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MATHEMATICAL SCIENCES

The Weibull Burr XII distribution in lifetime and income analysis

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Abstract: We study a five-parameter model called the Weibull Burr XII (WBXII) distribution, which extends several models, including new ones. This model is quite flexible in terms of the hazard function, which exhibits increasing, decreasing, upside-down bathtub, and bathtub shapes. Its density function allows different forms such as left-skewed, right-skewed, reversed-J, and bimodal. We aim to provide some general mathematical quantities for the proposed distribution, which can be useful to real data analysis. We develop a shiny application to provide interactive illustrations of the WBXII density and hazard functions. We estimate the model parameters using maximum likelihood and derive a profile log-likelihood for all members of the Weibull-G family. The survival analysis application reveals that the WBXII model is suitable to accommodate left-skewed tails, which are very common when the variable of interest is the time to failure of a product. The income application is related to player salaries within a professional sports league and it is peculiar because the mean of the player's salaries is much higher than for most professions. Both applications illustrate that the new distribution provides much better fits than other models with the same and less number of parameters.

Key words: Bimodal distribution, Burr XII distribution, profile log-likelihood, Weibull-G family.

INTRODUCTION

For the Burr XII (BXII) distribution, also known as the Singh-Maddala distribution (Singh & Maddala 1975, 1976) with shape parameters $d > 0$ and $c > 0$ and scale parameter $s > 0$, the cumulative distribution function (cdf) is

$$G(x; c, d, s) = 1 - \left[1 + \left(\frac{x}{s}\right)^c\right]^{-d}, \qquad x > 0. \tag{1}$$

For $d = 1$ and $s = m^{-1}$, we have the log-logistic (LL) distribution and, for $c = 1$, it reduces to the Lomax distribution. The probability density function (pdf) corresponding to equation (1) is

$$g(x; c, d, s) = c\, d\, s^{-c}\, x^{c-1} \left[1 + \left(\frac{x}{s}\right)^c\right]^{-d-1}. \tag{2}$$

This distribution is part of the Burr system of distributions (Burr 1942) and has extensive use in the context of income data. For recent examples, see Jäntti & Jenkins (2010), Brzeziński (2013), Tanak et al. (2015). Cirillo (2010) also applied this model for analyzing the size distribution of Italian firms by age.

Chotikapanich et al. (2013) considered it for calculating poverty measures in countries from South and Southeast Asia. Kumar et al. (2013) adopted the BXII distribution on reliability context.

Bourguignon et al. (2014) pioneered a family of univariate distributions generated by extending the Weibull (W) model applied to the odds ratio $G(x)/[1-G(x)]$. For any baseline cdf $G(x;\boldsymbol{\xi})$, which depends on a parameter vector $\boldsymbol{\xi}$, they defined the *Weibull-G* family (for $x \in \mathcal{D} \subseteq \mathbb{R}$) by the pdf and cdf

$$f(x;\alpha,\beta,\boldsymbol{\xi}) = \alpha\beta\, g(x;\boldsymbol{\xi}) \frac{G(x;\boldsymbol{\xi})^{\beta-1}}{1-G(x;\boldsymbol{\xi})^{\beta+1}} \exp\left\{-\alpha\left[\frac{G(x;\boldsymbol{\xi})}{1-G(x;\boldsymbol{\xi})}\right]^{\beta}\right\} \quad (3)$$

and

$$F(x;\alpha,\beta,\boldsymbol{\xi}) = \int_0^{\frac{G(x;\boldsymbol{\xi})}{1-G(x;\boldsymbol{\xi})}} \alpha\beta\, t^{\beta-1} \mathrm{e}^{-\alpha t^{\beta}} \mathrm{d}t = 1-\exp\left\{-\alpha\left[\frac{G(x;\boldsymbol{\xi})}{1-G(x;\boldsymbol{\xi})}\right]^{\beta}\right\}, \quad (4)$$

respectively. The *Weibull-G* family has the same parameters of the G distribution plus two shape parameters $\alpha > 0$ and $\beta > 0$. According to Bourguignon et al. (2014), these additional parameters are sought as a manner to furnish more flexible distributions. If $\beta = 1$, it gives the exponential generator (Gupta et al. 1998). Cordeiro et al. (2015) and Tahir et al. (2016c) introduced another two types of *Weibull-G* families.

Following the formulation by Bourguignon et al. (2014), we define the *Weibull-Burr XII* (WBXII) distribution and provide some of its mathematical quantities which were not addressed by Bourguignon et al. (2014). The new expressions can be helpful for those interested in applying this distribution to real life data.

In a similar approach, we can refer the reader to six other contributed works addressed to specific baselines of the *Weibull-G* family. These contributions are listed in Table I.

Table I. Contributed works on the Weibull-G family.

Distribution	Author(s)
Weibull exponential	Oguntunde et al. (2015)
Weibull Lomax	Tahir et al. (2015)
Weibull Rayleigh	Merovci & Elbatal (2015)
Weibull Pareto	Tahir et al. (2016a)
Weibull Dagum	Tahir et al. (2016b)
Weibull Fréchet	Afify et al. (2016)
Weibull Birnbaum-Saunders	Benkhelifa (2016)
Four-parameter Weibull Burr XII	Afify et al. (2018)

The WBXII distribution is obtained by inserting (1) and (2) in equations (3) and (4). Then, its pdf reduces to (for $x > 0$)

$$f(x) = \frac{\alpha\beta\, c\, d\, s^{-c} x^{c-1}}{[1+(x/s)^c]^{1-d}} \exp\left\{-\alpha\left[(1+(x/s)^c)^d - 1\right]^{\beta}\right\} \left[(1+(x/s)^c)^d - 1\right]^{\beta-1}, \quad (5)$$

where $\alpha > 0$, $\beta > 0$, $d > 0$ and $c > 0$ are shape parameters and $s > 0$ is a scale parameter. The corresponding cdf is

$$F(x) = 1 - \exp\left\{-\alpha\left[(1+(x/s)^c)^d - 1\right]^\beta\right\}. \tag{6}$$

Based on equation (6) we note that $z(x) = [1+(x/s)^c]^d$ is the inverse of the BXII survival function, which is identifiable. Then, $\alpha\,[z(x)-1]^\beta$ is identifiable and so the WBXII cdf.

Two interpretations of (6) are now presented. First, let T be a BXII random variable with cdf (1) describing a real life phenomenon. If the random variable X represents the *odds*, the risk that this stochastic mechanism following the lifetime T will not occur at time x is given by $G(x;c,d,s)/[1 - G(x;c,d,s)]$. If we model the randomness X of these odds by the Weibull density with scale parameter $\alpha > 0$ and shape parameter $\beta > 0$, the cdf of X is given by (6). For the second interpretation, we take a WBXII random variable X and a random variable T with the Weibull density (for $t > 0$) defined above. We can write $P(X \leq x) = F(x) = 1 - \exp\left\{-\alpha\left[(1+(x/s)^c)^d - 1\right]^\beta\right\} = P\left(T \leq G(x;c,d,s)/[1 - G(x;c,d,s)]\right)$. Since the function $\varkappa(x) = G(x;c,d,s)/[1-G(x;c,d,s)]$ is always monotonic and non–decreasing, we obtain $T = \varkappa(X)$, where the equality of random variables refers to equivalence of distributions. So, if X has the WBXII cdf (6), then $T = \varkappa(X)$ has a Weibull cdf with the above parameters.

If X is a random variable with density function (5), we write $X \sim \text{WBXII}(c,d,s,\alpha,\beta)$. The hazard rate function (hrf) of X reduces to

$$h(x) = \alpha\beta\, c\, d\, s^{-c} x^{c-1}\left[1+(x/s)^c\right]^{d-1}\left[(1+(x/s)^c)^d - 1\right]^{\beta-1}.$$

The main contributions of this paper are described below:

1. In the estimation section, we demonstrate that all members of the Weibull-G family present a semi-closed form for the maximum likelihood estimator (MLE) of α. Thus, the MLEs for any member of the Weibull-G family can also be determined from the profile log-likelihood function, which is much simpler.

2. Bourguignon et al. (2014) obtained general mathematical expressions for the Weibull-G family based on an infinite linear combination of exponentiated-G (exp-G) densities. We derive a new linear representation for the WBXII pdf in a simpler form based directly on the BXII model itself. Besides, we provide some important mathematical and statistical properties of the proposed distribution. These results are especially helpful for applications to real lifetime data.

3. Equation (5) has different forms, including left-skewed, right-skewed, reversed-J, decreasing-increasing-decreasing and bimodal. Plots of the WBXII density function for selected parameter values are displayed in Figure 1. We develop a shiny application that allows the reader to access dynamic plots of the WBXII pdf and hrf[1]. From those plots, we note that the WBXII density presents bimodality, or the unusual decreasing-increasing-decreasing shape when β is very small, and the baseline shape parameters c and d are large.

4. The proposed distribution overcomes a limitation of its baseline, whose hazard function presents only monotonic and unimodal shapes. The WBXII hrf admits the four main characteristics: decreasing, increasing, upside-down bathtub, and bathtub shaped. These are desirable

[1] https://newdists.shinyapps.io/WBXIIdist/.

properties for a lifetime distribution. Figure 2 provides plots of the hrf of *X* for selected parameter values.

5. Equation (5) extends at least twenty lifetime distributions, including new ones. In fact, if we combine the Weibull and its two sub-models (exponential and Rayleigh) with seven special cases (Lomax, Fisk, log-logistic, Weibull, exponential and Rayleigh) of the Burr XII distribution including this distribution itself, the WBXII model can generate twenty descendants. Note that the first three models listed in Table 1 published in 2015 are just special cases of the new distribution. For $\alpha = \beta = 1$, the power generalized Weibull (PGW) (Nikulin & Haghighi 2006, Dimitrakopoulou et al. 2007) also arises as a special model.

