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Probabilist Set Inversion using Pseudo-Intervals Arithmetic

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ABSTRACT. In this paper, one presents how to use a new interval arithmetic framework based on free algebra construction, called pseudo-intervals, which is associative and distributive and permits to build well-defined inclusion function for interval semi-group and for its associated vector space. One introduces the ψ-algorithm (Probabilist Set Inversion), which performs set inversion of functions and exhibits some numerical examples.

Keywords: Free algebra, pseudo-interval and interval arithmetic, set inversion, probability.

1 SET INVERSION

One of the most recurrent problem arising in sciences and engineering is to perform adjustments of a system in order to get the desired performances. For example, how to set-up a car engine so that some polluting gases ratio are less than a certain amount, or how to settle a robot to make it moving toward a desired target. Such kind of problem are dealing with the inversion of the relation between adjustments and desired performances.

Let us note $\mathcal{R} \subset \mathbb{R}^n$ the set of feasible adjustments, and $\mathcal{P} \subset \mathbb{R}^p$ the set of desired performance of a system. The mathematical modelling of the problem consists of the computation of $S = f^{-1}(\mathcal{P}) \cap \mathcal{R}$, as shown on Figure (1), where $f : \mathbb{R}^n \rightarrow \mathbb{R}^p$ is the function giving performances from adjustments. Since real number sets can be written as union of intervals, one has to perform this set inversion within the interval semi-group $\mathbb{IR}$ [1]. Some powerful set inversion methods are have been developed those last years, such as *SIvIA* [18] (Set Inversion *via* Interval Arithmetic) which is based on interval arithmetic [2, 6, 7, 8, 9, 10, 11, 12, 32, 33, 34, 35, 36].

The first mathematician who has used intervals was the famous Archimedes from Syracuse (287-212 b.C). He has proposed a two-sides bounding of π: $3 + \frac{10}{71} < \pi < 3 + \frac{1}{7}$ using polygons and a systematic method to improve it. In the beginning of the twentieth century, the mathematician and physicist Wiener, published two papers [3, 4], and used intervals to give an interpretation to the position and the time of a system. More papers on the subject were written [5, 6, 7, 36]

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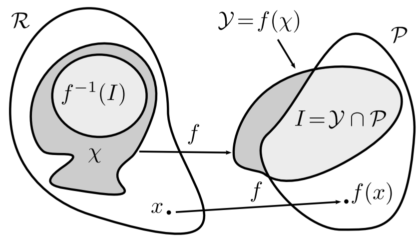

Figure 1: $\mathcal{R} \subset \mathbb{R}^n$ is the set of feasible adjustments, and $\mathcal{P} \subset \mathbb{R}^p$ the set of desired performance of a system. Set inversion computes $S = f^{-1}(\mathcal{P}) \cap \mathcal{R}$.

only after Second World War. Nowadays, we consider R.E. Moore [8, 9, 10, 11, 12] as the first mathematician who has proposed a framework for interval arithmetic and analysis. The interval arithmetic, or interval analysis has been introduced to compute very quickly range bounds (for example if a data is given up to an incertitude). Now interval arithmetic is a computing system which permits to perform error analysis by computing mathematics bounds. The extensions of the areas of applications are important: non linear problems, PDE, inverse problems. It finds a large place of applications in controllability, automatism, robotics, embedded systems, biomedical, haptic interfaces, form optimization, analysis of architecture plans, ...

Interval calculations are used nowadays as a powerful tool for global optimization and set inversion [8, 9, 10, 11, 13, 18, 36]. Several groups have developed some software and libraries to perform those new approaches such as *INTLAB* [19], *INTOPT90* and *GLOBSOL* [20], *Numerica* [21]. But their Achille's heel is the construction of the inclusion function from the natural one due to the lack of distributivity. Some approaches have developed methods to circumvent it with using boolean inclusion tests, series or limited expansions of the natural function where the derivatives are computed at a certain point of the intervals. Nevertheless, those transfers from real functions to functions defined on intervals are not systematic and not given by a formal process. This yields to the fact that the inclusion function definition has to be adapted to each problem with the risk to miss the primitive scope. Moreover, differential calculus and linear algebra need to be performed in the framework of vector space theory and not within semi-group one.

This article reminds first the definition and characteristics of the intervals semi-group $\mathbb{IR}$ and the construction of its associated vector space $\overline{\mathbb{IR}}$. After that it is explained how to get an associative and distributive arithmetic of intervals, called pseudo-intervals arithmetic, by embedding the vector space into a ***free algebra*** [2]. After that, one proposes a clear and simple scheme to build inclusion functions from the natural one for the semi-group and the vector space.

This permits to present a set inversion scheme, the ψ-algorithm (Probabilist Set Inversion), which is a *SIvIA* inspired scheme, and using probability calculations. One ends with numerical applications examples for set inversion in order to show how the pseudo-interval arithmetic efficient is.

2 AN ALGEBRAIC APPROACH FOR INTERVALS

An interval X is defined as a non-empty, closed and connected set of real numbers. One writes real numbers as intervals with same bounds, $\forall a \in \mathbb{R}$, $a \equiv [a, a]$. We denote by $\mathbb{IR} = \mathcal{P}_1$ the set of intervals of $\mathbb{R}$. The arithmetic operations on intervals, called ***Minkowski or classical operations***, are defined such that the result of the corresponding operation on elements belonging to operand intervals belongs to the resulting interval. That is, if $\diamond$ denotes one of the usual operations $+, -, *, /$, we have, if X and Y are closed intervals of $\mathbb{R}$,

$$X \diamond Y = \{x \diamond y \,/\, x \in X,\ y \in Y\}, \tag{2.1}$$

Although, $\overline{\mathbb{IR}}$ is provided with a pseudo-inverse operation, it does not satisfy $X - X = 0$, and hence a subtraction in the usual sense cannot be obtained. In many problems using interval arithmetic, that is the set $\mathbb{IR}$ with the Minkowski operations, there exists an informal transfers principle which permits, to associate with a real function f a function define on the set of intervals $\mathbb{IR}$ which coincides with f on the interval reduced to a point. But this transferred function is not unique. For example, if we consider the real function $f(x) = x^2 + x = x(x+1)$, we associate naturally the functions $\widetilde{f_1}$, $\widetilde{f_2} : \mathbb{IR} \longrightarrow \mathbb{IR}$ given by $\widetilde{f_1}(X) = X(X+1)$ and $\widetilde{f_2}(X) = X^2 + X$. These two functions do not coincide. Usually this problem is removed considering the most interesting transfers. But the qualitative "interesting" depends of the studied model and it is not given by a formal process.

In this section, we determine a natural extension $\overline{\mathbb{IR}}$ of $\mathbb{IR}$ provided with a vector space structure. The vectorial subtraction $X \setminus Y$ does not correspond to the semantic difference of intervals and the interval $\setminus X$ has no real interpretation. But these "negative" intervals have a computational role.

