the modulus of the

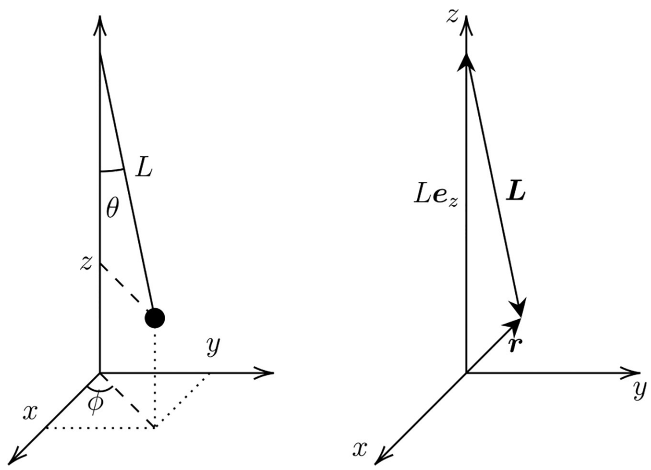

Figure 1: Representation of the pendulum (left) and vectors (right) that locate the points of interest: pendulum suspension point from the origin of the reference frame ($L\boldsymbol{e}_z$) and position of the bob evaluated from the reference system ($\boldsymbol{r}$) and also from the pendulum suspension point ($\boldsymbol{L}$).

tensile force can be approximated to $T = mg$ [1, 3, 5], leaving Eq. (1) in a simpler form

$$\ddot{\boldsymbol{r}} + \omega_0^2 \boldsymbol{r} = 2\dot{\boldsymbol{r}} \times \boldsymbol{\omega} \tag{3}$$

where $\omega_0^2 = g/L$. Eq. (3) is an equation of motion for a forced simple harmonic oscillator whose external agent is the non-inertial Coriolis force, which causes the observable effects on the Foucault pendulum [6]. The equation of motion (3) has the coordinates coupled due to the non-inertial term given by the cross product.

To decouple the equation of motion (3) it is necessary to eliminate the cross product. This procedure can be done by increasing the order of the differential equation. By calculating the successive derivatives of the equation of motion (3) until the sixth order and keeping all expressions in terms of the four cross products $\boldsymbol{r} \times \boldsymbol{\omega}$, $\dot{\boldsymbol{r}} \times \boldsymbol{\omega}$, $\boldsymbol{\omega} \times (\boldsymbol{r} \times \boldsymbol{\omega})$, and $\boldsymbol{\omega} \times (\dot{\boldsymbol{r}} \times \boldsymbol{\omega})$, a system of five linear vector equations if obtained, which can be solved exactly by considering $\boldsymbol{r}^{(6)}$ and all four different types of cross products as unknowns, resulting in the following decoupled equation of motion:

$$\boldsymbol{r}^{(6)} + (3\omega_0^2 + 4\omega^2)(\boldsymbol{r}^{(4)} + \omega_0^2 \ddot{\boldsymbol{r}}) + \omega_0^6 \boldsymbol{r} = 0 \tag{4}$$

which is the decoupled equation of motion of the Foucault pendulum within the Coriolis approximation. Its general solution is obtained from simple methods, as through the characteristic equation, leading to

$$\begin{aligned} \boldsymbol{r}(t) &= \boldsymbol{c}_0 \cos(\omega_0 t) + \boldsymbol{c}_1 \sin(\omega_0 t) + \boldsymbol{c}_2 \cos(\omega_1 t) \\ &\quad + \boldsymbol{c}_3 \sin(\omega_1 t) + \boldsymbol{c}_4 \cos(\omega_2 t) \\ &\quad + \boldsymbol{c}_5 \sin(\omega_2 t) \end{aligned} \tag{5}$$

where the $\boldsymbol{c}_j$ are arbitrary vector constants and $\omega_{1,2} = \sqrt{\omega_0^2 + \omega^2} \pm \omega$.

By using generic initial conditions $\boldsymbol{r}(0) = \boldsymbol{r}_0$ and $\dot{\boldsymbol{r}}(0) = \boldsymbol{v}_0$, after some vector algebra, the constants $\boldsymbol{c}_j$ are given by