6. A major advantage of fitting a wider model to real data is that we can easily verify, based on the likelihood ratio (LR) statistics, whether its sub-models (with fewer parameters) can be more properly to the data.

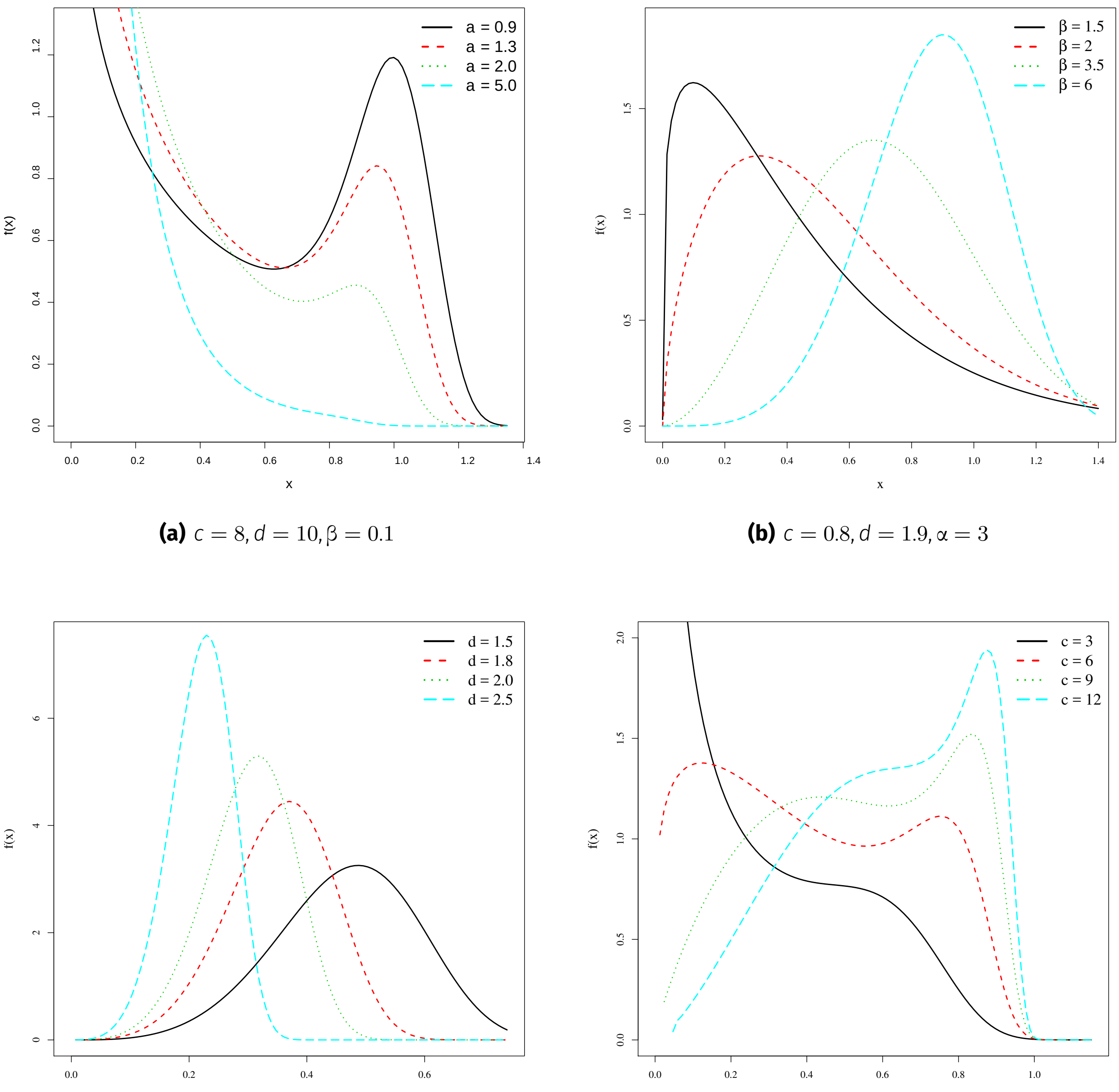
7. Equation (5) can be reduced to a four-parameter distribution by setting the scale parameter to one (Afify et al. 2018) and then it becomes a very competitive model to all well-known four-parameter lifetime distributions such as the beta Weibull, Kumaraswamy Weibull, Kumaraswamy gamma, beta Dagum, among several others.

8. Although the proposed model has five parameters, it can provide much better fits, based on Anderson-Darling and Kolmogorov-Smirnov statistics, than other models with the same and less number of parameters. This fact is proved empirically in applications to survival analysis and income distribution (see the application section). The survival analysis application illustrates the WBXII superiority to accommodate left-skewed tails, which are very common when the variable of interest is the time to failure of a product or component. The second data set is related to player salaries within a professional sports league. The salary distribution is typically heavy skewed to the right for several professions and also in the macro-economy. For the sports market, the salary distribution is no different from other occupations in terms of the shape. However, according to Rockerbie (2003), it is peculiar because their mean is much higher, and those markets are typically a natural monopoly. The WBXII model showed suitable to accommodate these feature as well.

An implementation in R language (R Core Team 2018) used to obtain plots, application and simulation for the five-parameter Weibull Burr XII distribution is available in the footnote[2].

The remainder of the paper is organized as follows. Two useful expansions for the WBXII cdf and pdf are derived. We investigate some of its mathematical properties such as the quantile function (qf), ordinary and incomplete moments, mean deviations, and generating function. We determine the order statistics. The maximum likelihood method is used to estimate the model parameters . Two applications to real data sets are addressed in the application section. Finally, we offer some concluding remarks.

[2]https://drive.google.com/open?id=1an1YbEeX1XHEbSvwIo8jbEAwkLxFJSxP.



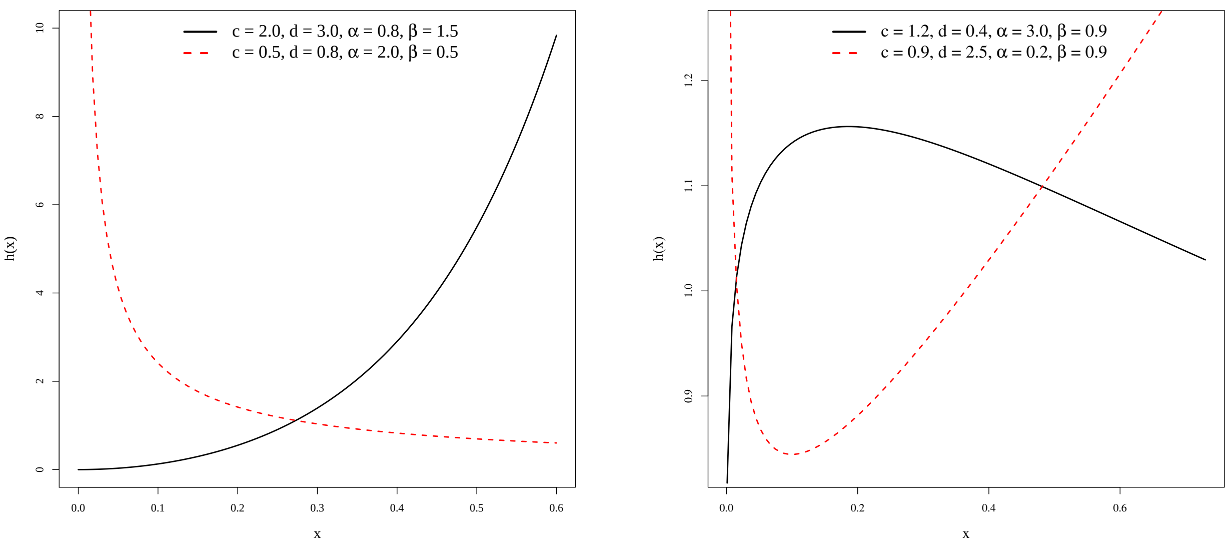
(a) $c = 8, d = 10, \beta = 0.1$

(b) $c = 0.8, d = 1.9, \alpha = 3$

(c) $c = 0.8, a = 1, \beta = 5$

(d) $d = 15, a = 1.2, \beta = 0.2$

Figure 1. Plots of the WBXII density with $s = 1$**.**



(a) **(b)**

Figure 2. Plots of the WBXII hrf with $s = 1$**.**

TWO USEFUL EXPANSIONS

Two useful expansions for equations (5) and (6) can be derived by using power series. It follows from Bourguignon et al. (2014) that the *Weibull-G* density function can be expressed as

$$f(x;\alpha,\beta,\boldsymbol{\xi}) = \sum_{j,k=0}^{\infty} \rho_{j,k}\, g(x;\boldsymbol{\xi}) G(x;\boldsymbol{\xi})^{(k+1)\beta+j-1},$$

where

$$\rho_{j,k} = \frac{(-1)^k \beta\, \alpha^{k+1}\, \Gamma((k+1)\beta+j+1)}{k!\, j!\, \Gamma((k+1)\beta+1)},$$

and $\Gamma(\cdot)$ is the gamma function. By replacing $G(x;\boldsymbol{\xi})$ for (1) and $g(x;\boldsymbol{\xi})$ for (2), we obtain

$$f(x;\alpha,\beta,\boldsymbol{\xi}) = c\, d\, s^{-c} \sum_{j,k=0}^{\infty} \rho_{j,k}\, x^{c-1} u^{-d-1} \left(1-u^{-d}\right)^{(k+1)\beta+j-1}, \tag{7}$$

where $u = 1 + \left(\frac{x}{s}\right)^c$. If $|z| < 1$ and $b > 0$ is a real non-integer, the power series holds

$$(1-z)^{b-1} = \sum_{r=0}^{\infty} \frac{(-1)^r \Gamma(b)}{r!\, \Gamma(b-r)}\, z^r. \tag{8}$$

Using the above expansion for $\left(1-u^{-d}\right)^{[(k+1)\beta+j-1]-1}$ in equation (7) and, after some algebraic manipulation, we obtain

$$f(x) = \sum_{r=0}^{\infty} w_r\, g(x; c, (r+1)d, s), \tag{9}$$

where (for $r = 0, 1, \ldots$)

$$w_r = \sum_{k,j=0}^{\infty} \frac{(-1)^r \rho_{j,k} \Gamma((k+1)\beta + j)}{\Gamma((k+1)\beta + j - r)(r+1)!} \tag{10}$$

and $g(x; c, (r+1)d, s)$ is the BXII density function with scale parameter s and shape parameters $(r+1)d$ and c. Equation (9) reveals that the WBXII density is an infinite linear combination of BXII densities. So, several structural properties of the WBXII distribution can be determined from those BXII properties. By integrating equation (9) gives

$$F(x) = \sum_{r=0}^{\infty} w_r \, G(x; c, (r+1)d, s). \tag{11}$$

Equations (9) and (11) are the main results of this section.