An algebraic extension of the classical interval arithmetic, called generalized interval arithmetic [13, 36] has been proposed first by M. Warmus [6, 7]. It has been followed in the seventies by H.-J. Ortolf & E. Kaucher [37, 42, 43, 44, 45]. In this former interval arithmetic, the intervals form a group with respect to addition and a complete lattice with respect to inclusion. In order to adapt it to semantic problems, Gardenes et al. have developed an approach called modal interval arithmetic [46, 47, 48, 49, 50, 51]. S. Markov and others investigate the relation between generalized intervals operations and Minkowski operations for classic intervals and propose the so-called directed interval arithmetic, in which Kaucher's generalized intervals can be viewed as classic intervals plus direction, hence the name directed interval arithmetic [32, 33]. In this arithmetic framework, proper and improper intervals are considered as intervals with sign [34]. Interesting relations and developments for proper and improper intervals arithmetic and for applications can be found in literature [38, 39, 40].

Our approach [2], that we remind below in this article, is similar to the previous ones in the sense that intervals are extended to generalized intervals; intervals and anti-intervals correspond respectively to the proper and improper ones. However we use a construction which is more canonical and based on the semi-group completion into a group, which permits then to build the associated real vector space, and to get an analogy with directed intervals.

In this section we present the set of intervals as a normed vector space with a Banach structure.

2.1 Interval semi-group

Let $\mathbb{IR}$ be the set of intervals. It is in one to one correspondence and can be represented as a point in the half-plane of $\mathbb{R}^2$, $\mathcal{P}_1 = \{(a,b) \in \mathbb{R}^2, a \leq b\}$. The set $\mathcal{P}_2 = \{(a,b) \in \mathbb{R}^2, a \geq b\}$ is the set of anti-intervals. $\mathbb{IR}$ is closed for the addition and endowed with a regular semi-group structure. The subtraction on $\mathbb{IR}$, which is not the symmetric operation of $+$, corresponds to the following operation on $\mathcal{P}_1$:

$$(a,b) - (c,d) = (a,b) + s_\Delta \circ s_0(c,d),$$

where s_0 is the symmetry with respect to 0, and s_Δ with respect to Δ. The multiplication $*$ is not globally defined. Consider the following subset of $\mathcal{P}_1$:

$$\begin{cases} \mathcal{P}_{1,1} = \{(a,b) \in \mathcal{P}_1, a \geq 0, b \geq 0\}, \\ \mathcal{P}_{1,2} = \{(a,b) \in \mathcal{P}_1, a \leq 0, b \geq 0\}, \\ \mathcal{P}_{1,3} = \{(a,b) \in \mathcal{P}_1, a \leq 0, b \leq 0\}. \end{cases}$$

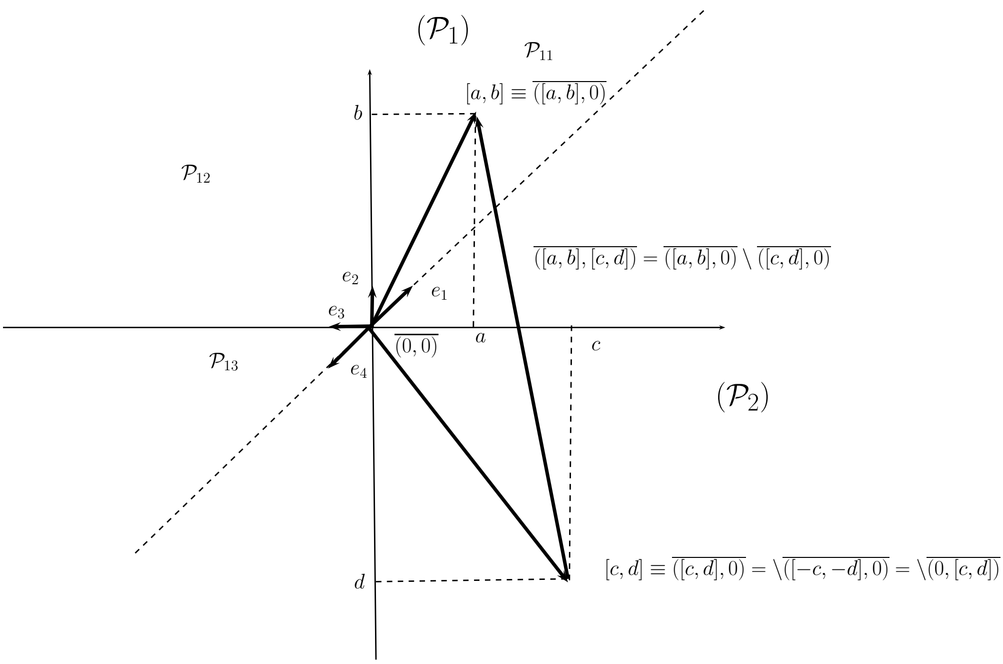

Figure 2: Representation of intervals in the half plane of $\mathbb{R}^2$.

We have the following cases:

1. If $(a,b), (c,d) \in \mathcal{P}_{1,1}$ the product is written $(a,b) * (c,d) = (ac, bd)$. The vectors $e_1 = (1,1)$ and $e_2 = (0,1)$ generate $\mathcal{P}_{1,1}$ that is any (x,y) in $\mathcal{P}_{1,1}$, can be decomposed as

$$(x,y) = xe_1 + (y-x)e_2, \text{ with } x > 0 \text{ and } y - x > 0.$$

The multiplication corresponds in this case to the following associative commutative algebra:

$$\begin{cases} e_1e_1 = e_1, \\ e_1e_2 = e_2e_1 = e_2e_2 = e_2. \end{cases}$$

2. Assume that $(a, b) \in \mathcal{P}_{1,1}$ and $(c, d) \in \mathcal{P}_{1,2}$ so $c \leq 0$ and $d \geq 0$. Thus we obtain $(a, b) * (c, d) = (bc, bd)$ and this product does not depend of a. Then we obtain the same result for any $a < b$. The product $(a, b) * (c, d) = (bc, bd)$ corresponds to

$$\begin{cases} e_1e_1 = e_2e_1 = e_1 \\ e_1e_2 = e_2e_2 = e_2 \end{cases}$$

This algebra is not commutative and it is different from the previous.

3. If $(a, b) \in \mathcal{P}_{1,1}$ and $(c, d) \in \mathcal{P}_{1,3}$ then $a \geq 0, b \geq 0$ and $c \leq 0, d \leq 0$ and we have $(a, b) * (c, d) = (bc, ad)$. Let $e_1 = (1, 1)$, $e_2 = (0, 1)$. This product corresponds to the following associative algebra:

$$\begin{cases} e_1e_1 = e_1, \\ e_1e_2 = e_2, \\ e_2e_1 = e_1 - e_2. \end{cases}$$

This algebra is not associative because $(e_2e_1)e_1 \neq e_2(e_1e_1)$. We have similar results for the cases $(\mathcal{P}_{1,2}, \mathcal{P}_{1,2})$, $(\mathcal{P}_{1,2}, \mathcal{P}_{1,3})$ and $(\mathcal{P}_{1,3}, \mathcal{P}_{1,3})$.

An objective of this paper is to present an associative algebra which contains all these results.

2.2 The real vector space $\overline{\mathbb{IR}}$

We recall briefly the construction proposed by Markov [14] to define a structure of abelian group. As $(\mathbb{IR}, +)$ is a commutative and regular semi-group, the quotient set, denoted by $\overline{\mathbb{IR}}$, associated with the equivalence relations:

$$(A, B) \sim (C, D) \Longleftrightarrow A + D = B + C,$$

for all $A, B, C, D \in \mathbb{IR}$, is provided with a structure of abelian group for the natural addition:

$$\overline{(A, B)} + \overline{(C, D)} = \overline{(A + C, B + D)}$$

where $\overline{(A, B)}$ is the equivalence class of (A, B). We denote by $\diagdown\overline{(A, B)}$ the inverse of $\overline{(A, B)}$ for the interval addition.