$$\begin{cases} \boldsymbol{c}_0 = \boldsymbol{r}_0 - \dfrac{\boldsymbol{\omega} \times (\boldsymbol{r}_0 \times \boldsymbol{\omega})}{\omega^2} \\ \boldsymbol{c}_1 = \dfrac{1}{\omega_0}\left[\boldsymbol{v}_0 - \dfrac{\boldsymbol{\omega} \times (\boldsymbol{v}_0 \times \boldsymbol{\omega})}{\omega^2}\right] \\ \boldsymbol{c}_2 = \dfrac{1}{\omega_1 + \omega_2} \\ \qquad \times \left[-\dfrac{\boldsymbol{v}_0 \times \boldsymbol{\omega}}{\omega} + \left(\dfrac{\omega_1 + \omega_2}{2} - \omega\right)\dfrac{\boldsymbol{\omega} \times (\boldsymbol{r}_0 \times \boldsymbol{\omega})}{\omega^2}\right] \\ \boldsymbol{c}_3 = \dfrac{1}{\omega_1 + \omega_2} \\ \qquad \times \left[\left(\dfrac{\omega_1 + \omega_2}{2} - \omega\right)\dfrac{\boldsymbol{r}_0 \times \boldsymbol{\omega}}{\omega} + \dfrac{\boldsymbol{\omega} \times (\boldsymbol{v}_0 \times \boldsymbol{\omega})}{\omega^2}\right] \\ \boldsymbol{c}_4 = \dfrac{1}{\omega_1 + \omega_2} \\ \qquad \times \left[\dfrac{\boldsymbol{v}_0 \times \boldsymbol{\omega}}{\omega} + \left(\dfrac{\omega_1 + \omega_2}{2} + \omega\right)\dfrac{\boldsymbol{\omega} \times (\boldsymbol{r}_0 \times \boldsymbol{\omega})}{\omega^2}\right] \\ \boldsymbol{c}_5 = \dfrac{1}{\omega_1 + \omega_2} \\ \qquad \times \left[-\left(\dfrac{\omega_1 + \omega_2}{2} + \omega\right)\dfrac{\boldsymbol{r}_0 \times \boldsymbol{\omega}}{\omega} + \dfrac{\boldsymbol{\omega} \times (\boldsymbol{v}_0 \times \boldsymbol{\omega})}{\omega^2}\right] \end{cases} \tag{6}$$

which uniquely determines the solution $\boldsymbol{r}(t)$ in terms of the initial conditions.

3. Rotation of the Oscillation Plane

The main phenomenon associated with the Foucault pendulum is the rotation of the plane of oscillation over time. With the exact solution of the Foucault pendulum problem given in (5) and (6), it is possible to precisely find the time dependence of this phenomenon and determine all the factors that influence it.

The investigation of the rotation of the plane oscillation rotation becomes simpler if we deal with the angular coordinates (θ, ϕ) given in Fig. 1, where θ is the angle that the string makes with the vertical and ϕ is the angle that the horizontal projection of $\boldsymbol{r}$ makes with the x-axis, that is, the angle of the plane of oscillation. The relationship of these angles with the Cartesian coordinates (x, y, z) of $\boldsymbol{r}$ is given by

$$\begin{cases} x = L\sin(\theta)\cos(\phi) \\ y = L\sin(\theta)\sin(\phi) \\ z = L(1 - \cos(\theta)) \end{cases} \tag{7}$$

To complete the description of the movement in terms of coordinates, it is necessary to determine the

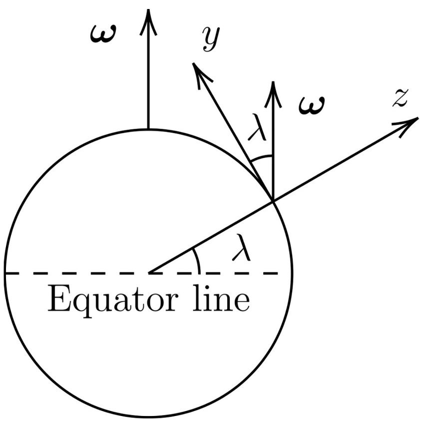

Figure 2: Representation of the Earth's rotation with the Cartesian axes of the non-inertial frame located on the planet's surface. The latitude λ is positive in the northern hemisphere.

components of $\boldsymbol{\omega}$. Figure 2 outlines the alignment of the Cartesian axes of the non-inertial frame located on the planet's surface: the x-axis always points in the direction of a parallel and the y-axis in the direction of a meridian, begin positive pointing to the North Pole. According to the representation in Fig. 2 of the Earth rotation, $\boldsymbol{\omega}$ is given by

$$\boldsymbol{\omega} = \omega\cos(\lambda)\boldsymbol{e}_y + \omega\sin(\lambda)\boldsymbol{e}_z \tag{8}$$

where λ is the latitude, evaluated from the center of the Earth and positive in the northern hemisphere.

According to Eqs. (7), $\phi(t) = \arctan(y(t)/x(t))$. However, even with the bob starting from rest, $\boldsymbol{v}_0 = \boldsymbol{0}$, the exact solution of this movement becomes too complicated due to the various vector terms in (6). Thus, some conditions need to be imposed to arrive at a treatable analytical solution such as

$$\theta \ll 1, \quad \omega \ll \omega_0 \tag{9}$$

where both have already been implicitly adopted to justify $T = mg$ [1, 3, 5].