MATHEMATICAL PROPERTIES

In this section, we obtain some mathematical quantities of the WBXII distribution including quantile and random number generation, ordinary and incomplete moments, moment generating function (mgf), mean deviations and Bonferroni and Lorenz curves. By determining analytical expressions for those quantities can be more efficient than computing them directly by numerical integration of the density function (5).

Density and hazard shapes

The WBXII density and hazard functions are quite flexible as can be noted in Figures 1 and 2. They can take various forms depending on the shape parameters α, β, c, and d. In this section, we illustrate the exact behavior of these functions for some parameter sets by analyzing their limiting behavior and derivatives of their logarithms with respect to x. In addition, we provide interactive plots that allow observing the behavior of these functions for several parameter combinations.

For the pdf (5), we note that

$$\lim_{x \to 0} f(x) = \begin{cases} \infty & \text{if } \beta, c < 1, \\ \alpha d s^{-1} & \text{if } \beta = c = 1, \\ 0 & \text{if } \beta, c > 1, \end{cases}$$

and $\lim_{x \to \infty} f(x) = 0$.

Some calculations show that

$$\frac{\mathrm{d} \log f(x)}{\mathrm{d}x} = \frac{1}{x} \left\{ c - 1 + \frac{c\,(x/s)^c}{1 + (x/s)^c} \left[d - 1 + \frac{d\,(\beta - 1)\,[1 + (x/s)^c]^d}{[1 + (x/s)^c]^d - 1} - \frac{\alpha \beta\, d\,[1 + (x/s)^c]^d}{\{[1 + (x/s)^c]^d - 1\}^{1-\beta}} \right] \right\}. \tag{12}$$

The critical points of the WBXII pdf are the roots of the above equation, and numerical software is required to solve it. Nevertheless, from (12), we can verify that the WBXII density is decreasing if $\beta \leq 1$, $c \leq 1$, and $d \leq 1$.

We can analyze the limiting behavior of the WBXII hrf for some parameter sets. We verify that

$$\lim_{x\to 0} h(x) = \begin{cases} \infty & \text{if } \beta, c < 1, \\ \alpha d s^{-1} & \text{if } \beta = c = 1, \\ 0 & \text{if } \beta, c > 1, \end{cases} \quad \text{and} \quad \lim_{x\to\infty} h(x) = \begin{cases} 0 & \text{if } \beta, c, d < 1, \\ \alpha s^{-1} & \text{if } \beta = c = d = 1, \\ \infty & \text{if } \beta, c, d > 1. \end{cases}$$

The critical point of $h(x)$ are obtained from

$$\frac{\mathrm{d}\log h(x)}{\mathrm{d}x} = \frac{1}{x}\left\{c - 1 + \frac{c\,(x/s)^c}{1+(x/s)^c}\left[d - 1 + \frac{d\,(\beta-1)\,[1+(x/s)^c]^d}{[1+(x/s)^c]^d - 1}\right]\right\} = 0.$$

From the last equation, we can note that: i) if $\beta < 1$, $c < 1$ and $d < 1$, then $\mathrm{d}\log h(x)/\mathrm{d}x < 0$ and the hrf is decreasing; ii) if $\beta = c = d = 1$, then $\mathrm{d}\log h(x)/\mathrm{d}x = 0$ and the hrf is constant in α/s; and iii) if $\beta > 1$, $c > 1$ and $d > 1$, then $\mathrm{d}\log h(x)/\mathrm{d}x > 0$ and the hrf is increasing; iv) the parameter α does not affect the hrf shapes; and v) numerical softwares are required to obtain the root of this equation.

Quantile function and random number generation

The qf of X follows by inverting (6) as

$$Q(u) = s\left\{\left[\left(\frac{-\log(1-u)}{\alpha}\right)^{1/\beta} + 1\right]^{1/d} - 1\right\}^{1/c}. \tag{13}$$

By setting $u = 0.5$ in (13) gives the median M of X. Different quantiles of interest can also be obtained from (13) by replacing appropriate values for u.

If U is a uniform variate on the unit interval $(0,1)$, then the random variable $X = Q(U)$ has pdf given by (5). Thus, the qf can be useful to generate observations from the WBXII distribution using the inverse transformation (see Section SIMULATION STUDY for an example). Another motivation to introduce this quantity is its applicability to obtain alternative expressions for the skewness and kurtosis. The Bowley's skewness (Kenney & Keeping 1962) based on quartiles is

$$B = \frac{Q(3/4) - 2Q(1/2) + Q(1/4)}{Q(3/4) - Q(1/4)}.$$

The Moors' kurtosis (Moors 1988) based on octiles is

$$K_M = \frac{Q(7/8) - Q(5/8) - Q(3/8) + Q(1/8)}{Q(6/8) - Q(2/8)}.$$

These measures are less sensitive to outliers and may exist even for distributions without moments. These quantile-based measures can be obtained for the WBXII model from (13). See the next section for illustrative examples.

Moments

The nth ordinary moment of X can be determined from (9) as

$$\mu'_n = E(X^n) = \sum_{r=0}^{\infty} w_r \int_0^{\infty} x^n\, g(x; d, (r+1)d, s)dx.$$

Using a result in Zimmer et al. (1998), we have (for $n < c\,d$)

$$\mu_n' = s^n\, d \sum_{r=0}^{\infty} (r+1) w_r\, B((r+1)d - n\,c^{-1}, 1 + n\,c^{-1}), \tag{14}$$

where $B(a,b)$ is the beta function. The central moments (μ_s), cumulants ($\varkappa_s$) and the skewness and kurtosis of X can be evaluated from (14) using well-known relationships.

Table II provides a small numerical study by computing the first three moments and the B and K_M coefficients for six scenarios, each one with a different parametrization for the WBXII distribution. Figure 3 displays plots of B and K_M for some parameter values. In fact, they indicate that the proposed distribution is quite flexible in terms of variation of the moments, skewness and kurtosis. It can accommodate positive and negative values for both skewness and kurtosis coefficients.

Table II. First three moments and B and K_M for some scenarios of the WBXII distribution.

Scen.	c	d	s	α	β	$E(X)$	$E(X^2)$	$E(X^3)$	B	K_M
1	0.1	0.4	2.5	3.0	1.5	3.3612	135.9916	7272.5887	0.9950	43.3541
2	1.5	3.0	0.2	2.0	0.5	0.0373	0.0031	0.0004	0.3309	0.8222
3	1.0	5.0	5.0	2.0	2.3	0.5158	0.3062	0.2001	-0.0009	0.0065
4	1.0	5.0	3.0	2.0	0.5	0.1898	0.1180	0.1150	0.5139	1.3401
5	0.4	0.2	1.8	3.0	4.0	3.0980	160.8452	9497.2515	0.4246	1.2157
6	0.8	1.2	0.2	0.9	5.0	0.1369	0.0201	0.0001	-0.0283	-0.0698

Incomplete moments

The hth incomplete moment of X, say $T_h(y) = \int_0^y x^h f(x)dx$, can be expressed as

$$T_h(y) = c\,d \sum_{r=0}^{\infty} (r+1)\, w_r \int_0^y x^{h-1} \left(\frac{x}{s}\right)^c \left[1 + \left(\frac{x}{s}\right)^c\right]^{-(r+1)d-1} dx.$$

By setting $t = \left[1 + \left(\frac{x}{s}\right)^c\right]^{-1}$ in the last equation, we obtain

$$T_h(y) = d\,s^h \sum_{r=0}^{\infty} (r+1)\, w_r \int_{s^c/(s^c+y^c)}^{1} t^{(r+1)d-\frac{h}{c}-1} (1-t)^{\frac{h}{c}} dt.$$

Hence, the hth incomplete moment of X reduces to (for $h < c\,d$)

$$T_h(y) = d\,s^h \sum_{r=0}^{\infty} (r+1)\, w_r\, B_{s^c/s^c+y^c}((r+1)d - h\,c^{-1}, 1 + h\,c^{-1}), \tag{15}$$

where $B_z(a,b) = \int_z^1 t^{a-1}(1-t)^{b-1}dt$ is the upper incomplete beta function.