We have $\diagdown\overline{(A, B)} = \overline{(B, A)}$. If $X = [a, a]$, $a \in \mathbb{R}$, then $\overline{(X, 0)} = \overline{(0, -X)}$ where $-X = [-a, -a]$, and $\diagdown\overline{(X, 0)} = \overline{(0, X)}$. In this case, we identify $X = [a, a]$ with a and we denote always by $\mathbb{R}$ the subset of intervals of type $[a, a]$.

Naturally, the group $\overline{\mathbb{IR}}$ is isomorphic to the additive group $\mathbb{R}^2$ by the isomorphism $(\overline{([a,b],[c,d])} \to (a-c, b-d)$ (Fig. 2). We find the notion of generalized interval and this yields immediately to the resulting result:

Proposition 1. *Let $\mathcal{X} = \overline{(X,Y)}$ be in $\overline{\mathbb{IR}}$. Thus*

(1) *If $l(Y) < l(X)$, there is an unique $A \in \mathbb{IR} \setminus \mathbb{R}$ such that $\mathcal{X} = \overline{(A,0)}$,*

(2) *If $l(Y) > l(X)$, there is an unique $A \in \mathbb{IR} \setminus \mathbb{R}$ such that $\mathcal{X} = \overline{(0,A)} = \setminus\overline{(A,0)}$,*

(3) *If $l(Y) = l(X)$, there is an unique $A = \alpha \in \mathbb{R}$ such that $\mathcal{X} = \overline{(\alpha,0)} = \overline{(0,-\alpha)}$.*

Any element $\mathcal{X} = \overline{(A,0)}$ with $A \in \mathbb{IR} - \mathbb{R}$ is said positive and we write $\mathcal{X} > 0$. Any element $\mathcal{X} = \overline{(0,A)}$ with $A \in \mathbb{IR} - \mathbb{R}$ is said negative and we write $\mathcal{X} < 0$. We write $\mathcal{X} \geq \mathcal{X}'$ if $\mathcal{X} \setminus \mathcal{X}' \geq 0$. For example if $\mathcal{X}$ and $\mathcal{X}'$ are positive, $\mathcal{X} \geq \mathcal{X}' \Longleftrightarrow l(\mathcal{X}) \geq l(\mathcal{X}')$. The elements $\overline{(\alpha,0)}$ with $\alpha \in \mathbb{R}^*$ are neither positive nor negative.

In [14], one defines on the abelian group $\overline{\mathbb{IR}}$, a structure of quasi linear space. Our approach is a little bit different. We propose to construct a real vector space structure. We consider the external multiplication:

$$\cdot : \mathbb{R} \times \overline{\mathbb{IR}} \longrightarrow \overline{\mathbb{IR}}$$

defined, for all $A \in \mathbb{IR}$, by

$$\begin{cases} \alpha \cdot \overline{(A,0)} &= \overline{(\alpha A,0)}, \\ \alpha \cdot \overline{(0,A)} &= \overline{(0,\alpha A)}, \end{cases}$$

for all $\alpha > 0$. If $\alpha < 0$ we put $\beta = -\alpha$. So we put:

$$\begin{cases} \alpha \cdot \overline{(A,0)} &= \overline{(0,\beta A)}, \\ \alpha \cdot \overline{(0,A)} &= \overline{(\beta A,0)}. \end{cases}$$

We denote $\alpha\mathcal{X}$ instead of $\alpha \cdot \mathcal{X}$. This operation satisfies

1. For any $\alpha \in \mathbb{R}$ and $\mathcal{X} \in \overline{\mathbb{IR}}$ we have:

$$\begin{cases} \alpha(\setminus\mathcal{X}) = \setminus(\alpha\mathcal{X}), \\ (-\alpha)\mathcal{X} = \setminus(\alpha\mathcal{X}). \end{cases}$$

2. For all $\alpha, \beta \in \mathbb{R}$, and for all $\mathcal{X}, \mathcal{X}' \in \overline{\mathbb{IR}}$, we have

$$\begin{cases} (\alpha+\beta)\mathcal{X} = \alpha\mathcal{X} + \beta\mathcal{X}, \\ \alpha(\mathcal{X}+\mathcal{X}') = \alpha\mathcal{X} + \alpha\mathcal{X}', \\ (\alpha\beta)\mathcal{X} = \alpha(\beta\mathcal{X}). \end{cases}$$

Theorem 1. *The triplet $(\overline{\mathbb{IR}}, +, \cdot)$ is a real vector space and the vectors $\mathcal{X}_1 = \overline{([0,1],0)}$ and $\mathcal{X}_2 = \overline{([1,1],0)}$ of $\overline{\mathbb{IR}}$ determine a basis of $\overline{\mathbb{IR}}$. So $\dim_{\mathbb{R}} \overline{\mathbb{IR}} = 2$.*

Proof. We have the following decompositions:

$$\begin{cases} \overline{([a,b],0)} = (b-a)\mathcal{X}_1 + a\mathcal{X}_2, \\ \overline{(0,[c,d])} = (c-d)\mathcal{X}_1 - c\mathcal{X}_2. \end{cases}$$

The linear map

$$\varphi : \overline{\mathbb{IR}} \longrightarrow \mathbb{R}^2$$

defined by

$$\begin{cases} \varphi(\,\overline{([a,b],0)}\,) = (b-a,a), \\ \varphi(\,\overline{(0,[c,d])}\,) = (c-d,-c) \end{cases}$$

is a linear isomorphism and $\overline{\mathbb{IR}}$ is canonically isomorphic to $\mathbb{R}^2$. The following map

$$||.|| : \overline{\mathbb{IR}} \longrightarrow \mathbb{R}^+ \tag{2.2}$$

$$\overline{(X,0)} \mapsto l(X) + |c(X)| \tag{2.3}$$

$$\text{or} \tag{2.4}$$

$$\overline{(0,X)} \mapsto l(X) + |c(X)| \tag{2.5}$$

with respectively $l(X)$ and $c(X)$ the width and the center of the interval X, is obviously a norm. Since $\overline{\mathbb{IR}}$ is isomorphic to $\mathbb{R}^2$ which is complete, this yields to the fact that this norm endows $\overline{\mathbb{IR}}$ with a Banach space structure. Thus, it is possible to perform differential calculus in $\overline{\mathbb{IR}}$ [22].

3 A 4-dimensional free algebra associated with $\overline{\mathbb{IR}}$

We define in this section a four-dimensional associative and distributive ***free algebra*** in which the real vector space is embedded.