By expanding $\phi(t) = \arctan(y(t)/x(t))$ in Taylor series to first order in both ω and $\theta_0 = \theta(0)$ and considering that the bob starts from rest, the following result is reached:

$$\Delta\phi(t) = \phi(t) - \phi_0 = -\left(\sin(\lambda) - \frac{\theta_0}{2}\sin(\phi_0)\cos(\lambda)\right)\omega t \tag{10}$$

where $\phi_0 = \phi(0)$. The bracket of (10) has one term related to the initial bob position, (θ_0, ϕ_0), and the latitude λ. An interesting parameter to obtain is the variation $\Delta\phi_{day}$ of the plane of oscillation angle in 24 h:

$$\Delta\phi_{day} = -2\pi\left(\sin(\lambda) - \frac{\theta_0\sin(\phi_0)\cos(\lambda)}{2}\right) \tag{11}$$

In the literature, $\Delta\phi_{day}$ is given only by $-2\pi\sin(\lambda)$, independent on the initial conditions [1–6]. The additional term for the angle variation in (11) shows that

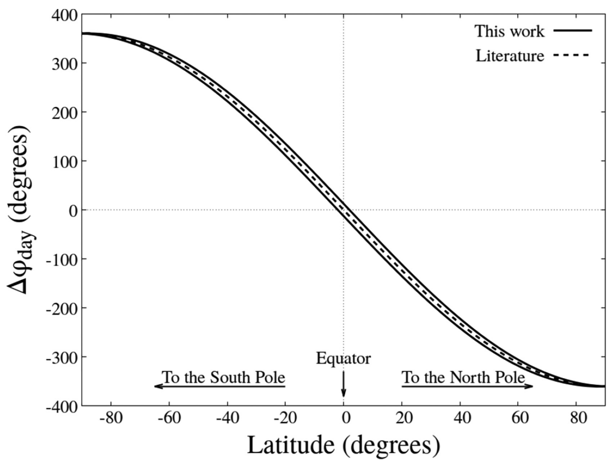

Figure 3: Variation of a Foucault pendulum plane of oscillation in 24 hours as a function of its latitude. Filled curves are given by Eq. (11) where the upper (lower) curve refers to a positive (negative) ϕ_0. The dashed curve is the expression for $\Delta\phi_{day}$ found in literature: $-2\pi\sin(\lambda)$. Such expressions only coincide at the Poles, where the Foucault pendulum executes a full 360° rotation of its oscillation plane per day.

the initial displacement of the plumb is decisive for the rotation of the oscillation plane [2], this difference being maximum if the bob is abandoned in such a way that the line that joins the bob and the equilibrium point of the pendulum lies exactly on a meridian ($\phi_0 = \pm\pi/2$). The dependence of $\Delta\phi_{day}$ with the latitude λ for this situation is represented by Fig. 3, where the starting angle was set as $\theta_0 = 4°$. For this θ_0 angle, the maximum difference between $\Delta\phi_{day}$ in Eq. (11) and the value predicted by the literature is $\approx 13°$, evaluated for $\lambda = 0$, the Equator. According to the literature, the plane of oscillation of a Foucault pendulum located exactly on the Equator line should not rotate whatsoever.

The only way to avoid rotation of the plane of oscillation is with the suspension point of the Foucault pendulum and the bob are exactly above the Equator ($\phi_0 = 0$ or $\phi_0 = \pi$) at the start of the movement. In this way, the bob never crosses the hemispheres during its movement.

4. Nonlinear Effects on the Oscillation Plane

The validity of Eq. (11) is restricted to first-order terms in ω obtained in the expansion of the components x and y of (5). By vanishing $\Delta\phi_{day}$ in (11) it is possible to determine a region of the planet where $\mathcal{O}(\omega^2)$ effects become relevant in the rotation of the plane of oscillation. By defining θ_{max} as the maximum θ_0 value where the small angle approximation $\sin(\theta) \approx \theta$ is valid in practical situations, the latitude where nonlinear effects become relevant is given by

$$|\lambda^{(nl)}| \leqslant \arctan\left(\frac{\theta_{max}}{2}\right) \tag{12}$$

For $\theta_{max} = 4°$, $\left|\lambda^{(nl)}\right| \leqslant 2°$. Therefore, for nonlinear effects in ω on the rotation of the plane of oscillation to be observable, the bob must be released within a region of the planet delimited by (12). Any Foucault pendulum located outside this latitude range will not exhibit noticeable $\mathcal{O}(\omega^2)$ effects on the rotation of the plane of oscillation.

Such nonlinear effects should be better observable if the Foucault pendulum bob is exactly abandoned with an orientation $\phi_0^{(nl)}$ given by

$$\phi_0^{(nl)} = \arcsin\left(\frac{2\tan(\lambda^{(nl)})}{\theta_{max}}\right) \tag{13}$$

obtained again from vanishing (11).

5. Conclusions

The movement of the Foucault pendulum was investigated. The decoupling of the equation of motion produced a sixth-order differential equation and an exact general vector solution was obtained in terms of generic initial conditions. The main phenomenon associated with the Foucault pendulum, the rotation of its oscillation plane, was evaluated according to the approximation of small angles for the pendulum and low Earth rotation compared to the frequency of oscillation of the Foucault pendulum for a bob starting from rest. The results differ from those found in the literature for a factor that depends on the initial conditions. Such a corrective factor implies the existence of rotation of the plane of oscillation even if the suspension point is located exactly on the Equator. This additional term is also responsible for delimiting regions of the planet where nonlinear terms in ω, the terrestrial rotation frequency, should be relevant for determining the rotation of the oscillation plane. Such a region is located around the Equator, with latitudes no greater than $\pm 2°$.

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