An important application of the first incomplete moment refers to the mean deviations about the mean and the median, namely

$$\delta_1 = 2\mu_1' F(\mu_1') - 2\,T_1(\mu_1') \qquad \text{and} \qquad \delta_2 = \mu_1' - 2\,T_1(M),$$

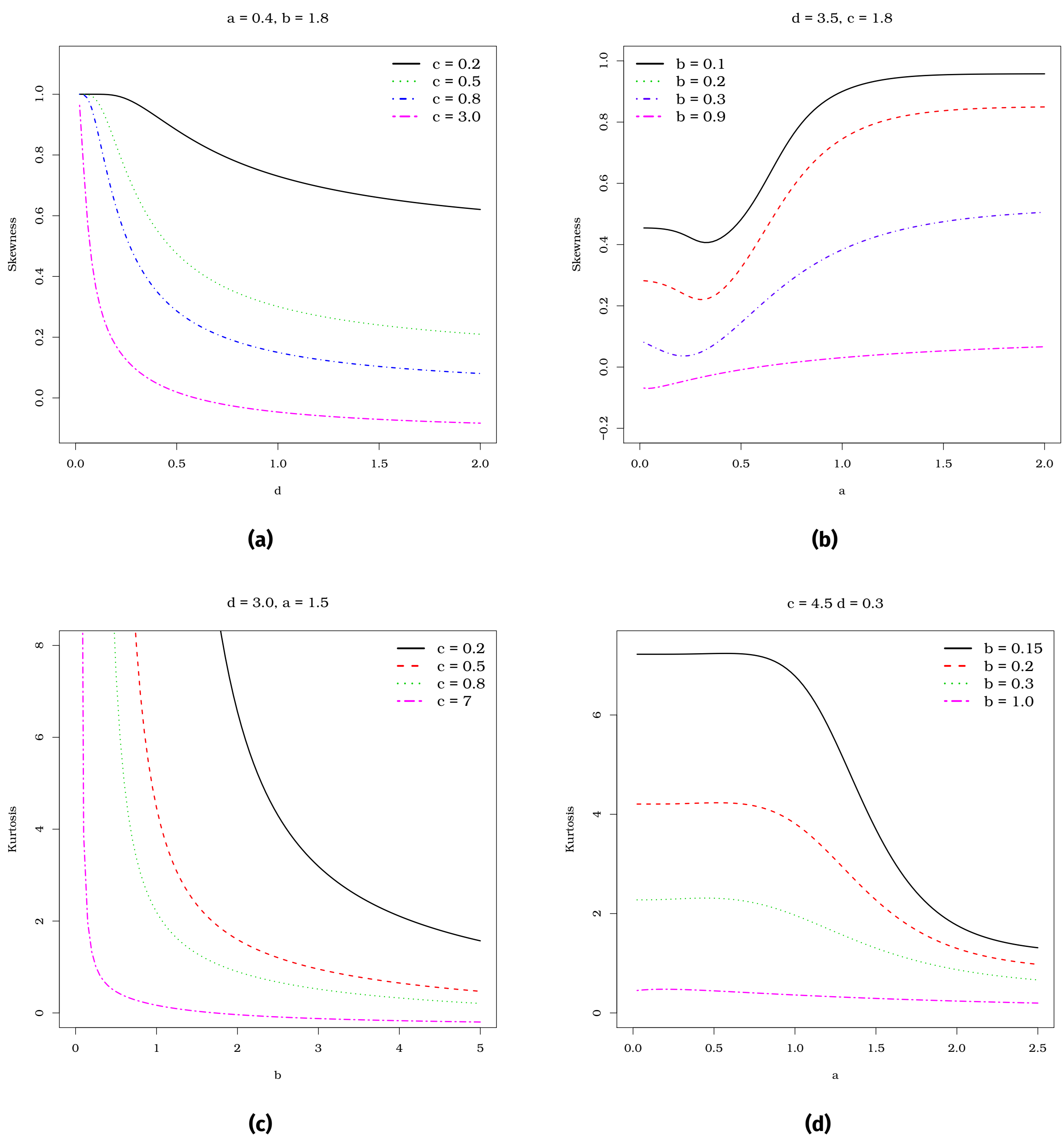


Figure 3. Skewness and kurtosis of X for some parameter values.

respectively, where $F(\mu_1')$ is easily obtained from (6), $\mu_1' = E(X)$, the median M of X follows from (13) as $M = Q(1/2)$, and (for $c\,d > 1$) $T_1(y)$ is the first incomplete moment given by (15) with $h = 1$. An alternative expression for $T_1(\cdot)$ comes from (9) as

$$T_1(y) = c\,d\,s \sum_{r=0}^{\infty}(r+1)w_r \int_0^y x^c \left[1+\left(\frac{x}{s}\right)^c\right]^{-(r+1)d-1} \mathrm{d}x.$$

Setting $z = (x/s)^c$, we obtain

$$T_1(y) = d\,s \sum_{r=0}^{\infty}(r+1)w_r \int_0^{\left(\frac{y}{s}\right)^c} z^{1/c}(1+z)^{-(r+1)d-1}\mathrm{d}z.$$

Thus,

$$T_1(y) = \frac{c\,d\,s\,y^{c+1}}{1+c} \sum_{r=0}^{\infty}(r+1)\,w_r\,{}_2F_1\left[1+\frac{1}{c},\ (r+1)d+1;\ 2+\frac{1}{c};\ -\left(\frac{y}{s}\right)^c\right],$$

where

$${}_2F_1(a,b;c;x) = \sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k}\frac{x^k}{k!}$$

is the hypergeometric function and $(a)_k = a(a+1)(a+k-1)$ is the (rising) Pochhammer symbol if $k > 1$ and $(a)_0 = 1$.

The above results are related to the Bonferroni and Lorenz curves. These curves are important in economics for studying income and poverty but can be useful in demography, reliability, insurance, medicine, and several other fields. For a given probability π, they are defined by $B(\pi) = T_1(q)/(\pi\mu_1')$ and $L(\pi) = T_1(q)/\mu_1'$, respectively, where $q = Q(\pi)$ is given by (13). If π is the proportion of units whose income is lower than or equal to q, the values of $L(\pi)$ yield fractions of the total income and $B(\pi)$ refers to the relative income levels.

Generating function

The mgf of X is defined by $M(t) = E(\mathrm{e}^{tX})$. Let $M_d(t)$ be the mgf of the BXII(c,d,s) distribution. We can write from (9)

$$M(t) = \sum_{r=0}^{\infty} w_r\,M_{(r+1)d}(t), \tag{16}$$

where $M_{(r+1)d}(t)$ is the BXII$(s,(r+1)d,c)$ generating function and $M_d(t)$ is given by

$$M_d(t) = cds^{-c}\int_0^{\infty} x^{c-1}\mathrm{e}^{tx}\left[1+\left(\frac{x}{s}\right)^c\right]^{-d-1}\mathrm{d}x. \tag{17}$$

Guerra et al. (2020) considered the following expansion in the above equation

$$\left[1+\left(\frac{x}{s}\right)^c\right]^{-d-1} = \sum_{j=0}^{\infty}\binom{-d-1}{j}\left[\left(\frac{x}{s}\right)^{cj}\mathbf{1}_{(0,s]}(x) + \left(\frac{x}{s}\right)^{-c(j+d+1)}\mathbf{1}_{(s,\infty)}(x)\right], \tag{18}$$

where $\mathbf{1}_A(x)$ denotes the indicator function over a given set of real numbers A, i.e., $\mathbf{1}_A(x) = 1$ if $x \in A$ and $\mathbf{1}_A(x) = 0$ elsewhere. Combining (17) and (18), and after some algebra, Guerra et al. (2020) expressed the BXII mgf as an infinite sum of incomplete gamma functions given by (for $t < 0$)

$$M_d(t) = c\,d \sum_{j=0}^{\infty} \binom{-d-1}{j} \Big[(-st)^{-(j+1)c}\,\gamma\left((j+1)c, -st\right) + (-st)^{(d+j)c}\,\Gamma\left(-(d+j)c, -st\right)\Big], \tag{19}$$

where $\gamma(a,z) = \int_0^z t^{a-1}\,\mathrm{e}^{-t}\mathrm{d}t$ and $\Gamma(a,z) = \int_z^\infty t^{a-1}\,\mathrm{e}^{-t}\mathrm{d}t$ are the lower and upper incomplete gamma functions, respectively. Hence, for $t < 0$, the mgf of X can be follows from (16) and (19) as a double summation

$$\begin{aligned} M(t) &= c\,d \sum_{i=0}^{\infty}(r+1)w_r \sum_{j=0}^{\infty} \binom{-(r+1)d-1}{j} \Big[(-st)^{-(j+1)c}\,\gamma\left((j+1)c, -st\right) \\ &+ (-st)^{c[(b+r)d+j]}\,\Gamma\left(-c\left[(r+1)d+j\right], -st\right)\Big]. \end{aligned} \tag{20}$$

Equation (20) is the main result of this section.

Stress-strength reliability

Let X_1 be the life of a component with a random strength that is subjected to a random stress X_2. We can define stress-strength reliability as $R = P(X_2 < X_1) = \int_0^\infty f_1(x)\,F_2(x)\mathrm{d}x$, i.e., the component fails when the stress applied to it exceeds the strength ($X_2 > X_1$); otherwise, the component will function well. This measure is very useful in reliability.