3.1 Definition of $\mathcal{A}_4$

In introduction, we have observed that the semi-group $\mathbb{IR}$ is identified to $\mathcal{P}_{1,1} \cup \mathcal{P}_{1,2} \cup \mathcal{P}_{1,3}$. Let us consider the following vectors of $\mathbb{R}^2$:

$$\begin{cases} e_1 = (1,1), \\ e_2 = (0,1), \\ e_3 = (-1,0), \\ e_4 = (-1,-1). \end{cases}$$

They correspond to the intervals $[1,1]$, $[0,1]$, $[-1,0]$, and $[-1,-1]$. Any point of $\mathcal{P}_{1,1} \cup \mathcal{P}_{1,2} \cup \mathcal{P}_{1,3}$ admits the decomposition

$$(a,b) = \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3 + \alpha_4 e_4$$

with $\alpha_i \geq 0$. The dependence relations between the vectors e_i are

$$\begin{cases} e_2 = e_3 + e_1 \\ e_4 = -e_1. \end{cases}$$

Thus there exists a unique decomposition of (a, b) in a chosen basis such that the coefficients are non negative. These basis are $\{e_1, e_2\}$ for $\mathcal{P}_{1,1}$, $\{e_2, e_3\}$ for $\mathcal{P}_{1,2}$, $\{e_3, e_4\}$ for $\mathcal{P}_{1,3}$. Let us consider the free algebra of basis $\{e_1, e_2, e_3, e_4\}$ whose products correspond to the Minkowski products. The multiplication table is

	e_1	e_2	e_3	e_4
e_1	e_1	e_2	e_3	e_4
e_2	e_2	e_2	e_3	e_3
e_3	e_3	e_3	e_2	e_2
e_4	e_4	e_3	e_2	e_1

This algebra is associative and its elements are called pseudo-intervals.

3.2 Pseudo-intervals product

Let $\varphi : \overline{\mathbb{IR}} \to \mathcal{A}_4$ the natural injective embedding, ψ the canonical embedding from $\mathcal{A}_4$ to $\mathcal{A}_4/F$ and $\varphi' = \psi \circ \varphi$. If we identify an interval with its image in $\mathcal{A}_4$, one has:

The application φ is not bijective. Its image on the elements $\mathcal{X} = \overline{(X, 0)} = \overline{([a, b], 0)}$ is:

$$\begin{cases} X = [a, b] \in \mathcal{P}_{1,1}, \varphi(\mathcal{X}) = ae_1 + (b-a)e_2 \quad (a \geq 0, b-a \geq 0) \\ X = [a, b] \in \mathcal{P}_{1,2}, \varphi(\mathcal{X}) = -ae_3 + be_2 \quad (-a \geq 0, b \geq 0) \\ X = [a, b] \in \mathcal{P}_{1,3}, \varphi(\mathcal{X}) = -be_4 + (b-a)e_3 \quad (-b \geq 0, b-a \geq 0). \end{cases}$$

Consider in $\mathcal{A}_4$ the linear subspace F generated by the vectors $e_1 - e_2 + e_3, e_1 + e_4$. As

$$\begin{aligned} (e_1 + e_4)(e_1 + e_4) &= 2(e_1 + e_4) \\ (e_1 + e_4)(e_1 - e_2 + e_3) &= e_1 + e_4 \\ (e_1 - e_2 + e_3)(e_1 - e_2 + e_3) &= e_1, \end{aligned}$$

F is not a sub-algebra of $\mathcal{A}_4$. Let us consider the map

$$\overline{\varphi} : \mathbb{IR} \to \mathcal{A}_4/F$$

defined from φ and the canonical projection on the quotient vector space $\mathcal{A}_4/F$. A vector $X = \sum \alpha_i e_i \in \mathcal{A}_4$ is equivalent to a vector of $\mathcal{A}_4$ with positive components if and only if

$$\alpha_2 + \alpha_3 \geq 0.$$

In this case, all the vectors equivalent to $x = \sum \alpha_i e_i$ with $\alpha_2 + \alpha_3 \geq 0$ correspond to the interval $[\alpha_1 - \alpha_3 - \alpha_4, \alpha_1 + \alpha_2 - \alpha_4]$ of $\mathbb{IR}$. Thus we have for any equivalent classes of $\mathcal{A}_4/F$ associated with $\sum \alpha_i e_i$ with $\alpha_2 + \alpha_3 \geq 0$ a pre-image in $\mathbb{IR}$. The map $\overline{\varphi}$ is injective. In fact, two intervals belonging to pieces $\mathcal{P}_{1,i}, \mathcal{P}_{1,j}$ with $i \neq j$, have distinguish images. Now if (a, b) and (c, d) belong to the same piece, for example $\mathcal{P}_{1,1}$, thus

$$\overline{\varphi}(a, b) = \{(a + \lambda + \mu, b - a - \lambda, \lambda, \mu), \lambda, \mu \in \mathbb{R}.\}$$

If $\overline{\varphi}(c, d) = \overline{\varphi}(a, b)$, there are $\lambda, \mu \in \mathbb{R}$ such that $(c, d) = (a + \lambda + \mu, b - a - \lambda, \lambda, \mu)$. This gives $a = c, b = d$. We have the same results for all the other pieces. Thus $\overline{\varphi} : \mathbb{IR} \rightarrow \mathcal{A}_4/F$ is bijective on its image, that is the hyperplane of $\mathcal{A}_4/F$ corresponding to $\alpha_2 + \alpha_3 \geq 0$.

Practically the multiplication of two intervals will so be made: let $X, Y \in \mathbb{R}$. Thus $X = \sum \alpha_i e_i$, $Y = \sum \beta_i e_i$ with $\alpha_i, \beta_j \geq 0$ and we have the product

$$X \bullet Y = \overline{\varphi}^{-1}(\varphi'(X).\varphi'(Y))$$

this product is well defined because $\overline{\varphi'(X).\varphi'(Y)} \in Im\overline{\varphi}$. This product is distributive because

$$\begin{aligned} X \bullet (Y + Z) &= \overline{\varphi}^{-1}(\varphi'(X).\varphi'(Y + Z)) \\ &= \overline{\varphi}^{-1}(\varphi'(X).(\varphi'(Y) + \varphi'(Z)) \\ &= \overline{\varphi}^{-1}(\varphi'(X).\varphi'(Y) + \varphi'(X).\varphi'(Z)) \\ &= X \bullet Y + X \bullet Z \end{aligned}$$

Remark. We have

$$\overline{\varphi}^{-1}(\varphi'(X).\varphi'(Y + Z)) \neq \overline{\varphi}^{-1}(\varphi'(X)).\overline{\varphi}^{-1}(\varphi'(Y + Z))).$$

We shall be careful not to return in $\mathbb{IR}$ during the calculations as long as the result is not found. Otherwise we find the semantic problems of the distributivity.

We extend naturally the map $\varphi : \mathbb{IR} \rightarrow \mathcal{A}_4$ to $\overline{\mathbb{IR}}$ by

$$\begin{cases} \varphi\overline{(A, 0)} = \varphi(A) \\ \varphi\overline{(0, A)} = -\varphi(A) \end{cases}$$

for every $A \in \mathbb{IR}$.

Theorem 2. *The multiplication*

$$\mathcal{X} \bullet \mathcal{Y} = \overline{\varphi}^{-1}(\varphi'(\mathcal{X}).\varphi'(\mathcal{Y}))$$

is distributive with respect the addition.

Proof. This is a direct consequence of the previous computations.