Let X_1 and X_2 have independent WBXII$(c, d_1, s, \alpha_1, \beta_1)$ and WBXII$(c, d_2, s, \alpha_2, \beta_2)$ distributions, respectively, with the same shape parameter c and scale parameter s. We can derive R using the results in (9) and (11). Note that the pdf of X_1 and cdf of X_2 can be expressed as

$$f_1(x) = \sum_{m=0}^{\infty} w_m\, g(x; c, (m+1)d_1, s) \quad \text{and} \quad F_2(x) = \sum_{n=0}^{\infty} w_n\, G(x; c, (n+1)d_2, s),$$

respectively, where w_m and w_n are given by (10). Thus, setting $u = 1 + \left(\frac{x}{s}\right)^c$, we obtain

$$R = \sum_{m,n=0}^{\infty} \frac{(n+1)d_2\, w_m\, w_n}{(m+1)d_1 + (n+1)d_2}.$$

ORDER STATISTICS

Order statistics are important tools in many areas of statistical theory and practice. Let $X_1, \cdots, X_n$ be a random sample of the Weibull-G family and $X_{i:n}$ the ith order statistic. The density $f_{i:n}(x)$ of $X_{i:n}$ has the form

$$f_{i:n}(x) = \frac{1}{B(i, n-i+1)} \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} f(x)\, F(x)^{i+j-1}. \tag{21}$$

Setting $u = 1 + (x/s)^c$ and using the power series in (8), we can rewrite $F(x)^{i+j-1}$ as

$$\left\{1 - \exp\left[-\alpha\,\frac{\left(1-u^{-d}\right)^{\beta}}{u^{-d\beta}}\right]\right\}^{i+j-1} = \sum_{k=0}^{\infty} \frac{(-1)^k\,(i+j-1)!}{k!\,(i+j-k-1)!} \exp\left\{-\alpha\,k\,\frac{\left(1-u^{-d}\right)^{\beta}}{u^{-d\beta}}\right\}.$$

Inserting the above expansion in (21) and after some algebra, we obtain

$$f_{i:n}(x) = \frac{\alpha\,\beta\,c\,d\,s^{-c}x^{c-1}\;u^{-d-1}}{B(i,n-i+1)}\,\frac{\left(1-u^{-d}\right)^{\beta-1}}{u^{-d(\beta+1)}} \sum_{k=0}^{\infty}\sum_{j=0}^{n-i}\binom{n-i}{j}\frac{(-1)^{j+k}\,(i+j-1)!}{k!\,(i+j-k-1)!} \tag{22}$$
$$\times\;\exp\left\{-\alpha\,(1+k)\,\frac{\left(1-u^{-d}\right)^{\beta}}{u^{-d\beta}}\right\}.$$

By expanding the exponential function in the last equation, rewriting $(u^{-d})^{\beta}$ as $\left[1-(1-u^{-d})\right]^{\beta}$, considering the power series given by (for $|z| < 1$ and $b > 0$ real non-integer)

$$(1-z)^{-b} = \sum_{j=0}^{\infty}\frac{\Gamma(b+j)}{j!\,\Gamma(b)}$$

and inserting both expansions in equation (22), we obtain

$$f_{i:n}(x) = \frac{\beta\,c\,d\,s^{-c}x^{c-1}\;u^{-d-1}}{B(i,n-i+1)} \sum_{k,l,m=0}^{\infty}\sum_{j=0}^{n-i}\binom{n-i}{j}\frac{(-1)^{j+k+l}\,\alpha^{l+1}\,(1+k)^l\,(i+j-1)!}{k!\,l!\,(i+j-k-1)!}$$
$$\times\;\frac{\Gamma((l+1)\beta+1+m)}{m!\,\Gamma((l+1)\beta+1)}\left(1-u^{-d}\right)^{(l+1)\beta+m-1}.$$

Finally, expanding $(1-u^{-d})^{(l+1)\beta+m-1}$ in the previous expression as in (8) and after some algebra, we can write

$$f_{i:n}(x) = \sum_{q=0}^{\infty} \upsilon_q\, g(x;c,(q+1)d,s), \tag{23}$$

where (for $q = 0, 1, \ldots$)

$$\upsilon_q = \sum_{k,l,m=0}^{\infty}\sum_{j=0}^{n-i}\binom{n-i}{j}\frac{(-1)^{j+k+l+q}\,\beta\,\alpha^{l+1}\,(1+k)^l\,(i+j-1)!}{k!\,l!\,m!\,(q+1)!\,B(i,n-i+1)\,(i+j-k-1)!}$$
$$\times\;\frac{\Gamma((l+1)\beta+1+m)\,\Gamma((l+1)\beta+m)}{\Gamma((l+1)\beta+1)\,\Gamma((l+1)\beta+m-q)}$$

and $g(x;c,(q+1)d,s)$ is the BXII density function with scale parameter s and shape parameters $(q+1)d$ and c. Equation (23) is the main result of this section. Based on this linear representation, we can obtain some structural properties of $X_{i:n}$ using a similar procedure as that one applied for the WBXII mathematical properties.

MAXIMUM LIKELIHOOD ESTIMATION

The maximum likelihood method is an important technique employed to estimate model parameters in distributions. Bourguignon et al. (2014) determined the MLEs for the *Weibull-G* parameters from the total log-likelihood function. In this section, we demonstrate alternatively that the MLEs of the*Weibull-G* family can be determined based on the profile log-likelihood function. We also provide the MLEs for the WBXII distribution.

The Weibull-G profile log-likelihood

Let $x_1, \cdots, x_n$ be observed values from the *Weibull-G* family with parameter vector $\boldsymbol{\Theta} = (\alpha, \beta, \boldsymbol{\xi}^\top)^\top$. As shown by Bourguignon et al. (2014), the total log-likelihood function for $\boldsymbol{\Theta}$ has the form

$$\begin{aligned}\ell(\boldsymbol{\Theta}) =& n \log(\alpha\beta) - \alpha \sum_{i=1}^{n} H(x_i;\boldsymbol{\xi})^{\beta} + \beta \sum_{i=1}^{n} \log[H(x_i;\boldsymbol{\xi})] \\ & - \sum_{i=1}^{n} \log[G(x_i;\boldsymbol{\xi})] - \sum_{i=1}^{n} \log[1 - G(x_i;\boldsymbol{\xi})],\end{aligned} \tag{24}$$

where $H(x;\boldsymbol{\xi}) = G(x;\boldsymbol{\xi}) / [1 - G(x;\boldsymbol{\xi})]$. Thus, the first component of the score vector $U(\boldsymbol{\Theta}) = \left(U_\alpha, U_\beta, U_{\boldsymbol{\xi}}^\top\right)^\top$ is

$$U_\alpha = \frac{n}{\alpha} - \sum_{i=1}^{n} H(x_i;\boldsymbol{\xi})^{\beta}.$$

For fixed β and $\boldsymbol{\xi}$, a semi-closed MLE for α follows from $U_\alpha = 0$ as

$$\hat{\alpha}(\beta, \boldsymbol{\xi}) = \frac{n}{\sum\limits_{i=1}^{n} H(x_i;\boldsymbol{\xi})^{\beta}}.$$

By replacing α by $\hat{\alpha}$ in (24), we obtain the *Weibull-G* profile log-likelihood for $\boldsymbol{\Theta_p} = (\beta, \boldsymbol{\xi})^\top$ as

$$\begin{aligned}\ell(\boldsymbol{\Theta_p}) = n \log(n\beta) + \beta \sum_{i=1}^{n} \log[H(x_i;\boldsymbol{\xi})] - \sum_{i=1}^{n} \log[G(x_i;\boldsymbol{\xi})] - \sum_{i=1}^{n} \log[1 - G(x_i;\boldsymbol{\xi})] \\ - n \log\left[\sum_{i=1}^{n} H(x_i;\boldsymbol{\xi})^{\beta}\right] - n.\end{aligned} \tag{25}$$

Hence, the MLE $\widehat{\boldsymbol{\Theta}}_{\boldsymbol{p}}$ of the parameter vector $\boldsymbol{\Theta_p}$ can be numerically found by maximizing (25) and the MLE of α is just $\hat{\alpha}(\widehat{\boldsymbol{\Theta}}_{\boldsymbol{p}})$. Note that the maximization of the profile log-likelihood might be simpler since it involves one less parameter.

The WBXII MLEs

Let $x_1, \cdots, x_n$ be a random sample of size n from the WBXII(c, d, s, α, β) distribution. Let $\boldsymbol{\Theta} = (c, d, s, \alpha, \beta)^T$ be the parameter vector. The log-likelihood function for $\boldsymbol{\Theta}$ follows as

$$\ell(\boldsymbol{\Theta}) = n \log(\alpha \beta\, c\, d\, s^{-1}) + (c-1)c^{-1} \sum_{i=1}^{n} \log(u_i - 1) + (d-1) \sum_{i=1}^{n} \log u_i - \alpha \sum_{i=1}^{n} \left(u_i^d - 1\right)^{\beta} + (\beta - 1) \sum_{i=1}^{n} \log\left(u_i^d - 1\right), \tag{26}$$

where $u_i = 1 + \left(\frac{x_i}{s}\right)^c$. The estimates of the model parameters can be obtained by maximizing (26).

Alternatively, we can be differentiating (26) and solving the resulting nonlinear likelihood equations. The components of the score vector $\boldsymbol{U(\Theta)}$ are

$$\begin{aligned}
U_c(\boldsymbol{\Theta}) = {} & n\, c^{-1} + c^{-1} \sum_{i=1}^{n} \log(u_i - 1) + (d-1)\, c^{-1} \sum_{i=1}^{n} (u_i - 1) \log(u_i - 1)\, u_i^{-1} \\
& - \alpha \beta\, d\, (c\, s)^{-1} \sum_{i=1}^{n} (u_i - 1) \log(u_i - 1)\, u_i^{d-1} \left(u_i^d - 1\right)^{\beta - 1} \\
& + d\, (\beta - 1)\, (c\, s)^{-1} \sum_{i=1}^{n} (u_i - 1) \log(u_i - 1)\, u_i^{d-1} \left(u_i^d - 1\right)^{-1}, \\
U_d(\boldsymbol{\Theta}) = {} & n\, d^{-1} + \sum_{i=1}^{n} \log u_i + (\beta - 1) \sum_{i=1}^{n} u_i^d \log u_i \left(u_i^d - 1\right)^{-1} \\
& - \alpha\beta \sum_{i=1}^{n} u_i^d \log u_i \left(u_i^d - 1\right)^{\beta - 1}, \\
U_s(\boldsymbol{\Theta}) = {} & -c\, n\, s^{-1} + c(d-1)s^{-1} \sum_{i=1}^{n} (u_i - 1) u_i^{-1} \\
& - c\, d\, (\beta - 1)\, s^{-1} \sum_{i=1}^{n} (u_i - 1) u_i^{d-1} \left(u_i^d - 1\right)^{-1} \\
& + \alpha \beta\, c\, d\, s^{-1} \sum_{i=1}^{n} (u_i - 1) u_i^{d-1} \left(u_i^d - 1\right)^{\beta - 1}, \\
U_\alpha(\boldsymbol{\Theta}) = {} & n\, \alpha^{-1} - \sum_{i=1}^{n} \left(u_i^d - 1\right)^{\beta}
\end{aligned}$$

and

$$U_\beta(\boldsymbol{\Theta}) = n\, \beta^{-1} + \sum_{i=1}^{n} \log\left(u_i^d - 1\right) - \alpha \sum_{i=1}^{n} \left(u_i^d - 1\right)^{\beta} \log\left(u_i^d - 1\right).$$

Setting these expressions to zero, $\boldsymbol{U(\Theta)} = \mathbf{0}$, and solving them simultaneously yields the MLEs of the five parameters. These equations cannot be solved analytically, but some statistical softwares can be used to solve them numerically using iterative methods such as the Newton-Raphson type algorithms.

For fixed c, d, s and β, the MLE of α is

$$\hat{\alpha}(c,d,s,\beta) = \frac{n}{\sum_{i=1}^{n}(u_i^d - 1)^{\beta}}. \tag{27}$$

By fixing $x_1, \cdots, x_n$, it is easy to verify from (27) that

- $\hat{\alpha} \to 1$ when $\beta \to 0^+$;
- $\hat{\alpha} \to \infty$ when $s \to \infty$;
- $\hat{\alpha} \to 0^+$ when $s \to 0^+$;
- $\hat{\alpha} \to 0^+$ when $d \to \infty$;
- $\hat{\alpha} \to \infty$ when $d \to 0^+$.

By replacing α by (27) in equation (26) and letting $\boldsymbol{\Theta_p} = (c, d, s, \beta)$, the profile log-likelihood function for $\boldsymbol{\Theta_p}$ has the form

$$\begin{aligned}\ell(\boldsymbol{\Theta_p}) = {} & n\log(n\,\beta\,c\,d\,s^{-1}) + (c-1)c^{-1}\sum_{i=1}^{n}\log(u_i - 1) + (d-1)\sum_{i=1}^{n}\log u_i \\ & - n\log\sum_{i=1}^{n}\left(u_i^d - 1\right)^{\beta} + (\beta-1)\sum_{i=1}^{n}\log\left(u_i^d - 1\right) - n.\end{aligned} \tag{28}$$

The components of the score vector $\boldsymbol{U(\Theta_p)}$ of (28) are

$$\begin{aligned}U_c(\boldsymbol{\Theta_p}) = {} & n\,c^{-1} + c^{-1}\sum_{i=1}^{n}\log(u_i - 1) + (d-1)c^{-1}\sum_{i=1}^{n}(u_i - 1)\log(u_i - 1)u_i^{-1} \\ & - n\,\beta\,d\,c^{-1}\left[\sum_{i=1}^{n}\left(u_i^d - 1\right)^{\beta}\right]^{-1}\sum_{i=1}^{n}(u_i - 1)u_i^{d-1}\left(u_i^d - 1\right)^{\beta-1}\log(u_i - 1) \\ & + d\,(\beta-1)\,c^{-1}\sum_{i=1}^{n}(u_i - 1)u_i^{d-1}\left(u_i^d - 1\right)^{-1}\log(u_i - 1),\end{aligned}$$

$$U_d(\boldsymbol{\Theta_p}) = n\,d^{-1} + \sum_{i=1}^{n} \log u_i + (\beta-1)\sum_{i=1}^{n} u_i^d \left(u_i^d - 1\right)^{-1} \log u_i$$

$$- n\beta \left[\sum_{i=1}^{n}\left(u_i^d - 1\right)^{\beta}\right]^{-1} \sum_{i=1}^{n} u_i^d \left(u_i^d - 1\right)^{\beta-1} \log u_i,$$

$$U_s(\boldsymbol{\Theta_p}) = -n\,c\,s^{-1} + c(d-1)s^{-1}\sum_{i=1}^{n}(u_i - 1)u_i^{-1}$$

$$- c\,d\,(\beta-1)\,s^{-1}\sum_{i=1}^{n}(u_i-1)\,u_i^{d-1}\left(u_i^d - 1\right)^{-1}$$

$$+ n\,\beta\,c\,d\,s^{-1}\left[\sum_{i=1}^{n}\left(u_i^d - 1\right)^{\beta}\right]^{-1}\sum_{i=1}^{n}(u_i-1)\,u_i^{d-1}\left(u_i^d - 1\right)^{\beta-1}$$

and

$$U_\beta(\boldsymbol{\Theta_p}) = n\,\beta^{-1} + \sum_{i=1}^{n}\log\left(u_i^d - 1\right) - n\left[\sum_{i=1}^{n}\left(u_i^d - 1\right)^{\beta}\right]^{-1}\sum_{i=1}^{n}\left(u_i^d - 1\right)^{\beta}\log\left(u_i^d - 1\right).$$

Solving the equations $\boldsymbol{U(\Theta_p)} = \mathbf{0}$ simultaneously yields the MLEs of c, d, s and β. The MLE of α is just $\hat{\alpha}(\hat{c}, \hat{d}, \hat{s}, \hat{\beta})$. The maximization of the profile log-likelihood might be simpler since it involves only four parameters.

For interval estimation of the model parameters, we require the observed information matrix $\boldsymbol{J(\Theta)}$, whose elements can be obtained from the authors upon request. Under standard regularity conditions, the approximate confidence intervals for the model parameters can be constructed based on the multivariate normal $N_5(0, \boldsymbol{J(\widehat{\Theta})}^{-1})$ distribution.

A major advantage of fitting the proposed distribution to a real data set is that we can easily verify, based on the likelihood ratio (LR) statistics, whether any of its sub-models (with fewer parameters) can be preferred to these data.

The maximized (unrestricted and restricted) log-likelihoods are useful to compute LR statistics to verify if WBXII sub-models (with fewer parameters) can be preferred for fitting a given data set. This is a major advantage since the WBXII model extends at least twenty lifetime distributions, including new ones. Let $\boldsymbol{\Theta_0}$ be the restricted parameter vector for a given WBXII sub-model. Thus, hypothesis tests of the type $H_0 : \boldsymbol{\Theta} = \boldsymbol{\Theta}_0$ versus $H : \boldsymbol{\Theta} \neq \boldsymbol{\Theta}_0$ can be performed using LR statistics. For example, the LR statistic for testing $H_0 : \alpha = \beta = 1$ (versus $H : H_0$ is not true), thus comparing the WBXII and PGW distributions, is

$$w = 2\{l(\widehat{c}, \widehat{d}, \widehat{s}, \widehat{\alpha}, \widehat{\beta}) - l(\widetilde{c}, \widetilde{d}, \widetilde{s}, 1, 1)\} \xrightarrow{d} \chi_2^2,$$

where $\widehat{s}$, $\widehat{d}$, $\widehat{c}$, $\widehat{\alpha}$, and $\widehat{\beta}$ are the MLEs under H, $\widetilde{c}$, $\widetilde{d}$, and $\widetilde{s}$ are the estimates under H_0 and $\boldsymbol{\Theta_0} = (c, d, s, 1, 1)^\top$.

SIMULATION STUDY

In this section, we evaluate the performance of the MLEs of the parameters of the WBXII distribution. We conduct Monte Carlo simulations based on 10,000 replications under five different parameter combinations and sample size n =100, 250 and 500. The simulation study is performed using the `optim` subroutine and SANN algorithm in `R` software for maximizing the log-likelihood in (26). Table III reports the empirical mean estimates and corresponding root mean squared errors (RMSEs). For all parameter combinations, we note that the empirical biases and RMSEs decrease when the sample size increases in agreement with the first-order asymptotic theory.

Table III. Mean estimates and RMSEs of the WBXII distribution.

		Mean					RMSE				
$\boldsymbol{\Theta}$	n	$\hat{c}$	$\hat{d}$	$\hat{s}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{c}$	$\hat{d}$	$\hat{s}$	$\hat{\alpha}$	$\hat{\beta}$
(0.1, 0.4, 2.5,3,1.5)	100	0.216	0.586	2.763	2.379	1.376	0.285	0.480	1.635	1.631	0.830
	250	0.136	0.466	2.569	2.666	1.472	0.138	0.193	1.072	1.150	0.547
	500	0.109	0.430	2.526	2.832	1.511	0.058	0.110	0.761	0.824	0.366
(1.5,3,0.2,2,5)	100	1.409	4.700	0.907	2.557	0.835	0.928	2.892	1.452	1.880	0.799
	250	1.537	3.338	0.231	2.121	5.121	0.362	1.136	0.122	1.039	1.059
	500	1.525	3.209	0.217	2.067	5.073	0.292	0.909	0.084	0.862	0.848
(1,5,5,2,2.3)	100	2.117	5.968	3.756	3.468	1.823	2.011	3.394	2.905	3.225	1.297
	250	1.608	5.472	4.137	3.066	2.022	1.288	2.597	2.450	2.539	1.042
	500	1.362	5.275	4.380	2.755	2.122	0.900	2.031	2.072	2.030	0.860
(1,5,3,2,0.5)	100	1.430	5.925	3.203	2.184	0.590	1.029	2.822	2.336	1.327	0.583
	250	1.288	5.575	3.125	2.080	0.546	0.759	2.206	1.918	0.916	0.447
	500	1.183	5.318	3.089	2.037	0.517	0.561	1.770	1.622	0.714	0.314
(0.4,0.2,1.8,3,4)	100	0.674	0.199	2.198	2.833	4.097	0.581	0.125	1.333	1.488	0.803
	250	0.543	0.198	1.953	2.860	4.093	0.379	0.098	0.999	1.144	0.584
	500	0.475	0.197	1.887	2.913	4.062	0.247	0.067	0.835	0.926	0.428

APPLICATIONS

In this section, we illustrate the usefulness of the WBXII distribution for modeling income and lifetime data. The first data set represents the times to failure (10^3 h) of 40 suits of turbochargers in one type of diesel engine (Xu et al. 2003). These data were previously considered by Benkhelifa (2016). The second data set consists in annual salaries of 862 professional baseball players of the Major League Baseball for the season 2016. The data are measured in American dollars and are available for download at https://www.usatoday.com/sports/mlb/salaries/2016/player/all/. Both data sets are available in the appendix.