3.3 Pseudo-intervals division

Division between intervals can also be defined with solving $X \cdot Y = (1, 0, 0, 0)$ in $\mathcal{A}_4$ or in a isomorphic algebra. In $\mathcal{A}_4$ we consider the change of basis

$$\begin{cases} e'_1 = e_1 - e_2 \\ e'_i = e_i, i = 2, 3 \\ e'_4 = e_4 - e_3. \end{cases}$$

This change of basis shows that $\mathcal{A}_4$ is isomorphic to $\mathcal{A}'_4$

	e'_1	e'_2	e'_3	e'_4
e'_1	e'_1	0	0	e'_4
e'_2	0	e'_2	e'_3	0
e'_3	0	e'_3	e'_2	0
e'_4	e'_4	0	0	e'_1

.

The unit of $\mathcal{A}'_4$ is the vector $e'_1 + e'_2$. This algebra is a direct sum of two ideals: $\mathcal{A}'_4 = I_1 + I_2$ where I_1 is generated by e'_1 and e'_4 and I_2 is generated by e'_2 and e'_3. It is not an integral domain, that is, we have divisors of 0. For example $e'_1 \cdot e'_2 = 0$.

Proposition 2. *The multiplicative group $\mathcal{A}_4^*$ of invertible elements of $\mathcal{A}_4$ is the set of elements $x = (x_1, x_2, x_3, x_4)$ such that*

$$\begin{cases} x_4 \neq \pm x_1, \\ x_3 \neq \pm x_2. \end{cases}$$

This means that the invertible intervals do not contain 0. If $x \in \mathcal{A}_4^*$ we have:

$$x^{-1} = \left(\frac{x_1}{x_1^2 - x_4^2}, \frac{x_2}{x_2^2 - x_3^2}, \frac{x_3}{x_2^2 - x_3^2}, \frac{x_4}{x_1^2 - x_4^2} \right).$$

3.4 Monotony property

Let us compute the product of intervals using the product in $\mathcal{A}_4$ and compare it with the Minkowski product. Let $X = [a, b]$ and $Y = [c, d]$ two intervals.

Lemma 1. *If X and Y are not in the same piece $\mathcal{P}_{1,i}$, then $X \bullet Y$ corresponds to the Minkowski product.*

Proof. i) If $X \in \mathcal{P}_{1,1}$ and $Y \in \mathcal{P}_{1,2}$ then $\varphi(X) = (a, b - a, 0, 0)$ and $\varphi(Y) = (0, d, -c, 0)$. Thus

$$\begin{aligned} \varphi(X)\varphi(Y) &= (ae_1 + (b - a)e_2)(de_2 - ce_3) \\ &= bde_2 - cbe_3 \\ &= (0, bd, -cb, 0) \\ &= \varphi([cb, bd]). \end{aligned}$$

ii) If $X \in \mathcal{P}_{1,1}$ and $Y \in \mathcal{P}_{1,3}$ then $\varphi(X) = (a, b - a, 0, 0)$ and $\varphi(Y) = (0, 0, d - c, -d)$. Thus

$$\begin{aligned} \varphi(X)\varphi(Y) &= (ae_1 + (b - a)e_2)((d - c)e_3 - de_4) \\ &= (ad - bc)e_3 - ade_4 \\ &= (0, 0, ad - cb, -ad) \\ &= \varphi([bc, ad]). \end{aligned}$$

iii) If $X \in \mathcal{P}_{1,2}$ and $Y \in \mathcal{P}_{1,3}$ then $\varphi(X) = (0, b, -a, 0)$ and $\varphi(Y) = (0, 0, d - c, -d)$. Thus

$$\begin{aligned}\varphi(X)\varphi(Y) &= (be_2 - ae_3)((d-c)e_3 - de_4)\\ &= ace_2 - bce_3\\ &= (0, ac, -cb, 0)\\ &= \varphi([bc, ad]).\end{aligned}$$

Lemma 2. *If X an Y are both in the same piece $\mathcal{P}_{1,1}$ or $\mathcal{P}_{1,3}$, then the product $X \bullet Y$ corresponds to the Minkowski product.* The proof is analogous to the previous.

Let us assume that $X = [a, b]$ and $Y = [c, d]$ belong to $\mathcal{P}_{1,2}$. Thus $\varphi(X) = (0, b, -a, 0)$ and $\varphi(Y) = (0, d, -c, 0)$. We obtain

$$XY = (be_2 - ae_3)(de_2 - ce_3) = (bd + ac)e_2 + (-bc - ad)e_3.$$

Thus

$$[a, b][c, d] = [bc + ad, bd + ac].$$

This result is greater that all the possible results associated with the Minkowski product. However, we have the following property:

Proposition 3. ***Monotony property:*** *Let $\mathcal{X}_1, \mathcal{X}_2 \in \overline{\mathbb{IR}}$. Then*

$$\begin{cases}\mathcal{X}_1 \subset \mathcal{X}_2 \Longrightarrow \mathcal{X}_1 \bullet \mathcal{Z} \subset \mathcal{X}_2 \bullet \mathcal{Z} \text{ for all } \mathcal{Z} \in \overline{\mathbb{IR}}.\\ \overline{\varphi}(\mathcal{X}_1) \le \overline{\varphi}(\mathcal{X}_2) \Longrightarrow \overline{\varphi}(\mathcal{X}_1 \bullet \mathcal{Z}) \le \overline{\varphi}(\mathcal{X}_2 \bullet \mathcal{Z})\end{cases}$$

The order relation on $\mathcal{A}_4$ that ones uses here is

$$\begin{cases}(x_1, x_2, 0, 0) \le (y_1, y_2, 0, 0) \Longleftrightarrow y_1 \le x_1 \text{ and } x_2 \le y_2,\\ (x_1, x_2, 0, 0) \le (0, y_2, y_3, 0) \Longleftrightarrow x_2 \le y_2,\\ (0, x_2, x_3, 0) \le (0, y_2, y_3, 0) \Longleftrightarrow x_3 \le y_3 \text{ and } x_2 \le y_2,\\ (0, 0, x_3, x_4) \le (0, y_2, y_3, 0) \Longleftrightarrow x_3 \le y_3,\\ (0, 0, x_3, x_4) \le (0, 0, y_3, y_4) \Longleftrightarrow x_3 \le y_3 \text{ and } y_4 \le x_4.\end{cases}$$

Proof. Let us note that the second property is equivalent to the first. It is its translation in $\overline{\mathcal{A}_4}$. We can suppose that $\mathcal{X}_1$ and $\mathcal{X}_2$ are intervals belonging moreover to $\mathcal{P}_{1,2}$: $\varphi(\mathcal{X}_1) = (0, b, -a, 0)$, $\varphi(\mathcal{X}_2) = (0, d, -c, 0)$. If $\varphi(\mathcal{Z}) = (z_1, z_2, z_3, z_4)$, then

$$\begin{cases}\overline{\varphi}(\mathcal{X}_1 \bullet \mathcal{Z}) = (0, bz_1 + bz_2 - az_3 - az_4, -az_1 + bz_3 - az_2 + bz_4, 0),\\ \overline{\varphi}(\mathcal{X}_2 \bullet \mathcal{Z}) = (0, dz_1 + dz_2 - cz_3 - cz_4, -cz_1 + dz_3 - cz_2 + dz_4, 0).\end{cases}$$

Thus

$$\overline{\varphi}(\mathcal{X}_1 \bullet \mathcal{Z}) \le \overline{\varphi}(\mathcal{X}_2 \bullet \mathcal{Z}) \Longleftrightarrow \begin{cases}(b-d)(z_1 + z_2) - (a-c)(z_3 - z_4) \le 0,\\ -(a-c)(z_1 + z_2) + (b-d)(z_3 = z_4) \le 0.\end{cases}$$

But $(b - d)$, $-(a - c) \le 0$ and $z_2, z_3 \ge 0$. This implies $\overline{\varphi}(\mathcal{X}_1 \bullet \mathcal{Z}) \le \overline{\varphi}(\mathcal{X}_2 \bullet \mathcal{Z})$.

4 THE ALGEBRAS $\mathcal{A}_n$ AND AN BETTER RESULT OF THE PRODUCT

We can refine our result of the product to come closer to the result of Minkowski. Consider the one dimensional extension $\mathcal{A}_4 \oplus \mathbb{R}e_5 = \mathcal{A}_5$, where e_5 is a vector corresponding to the interval $[-1, 1]$ of $\mathcal{P}_{1,2}$. The multiplication table of $\mathcal{A}_5$ is

	e_1	e_2	e_3	e_4	e_5
e_1	e_1	e_2	e_3	e_4	e_5
e_2	e_2	e_2	e_3	e_3	e_5
e_3	e_3	e_3	e_2	e_2	e_5
e_4	e_4	e_3	e_2	e_1	e_5
e_5	e_5	e_5	e_5	e_5	e_5

The piece $\mathcal{P}_{1,2}$ is written $\mathcal{P}_{1,2} = \mathcal{P}_{1,2,1} \cup \mathcal{P}_{1,2,1}$ where $\mathcal{P}_{1,2,1} = \{[a,b], -a \leq b\}$ and $\mathcal{P}_{1,2,2} = \{[a,b], -a \geq b\}$. If $X = [a,b] \in \mathcal{P}_{1,2,1}$ and $Y = [c,d] \in \mathcal{P}_{1,2,2}$, thus

$$\varphi(X).\varphi(Y) = (0, b+a, 0, 0, -a).(0, 0, -c-d, 0, d) = (0, -(a+b)(c+d), 0, 0, a(c+d)+bd).$$

Thus we have

$$X \bullet Y = [-bd - ac - ad, -bc].$$

Example. Let $X = [-2, 3]$ and $Y = [-4, 2]$. We have $X \in \mathcal{P}_{1,2,1}$ and $Y \in \mathcal{P}_{1,2,2}$. The product in $\mathcal{A}_4$ gives

$$X \bullet Y = [-16, 14].$$

The product in $\mathcal{A}_5$ gives

$$X \bullet Y = [-12, 10].$$

The Minkowski product is

$$[-2, 3].[-4, 2] = [-12, 8].$$

Thus the product in $\mathcal{A}_5$ is better.

Conclusion. Considering a partition of $\mathcal{P}_{1,2}$, we can define an extension of $\mathcal{A}_4$ of dimension n, the choice of n depends on the approach wanted of the Minkowski product. For example, let us consider the vector e_6 corresponding to the interval $[-1, \frac{1}{2}]$. Thus the Minkowsky product gives $e_6 \cdot e_6 = e_7$ where e_7 corresponds to $[-\frac{1}{2}, 1]$. This yields to the fact that $\mathcal{A}_6$ is not an associative algebra but it is the case for $\mathcal{A}_7$ whose table of multiplication is

	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_2	e_2	e_2	e_3	e_3	e_5	e_6	e_7
e_3	e_3	e_3	e_2	e_2	e_5	e_7	e_6
e_4	e_4	e_3	e_2	e_1	e_5	e_7	e_6
e_5	e_5	e_5	e_5	e_5	e_5	e_5	e_5
e_6	e_6	e_6	e_7	e_7	e_5	e_7	e_6
e_7	e_7	e_7	e_6	e_6	e_5	e_6	e_7

Example. Let $X = [-2, 3]$ and $Y = [-4, 2]$. The decomposition on the basis $\{e_1, \cdots, e_7\}$ with positive coefficients writes

$$X = e_5 + 2e_7, \quad Y = 2e_6.$$

$$X \bullet Y = (e_5 + 2e_7)(4e_6) = 4e_5 + 8e_6 = [-12, 8].$$

We obtain now the Minkowski product.

5 INCLUSION FUNCTIONS

It is necessary for some problems to extend the definition of a function defined for real numbers $f : \mathbb{R} \to \mathbb{R}$ to function defined for intervals $[f] : \mathbb{IR} \to \mathbb{IR}$ such as $[f]([a, a]) = f(a)$ for any $a \in \mathbb{R}$. It will be convenient to have the same formal expression for f and $[f]$. Usually the lack of distributivity in Minkowski arithmetic doesn't give the possibility to get the same formal expressions. But with the pseudo-intervals arithmetic we have presented, there is no data dependency any more and one can define easily inclusion functions from the natural one. For example, let's extend to intervals the real functions $f_0(x) = x^2 - 2x + 1$, $f_1(x) = (x-1)^2$, $f_2(x) = x(x-2)+1$. Usually, with the Minkwoski operations, the three expressions of this same function for the interval $X = [3, 4]$ are $[f]_0(X) = [2, 11]$, $[f]_1(X) = [4, 9]$ and $[f]_2(X) = [6, 12]$. Data dependency occurs when the variable appears more than once in the function expression. The deep reason of that is the lack of distributivity in Minkowski arithmetic. But within the arithmetic developed in $\mathcal{A}_4$ or higher dimension free algebras [2], this problem vanishes. For example: with $X = [3, 4]$ and since $X \in \mathcal{P}_{11}$,

$$\varphi(X) = (3, 4 - 3, 0, 0) = (3, 1, 0, 0) = 3e_1 + e_2. \tag{5.1}$$

Since $e_1 = (1, 0, 0, 0)$, $e_2 = (0, 1, 0, 0)$ and with means of product table, one has

$$\begin{aligned} \varphi([f]_0(X)) &= (3e_1 + e_2)^2 - 2(3e_1 + e_2) + 1 \\ &= 9e_1^2 + 2 \cdot 3e_1e_2 + e_2^2 - 2 \cdot 3e_1 - 2e_2 + 1 \\ &= 9e_1 + 6e_2 + e_2 - 6e_1 - 2e_2 + e_1 = 4e_1 + 5e_2 \\ &= \varphi([4, 9]), \end{aligned} \tag{5.2}$$

$$\begin{aligned} \varphi([f]_1(X)) &= (3e_1 + e_2 - 1)^2 \\ &= (2e_1 + e_2)^2 \\ &= 4e_1^2 + 4e_1e_2 + e_2^2 \\ &= 4e_1 + 4e_2 + e_2 \\ &= 4e_1 + 5e_2 \\ &= \varphi([4, 9]), \end{aligned} \tag{5.3}$$

and

$$\begin{aligned}
\varphi([f]_2(X)) &= (3e_1 + e_2) \cdot (3e_1 + e_2 - 2) + 1 \\
&= 9e_1 + 3e_1e_2 - 6e_1 + 3e1e_2 + e_2^2 - 2e_2 + e_1 \\
&= 4e_1 + 3e_2 + 3e_2 + e_2 - 2e_2 \\
&= 4e_1 + 5e_2 \\
&= \varphi([4, 9]). \qquad (5.4)
\end{aligned}$$

Thus, $[f]_0(X) = [f]_1(X) = [f]_2(X) = [4, 9]$ and the inclusion function is defined univocally regardless the way to write the original one.