We use these two data sets to compare the fits of the WBXII distribution with other six related models, i.e., the beta Burr XII (BBXII), Kumaraswamy Burr XII (KwBXII), BXII, LL, PGW, and W distributions. The seven competitive models are defined as follows. The BBXII pdf is

$$f(x) = \frac{cd(x)^{c-1}}{s^c B(\alpha,\beta)} \left\{1 - \left[1 + (x/s)^c\right]^{-d}\right\}^{\alpha-1} \left[1 + (x/s)^c\right]^{-(d\beta+1)}, \qquad x > 0,$$

where $\alpha > 0$ and $\beta > 0$ are shape parameters; the KwBXII density is

$$\begin{aligned} f(x) &= \alpha\beta\, c\, d\, s^{-c} x^{c-1} \left[1 + \left(\frac{x}{s}\right)^c\right]^{-d-1} \left\{1 - \left[1 + \left(\frac{x}{s}\right)^c\right]^{-d}\right\}^{\alpha-1} \times \\ & \left[1 - \left\{1 - \left[1 + \left(\frac{x}{s}\right)^c\right]^{-d}\right\}^{\alpha}\right]^{\beta-1}, \quad x > 0, \end{aligned}$$

where $\alpha > 0$ and $\beta > 0$ are shape parameters; the BXII density is given by equation (2); the LL is defined by taking $s = m^{-1}$ and $d = 1$ in (2); and the PGW and W distributions are sub-models of (5) by taking $\alpha = \beta = 1$ and $\alpha = \beta = d = 1$, respectively.

In each case, the parameters are estimated by maximum likelihood using the `AdequacyModel` script in the `R` software (Marinho et al. 2016). We report the MLEs and their corresponding standard errors. We present the following goodness-of-fit statistics: the Akaike information criteria (AIC), consistent Akaike information criteria (CAIC), Hannan-Quinn information criteria (HQIC), corrected Anderson-Darling statistic (A^*) (Chen & Balakrishnan 1995) and Kolmogorov-Smirnov (KS) statistic. The lower values of these statistics are associated with better fits. We also compute the LR statistics for testing WBXII sub-models.

Turbochargers failure time

Table IV provides some descriptive statistics of the turbochargers failure time data. Note that these data present negative skewness (S) and kurtosis (K) and have an amplitude of 7.4. We also have close values for the mean and median. This descriptive summary indicates that the turbochargers data follow a power-law distribution with a left-skewed tail.

Table IV. Descriptive statistics for turbochargers data.

Mean	Median	SD	Variance	S	K	Min.	Max.
6.25	6.50	1.96	3.82	-0.66	-0.36	1.60	9.00

Table V lists the MLEs for the fitted models to these data and their corresponding standard errors. For all fitted models, the parameter estimates are significant. Table VI gives the goodness-of-fit statistics. The WBXII distribution has the lowest values for all statistics. Note that the WBXII is quite competitive with the W distribution. However, the W model may not be an effective alternative for modeling left-skewed data. Table VII provides the LR statistics for the PGW and W fitted models. By considering a significance level of 10%, we may reject both sub-models in favor of the WBXII distribution. It is another clear evidence of the WBXII superiority for modeling these data. Figure 4 displays the histogram and the estimated densities with lower values for goodness-of-fit statistics. We note that the WBXII yields a good adjustment to the current data. In fact, the wider model is more accurate than the W distribution for modeling the right tail and is quite competitive with the BBXII distribution. Thus, we can conclude from Figure 4 and Tables VI and VII that the WBXII model provides the best fit to the turbocharges failure time data.

Table V. MLEs of the model parameters and their standard errors in parentheses.

	c	d	s	α	β
WBXII	13.4956	7.5404	8.8931	1.1128	0.2216
	(2.7613)	(3.6805)	(0.7060)	(0.3671)	(0.0576)
BBXII	15.4893	11.1316	11.2702	0.1666	4.5249
	(0.0395)	(0.1854)	(0.1994)	(0.0282)	(2.0589)
KwBXII	15.1758	6.2322	9.2966	0.1559	0.7550
	(0.1931)	(0.9355)	(0.3827)	(0.0398)	(0.2306)
BXII	3.8290	3.9620	9.6190		
	(0.5506)	(1.8934)	(1.5960)		
PGW	3.5830	1.3300	7.7010		
	(0.5466)	(0.6152)	(1.4125)		
W	3.8740		6.9230		
	(0.5177)		(0.2948)		
	c	m			
LL	4.8480	6.2230			
	(0.6544)	(0.3476)			

Baseball players salaries

Some descriptive statistics for the baseball players data are provided in Table VIII. These data present positive values for the S and K coefficients, thus indicating right-skew data. We have a high amplitude, variance, and SD. We also note that the mean and median are not so close. This behavior is quite common in income data sets.

Table IX provides the MLEs and their standard errors for the seven models fitted to the baseball players data. We have significant estimates for all parameters of these models. Table X lists some goodness-of-fit measures for the fitted models. The WBXII distribution presents the lowest values for all statistics. These results indicate that the WBXII distribution yields a better fit than the other fitted models to the baseball players data. The results for the LR tests are given in Table XI. Clearly, we reject the PGW and W distributions in favor of the wider model. So, there is a strong evidence of the potential need for the extra shape parameters of the WBXII in the second application. Figure 5 displays the fitted WBXII, BBXII and KwBXII densities and the histogram for the baseball players data. They confirm that

Table VI. Goodness-of-fit statistics for the fits to the turbochargers failure time data.

	AIC	CAIC	HQIC	A^*	KS
WBXII	**165.8103**	**167.5750**	**168.8635**	**0.1186**	**0.0532**
BBXII	166.9631	168.7278	170.0163	0.1240	0.0744
KwBXII	167.0753	168.8400	170.1286	0.1241	0.0579
BXII	174.8080	175.4746	176.6399	0.8475	0.1029
PGW	169.6197	170.2864	171.4516	0.4962	0.1066
W	168.9511	169.2754	170.1724	0.5730	0.1072
LL	181.4134	181.7377	182.6347	1.4072	0.1432

Table VII. LR statistics for the fits to the turbochargers failure time data.

Models	Θ_0	Statistic w	p-value
WBXII *vs* PGW	$(c, d, s, 1, 1)^\top$	7.80889	0.02015
WBXII *vs* W	$(c, 1, s, 1, 1)^\top$	9.14138	0.05013

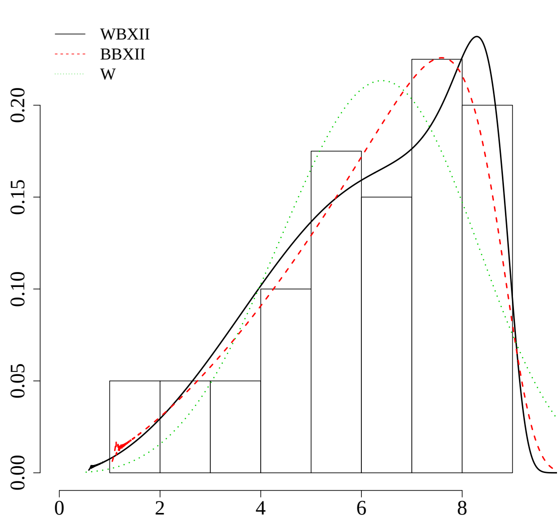


Figure 4. Histogram and estimated densities of the WBXII, BBXII and W models for the turbochargers failure time data.

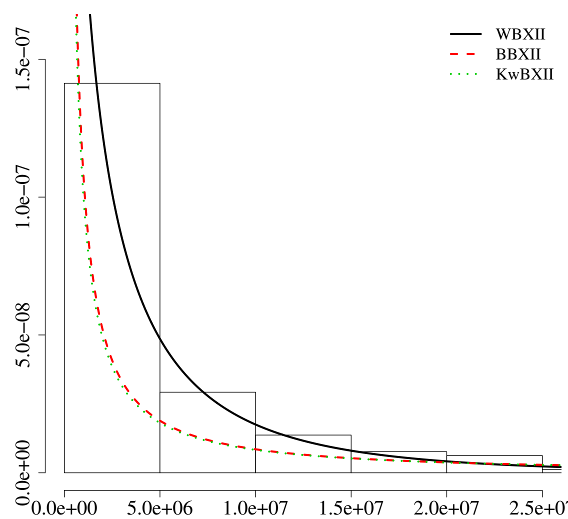


Figure 5. Histogram and estimated densities of the WBXII, BBXII and KwBXII models for the baseball players data.

the WBXII model yields the best fit. Finally, we can conclude that the WBXII is an effective alternative to modeling lifetime (see the first data set) and income (see the second data set) data, especially when they present power-law tails. It is quite competitive to the classical Weibull distribution and other BXII generalizations.

Table VIII. Descriptive statistics for baseball players data.

Mean	Median	SD	Variance	S	K	Min.	Max.
4,529,859.69	1.5×10^6	6,070,096	3.7×10^{13}	1.98	3.74	507,500	34,416,666

Table IX. MLEs of the model parameters and corresponding standard errors in parentheses.

	c	*d*	*s*	α	β
WBXII	0.5527	0.0796	1.8716	2.4141	7.4298
	(0.0686)	(0.0115)	(0.8292)	(1.0993)	(0.3079)
BBXII	1.8134	0.0487	5.7723	12.3094	6.2716
	(0.1742)	(0.0046)	(0.7939)	(0.6308)	(0.5546)
KwBXII	3.92390	0.03251	2.59116	9.16545	4.0435
	(0.2031)	(0.0016)	(0.4065)	(0.5123)	(0.2463)
BXII	6.8459	0.0102	2.6500		
	(0.6858)	(0.0010)	(0.3534)		
PGW	1.6123	0.0400	10.1363		
	(0.1926)	(0.0047)	(1.2913)		
W	0.0646		9.9377		
	(0.0015)		(1.2611)		
	c	*m*			
LL	0.1289	14.2324			
	(0.0036)	(1.7615)			

Table X. Goodness-of-fit statistics for the fitted models for baseball players data.

	AIC	CAIC	HQIC	A^*	KS
WBXII	**28023.5004**	**28023.5705**	**28032.6096**	**44.3602**	**0.2168**
BBXII	28977.3246	28977.3947	28986.4338	45.9851	0.3812
KwBXII	29077.1495	29077.2196	29086.2586	45.7928	0.3858
BXII	31195.1989	31195.2268	31200.6643	45.8619	0.5745
PGW	30421.9845	30422.0125	30427.4500	45.2787	0.6359
W	32146.5307	32146.5447	32150.1744	44.9291	0.8667
LL	31825.4901	31825.5040	31829.1337	45.4325	0.7943

Table XI. Likelihood ratio statistics for the fits to the baseball players data.

Models	Θ_0	Statistic *w*	p-value
WBXII *vs* PGW	$(c, d, s, 1, 1)^\top$	2402	< 0.0001
WBXII *vs* W	$(c, 1, s, 1, 1)^\top$	4129	< 0.0001

CONCLUDING REMARKS

The five-parameter *Weibull Burr XII* distribution is introduced and studied in detail. The proposed model extends at least twenty lifetime distributions, including new ones. Its hazard rate function can be increasing, decreasing, upside-down bathtub, and bathtub-shaped. It is also very flexible in terms of the density function, which has several forms including left-skewed, right-skewed, reversed-J, and bimodal. We emphasize that a shiny application is developed to provide interactive plots and illustrate the behavior of those functions for several parameter combinations. Some mathematical properties of the proposed model are presented, including the ordinary and incomplete moments, quantile and generating functions, mean deviations, stress-strength reliability, and order statistics. We estimate the model parameters using maximum likelihood, present the components of the score vector and the profile log-likelihood. We also derive a general result for the Weibull-G family, which presents a semi-closed form for the maximum likelihood estimator of the parameter α. We provide applications to real lifetime and income data sets. They illustrate the usefulness of the proposed distribution for modeling these kinds of data and also prove empirically that the WBXII distribution is quite competitive to other known Burr XII and Weibull generalizations.

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Author contributions

The participation of the authors in the production of the manuscript is as follows: Renata Rojas Guerra - conceptualization and characterization of the new distribution, mathematical properties and implementation of computational routines. Fernando A. Peña-Ramírez - application, simulation studies, writing the original draft, and computational routines. Gauss M. Cordeiro - review and general correction of the paper.

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APPENDIX A

A.1. First data set

Table A.I. Turbochargers failure time data.

1.60	3.50	4.80	5.40	6.00	6.50	7.30	7.70	8.10	8.50
2.00	3.90	5.00	5.60	6.10	6.70	7.30	7.80	8.30	8.70
2.60	4.50	5.10	5.80	6.30	7.00	7.30	7.90	8.40	8.80
3.00	4.60	5.30	6.00	6.50	7.10	7.70	8.00	8.40	9.00

A.2. Second data set

Table A.II. Baseball players salaries data.

30714286	8400000	6500000	4000000	2275000	1050000	530500	518000	511200
34416666	16000000	6750000	4000000	5125000	1050000	530000	518000	511000
31000000	18000000	6725000	4000000	3000000	7081428	530000	517800	511000
29200000	12250000	7333333	4000000	2750000	1000000	530000	517700	511000
25714285	11666667	6575000	4000000	2175000	1000000	530000	517500	510500
25000000	25000000	8666666	4000000	2150000	1000000	530000	517500	510500
25000000	11333333	8750000	3750000	1800000	1000000	529600	517500	510500
24000000	10000000	6500000	3900000	2100000	1000000	529000	517500	510500
25833333	14325000	6250000	3900000	2100000	1000000	529000	517500	510200
25000000	13000000	6500000	3900000	2075000	1000000	528700	517300	510200
24400000	11500000	6425000	3900000	9083333	1000000	528600	517246	510120
24000000	10357142	5543750	3800000	3625000	1000000	528200	517000	510000
23777778	10000000	5500000	3750000	2500000	1000000	528000	517000	510000
25000000	11500000	6250000	3750000	2000000	1000000	527600	516700	510000
22500000	11000000	6250000	3583333	2000000	1000000	527500	516700	510000
23000000	10500000	5487500	3700000	2000000	1000000	527500	516650	510000
22000000	10000000	6225000	3125000	2000000	1000000	527000	516500	510000
24000000	11000000	6200000	3300000	2000000	1000000	527000	516500	510000
30000000	10976096	6170000	3333333	2000000	987500	527000	516500	510000
22125000	10936574	8750000	4000000	2000000	975000	526400	516100	510000
22142857	18000000	6125000	5125000	2000000	975000	526014	516100	510000
17666666	10700000	6125000	3500000	2000000	950000	525500	516100	509700
18000000	10650000	12500000	3500000	2000000	925000	525500	516000	509700
20000000	10550000	8285714	3500000	2000000	2666666	525500	516000	509675

23000000	14000000	6416666	5400000	2000000	900000	525300	515900	509600
21857142	10400000	6000000	2900000	2000000	900000	525270	515900	509500
27500000	9333333	6000000	3450000	2000000	900000	525000	515800	509500
22000000	12000000	6000000	3400000	2000000	897500	525000	515750	509500
18750000	5700000	6000000	3400000	2000000	895000	525000	515500	509500
21250000	10000000	7750000	3375000	1925000	850000	525000	515400	509500
18555555	7000000	6400000	5166666	1825000	850000	524900	515000	509500
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15500000	9625000	5731704	3300000	1750000	807500	524500	515000	509500
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20625000	10000000	7150000	10000000	1700000	800000	524500	515000	509500
22500000	11325000	5600000	3857142	1697500	800000	524100	515000	509500
15775000	11600000	5600000	3200000	1650000	765000	524100	515000	509500
18571429	6500000	5500000	3125000	1650000	725000	524000	515000	509300
23000000	9150000	5250000	3125000	1600000	660000	523900	514875	509200
19500000	25000000	5000000	3150000	1600000	652000	523700	514500	509000
17250000	9000000	4250000	3125000	1600000	650000	523500	514500	508900
11538461	5103900	5500000	3100000	2100000	650000	523500	514500	508800
18000000	8000000	4200000	3025000	1550000	625000	523400	514500	508800
22000000	10000000	6000000	5000000	1525000	607000	523000	514400	508750
17500000	9000000	5350000	3900000	1875000	606000	522900	514400	508600
17000000	8500000	5312000	3500000	1750000	600000	522700	514250	508600
17000000	7500000	5300000	3000000	1500000	600000	522500	514200	508500
18000000	11000000	5857142	3000000	1500000	587500	522500	514000	508500
19000000	8750000	6825000	3000000	1500000	575000	522500	514000	508500
21666666	8000000	5250000	3000000	1500000	575000	522400	513900	508500
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16000000	9250000	5100000	2950000	1500000	566000	521700	513308	508500
14250000	6950000	7000000	2925000	1500000	563750	521600	513300	508500
24083333	6000000	10333333	2925000	1500000	556000	521300	513000	508500
15000000	8250000	5000000	4250000	1500000	550000	521300	513000	508500
16000000	7833333	4333333	2900000	1490314	550000	521200	513000	508450
16000000	14285714	5000000	2875000	1475000	550000	521100	513000	508200
15800000	12500000	3750000	3500000	1475000	548000	521000	512900	508200
15800000	10000000	5000000	2800000	1475000	547500	521000	512500	508000

15800000	9166666	5000000	2800000	1450000	546500	521000	512500	508000
13000000	7000000	5000000	2800000	1400000	546250	520700	512500	508000
14500000	9000000	5000000	2800000	1400000	545000	520500	512500	508000
15000000	8000000	5000000	2800000	1400000	545000	520500	512500	507500
16400000	8000000	4800000	4700000	1387500	545000	520500	512500	507500
15000000	8000000	4800000	3000000	1375000	543400	520300	512500	507500
12000000	8000000	4750000	2750000	1500000	542604	520200	512500	507500
16000000	8000000	7700000	2725000	1350000	542500	520200	512500	507500
14250000	8000000	5500000	2700000	1350000	541000	520000	512500	507500
15000000	8000000	5000000	2650000	3466666	540300	520000	512500	507500
16666666	8571428	4500000	2625000	1300000	540000	520000	512500	507500
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12000000	7562500	4400000	2600000	1275000	539000	520000	512500	507500
14000000	14000000	4350000	2600000	1275000	539000	520000	512500	507500
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13750000	7666666	5750000	2075000	1250000	537500	520000	512500	507500
13000000	7333333	4250000	2525000	1250000	537500	520000	512500	507500
8583333	7500000	4250000	4710739	1250000	536500	520000	512500	507500
12083333	6500000	4250000	2750000	1250000	536200	519700	512100	507500
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13000000	7250000	4200000	2500000	1250000	535000	519500	512000	507500
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13600000	7150000	4125000	2500000	1200000	535000	519100	511500	507500
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13250000	7000000	4100000	2500000	1150000	532900	518500	511500	507500
12000000	7000000	10416666	2500000	975000	532500	518425	511500	507500
12500000	7000000	5918483	2425000	1100000	532500	518200	511400	507500
16875000	7000000	5000000	2400000	1065000	532500	518100	511360	507500
12500000	9250000	5000000	2375000	1050000	532000	518000	511250	507500
13333333	6166666	4000000	2350000	1050000	532000	518000	511200	507500
507500	507500	507500	507500	507500	507500	507500		