On the other hand, the construction of the inclusion function depends on the type of problem one deals with. If one aims to perform set inversion for example, it has to be done in the semi-group $\mathbb{IR}$. But, the subtraction is not defined in $\mathbb{IR}$. This problem can be circumvented by replacing it with an addition and a multiplication with the interval $e_4 = [-1, -1]$. This maintains the associativity and distributivity of arithmetic and permits to introduce a pseudo-subtraction. For example: if $f(x) = x^2 - x = x(x-1)$ for real numbers, one defines $[f](X) = X^2 + e_4 \cdot X$. One reminds the product $[-1, -1] \cdot [a, b]$ is equal to $[-b, -a]$. Due to the fact that the arithmetic is now associative and distributive, one doesn't have data dependency anymore and $[f](X) = X^2 + e_4 \cdot X = X \cdot (X + e_4)$. The last term corresponds to the transfer of $x(x-1)$. Division can be transferred to the semi-group in the same way by replacing $\frac{1}{x} = x^{-1}$ with X^{e_4}.

Taylor polynomial expansions, differential calculus and linear algebra operations are defined only in a vector space. Therefore the transfer for the vector space is done directly. This permits to get infinitesimal intervals with the subtraction and to compute derivatives. This is of course not allowed and not possible into the semi-group. From $\mathbb{IR}$ to the vector space $\overline{\mathbb{IR}}$, $f : x \mapsto -x$ is transferred to $[f] : X \equiv \overline{(X, 0)} \mapsto \backslash\overline{(X, 0)} \equiv \backslash X$. This means that $[a, b]$ subtraction is the anti-interval $[-a, -b]$ addition. One of the most important consequence is that it is possible to transfer some functions directly to the pseudo-intervals. For example, it is easy to prove analytically in $\overline{\mathbb{IR}}$ that $[\exp](\overline{([a, b], 0)}) = \overline{([\exp(a), \exp(b)], 0)}$ with means of Taylor expansion.

6 PROBABILIST SET INVERSION: ψ-algorithm

6.1 Flowchart

One presents an efficient set inversion method whose flowchart is very simple. One of the powerful application of interval calculus is the set inversion of a real-valued function defined on real numbers. As mentioned in the first section, the mathematical modelling of this problem is the following as shown on Figure 1: let's note $f : \mathbb{R}^n \mapsto \mathbb{R}^p$ a function for a physical system, which is required to be surjective only, $\mathcal{R} \subset \mathbb{R}^n$ the set of adjustments, and $\mathcal{P} \subset \mathbb{R}^p$ the set of performance of a system. Set inversion consists of the computation of $\mathcal{S} = f^{-1}(\mathcal{P}) \cap \mathcal{R}$, and one has to perform it within the semi-group $\mathbb{IR}$. Some interesting and powerful methods using intervals have been developed those last years, especially SIVIA [18], Set Inversion *Via*

Interval Analysis. But the inclusion function being not well defined in the semi-group with the Minkowski arithmetic, SIVIA uses boolean inclusion tests and finds accepted, rejected and "uncertain" domains.

With the algebraic arithmetic, one doesn't need boolean tests since the inclusion functions are well-defined. Thus, we propose the ψ-algorithm (Probabilist Set Inversion) inspired from SIVIA but without boolean tests and with a conditional probability calculation and domain bisections. This yields to accepted or rejected domains only. We are interested to compute the following conditional probability

$$\begin{aligned} p(\mathcal{X}) &= p([f](\mathcal{X}) \subset \mathcal{P} \mid f(x) \in [f](\mathcal{X}),\ \forall x \in \mathcal{X}) \\ &= \frac{mes([f](\mathcal{X}) \cap \mathcal{P})}{mes([f](\mathcal{X}))} \\ &= \frac{mes(\mathcal{Y} \cap \mathcal{P})}{mes(\mathcal{Y})} = \frac{mes(\mathcal{I})}{mes(\mathcal{Y})} \end{aligned} \tag{6.1}$$

where mes is the Lebesgue measure in $\mathbb{R}^p$ (length, surface, ...). If this probability equals 1 then the set is added to the list of solutions. If it is zero the set is rejected and removed from the list of interval candidates. If the probability is such as $p(\mathcal{X}) \in]0,1[$, then $\mathcal{X}$ is bisected and ψ-algorithm applies the same procedure recursively for the resulting intervals until the size is lower than a fixed size resolution of the intervals or until the sets are accepted or rejected. Since ψ-algorithm creates sequences of decreasing intervals which are compact sets, it is obvious that ψ-algorithm converges to fixed points probabilities which are simply 0 and 1. In fact, we consider a sequence of compact sets $\{K_n\}_{n\in\mathbb{N}}$ satisfying $d(K_n) > d(K_{n+1})$ where d is the diameter of the compact set. If $K_n \cap K_m = \emptyset$, for any $n \neq m$, the sequence is convergent and the limit is the empty set. Then, one has just to consider the sequence of bisected sets.

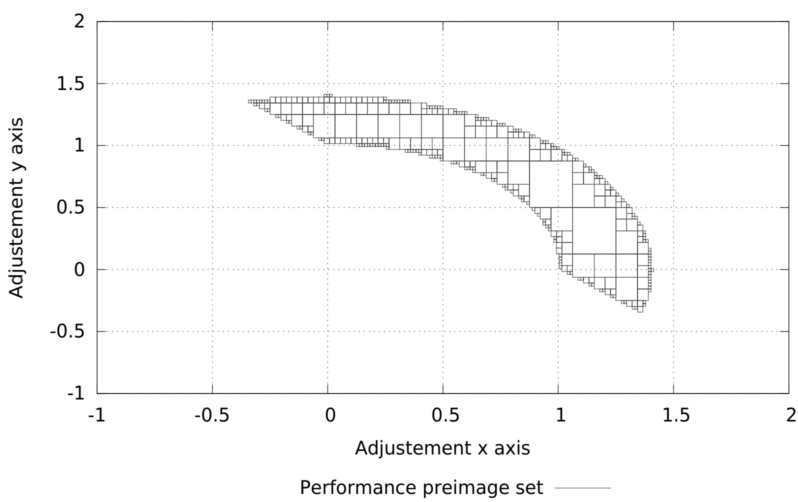

Figure 3: ψ-algorithm for f_1.

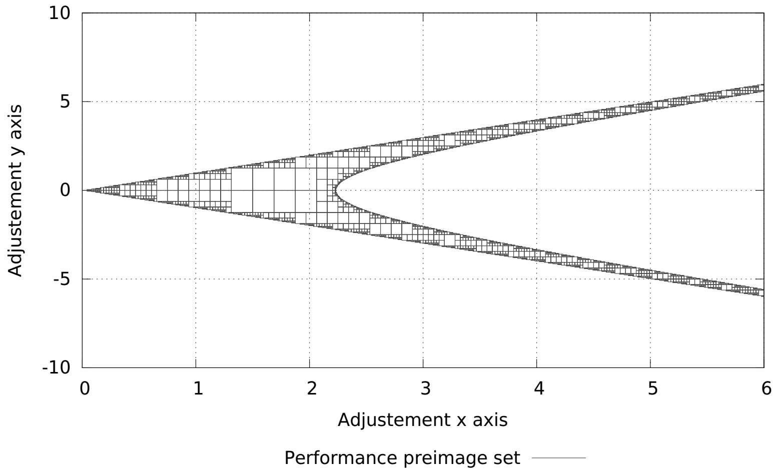

Figure 4: ψ-algorithm for f_2.

6.2 Numerical applications

We have developed a numerical library for *python* environment [27] called yet *typhon*. It is a pure numerical implementation performing the basic arithmetic presented above [2]. This library aims to give simple and optimized routines to perform interval calculations based on the algebraic arithmetic. One gives in this section some numerical application examples of ψ-algorithm in order to illustrate how it can treat usual inversion problems and build well-defined inclusion functions.

Let's define the non-linear functions $f_i : \mathbb{R}^2 \to \mathbb{R}^2$, $i = 1, 2$, with respectively adjustments and performances sets $\mathcal{R}_i, \mathcal{P}_i$:

$$\begin{aligned} f_1(x, y) &= (x^2 + y^2, x + y),\ \mathcal{R}_1 = [-1, 2]^2,\ \mathcal{P}_1 = [1, 2] \times [1, 4] \\ f_2(x, y) &= (x^2 - y^2, \frac{y}{1 + x}),\ \mathcal{R}_2 = [0, 6] \times [-10, 10],\ \mathcal{P}_2 = [0, 5] \times [-4, 4]. \end{aligned} \tag{6.2}$$

Those examples have been chosen to give examples of addition, subtraction, product and division transfers from $\mathbb{R}$ to $\mathbb{IR}$, and to exhibit the difference between the usual Minkowski arithmetic and the algebraic one [2]. The calculations with the ψ-algorithm are shown on Figures 3, 4 and 5. The convergence to 0 or 1 probabilities only, shows that inclusion functions are well constructed and that the pseudo-interval arithmetic is robust. The following example

$$f_3(x, y) = (x^2 - y^2 \cdot \exp(x) + x \cdot \exp(y), x \cdot (x + y) - y^2),\ \mathcal{R}_3 = \mathcal{P}_3 = [-5, 5]^2 \tag{6.3}$$

presented on Figure 5 shows clearly that the ψ-algorithm implemented in the algebraic arithmetic we use is not data dependant. The variables appear more than once in the formal expression of the function f_3. The CPU time for this inversion is about 255 seconds on a simple 1.67 Ghz Intel processor for a spatial surface resolution of 10^{-4}.

There is no limitation for the dimensions of the adjustments and performances sets as shown on Figure 6 for the function $f_4 : \mathbb{R}^3 \longrightarrow \mathbb{R}^4$.

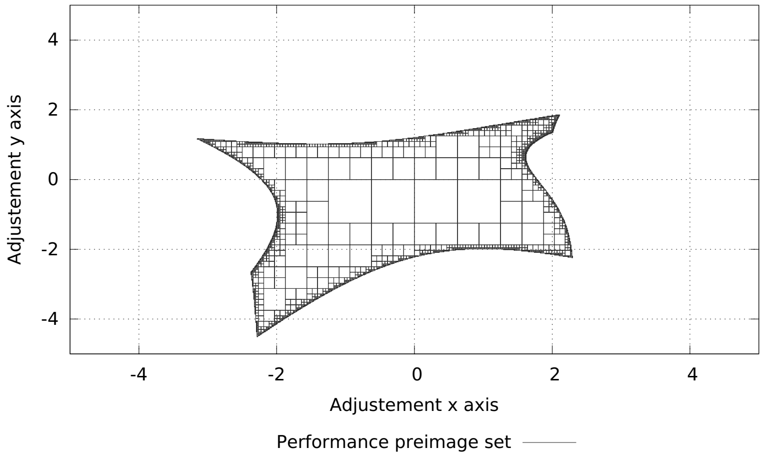

Figure 5: ψ-algorithm for f_3.

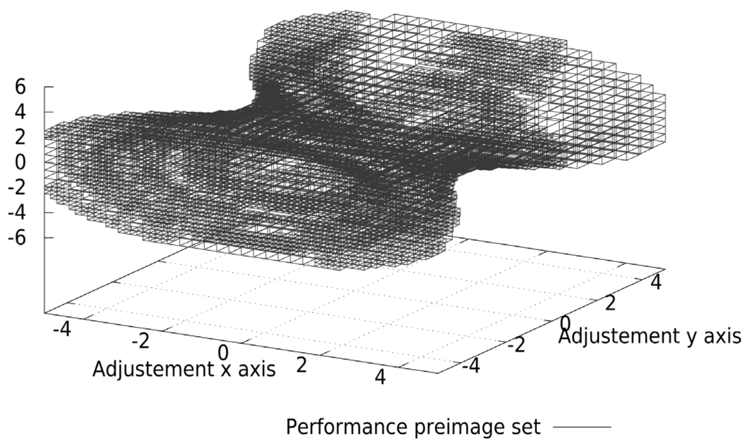

Figure 6: ψ-algorithm for $f_4(x, y, z) = (x, y, z, x^2 - y^2 + z^2)$ with $\mathcal{R}_4 = [-5, 5]^3$ and $\mathcal{P}_4 = [-10, 10]^4$.

Due to the bisection, the algorithm computational complexity is exponential according to the iterations N, and it is not improved compared to SIVIA one. In our scheme, computational time is defined as

$$T_{comp} = \mathcal{O}(N) = k \cdot 2^N. \tag{6.4}$$

However, if the native function is differentiable on $\mathbb{R}^n$, it is possible to define an adaptive mesh, with bisection spanned only in the space directions where the derivative magnitude is larger than a certain fixed value, because it is not useful to bisect in flat directions. This will obviously decrease the computational complexity constant k. It is possible to decrease the computational time constant with mean of paralleling using domain decomposition [18]. The adjustment set is divided on the first axis, and each processor performs the ψ-algorithm on one of those subdomains. The master processor collects all the results at the end of the calculations.

7 CONCLUSION

A new algebraic approach for interval arithmetic, called pseudo-interval arithmetic has been proposed. It is based on free algebra build from Minkowski products of basis intervals and with dimension higher or equal to 4. One has identified intervals with the elements of this associative algebra and showed that their product is distributive with respect to their addition. Increasing the dimension will give pseudo-intervals product closer to Minkowski one's.

One has presented also a heuristic way to transfer real functions to inclusion ones depending on the space needed (semi-group or vector space). This permits to define a simple but very efficient algorithm for set inversion, the ψ-algorithm, which uses pseudo-intervals arithmetic and probability calculations. The convergence of this algorithm is guaranteed, and it offers several possibilities of applications, such as solving algebraic equations, differential equations, probability law of random variables calculations (discrete or continuous), topological analysis, numerical Lebesgue integrals computations, data analysis such as principal components analysis, and parameters identification.

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RESUMO. Neste artigo, apresenta-se como usar um novo arcabouço de aritmética intervalar com base na construção de álgebra livre, chamado pseudo intervalos, que é associativa e distributiva e permite a construção da inclusão de função bem definida para semi-grupo intervalar e para seu espaço vetorial associado. Apresenta-se o ψ-algoritmo (Inversão Probabilística de Conjuntos), que realiza a inversão de funções e exibe-se alguns exemplos numéricos.

Palavras-chave: Álgebra livre, Pseudo-intervalos e aritmética intervalar, conjunto de inversão, probabilidade.

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