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A Metric property of Umbilic points

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ABSTRACT

In the space of cubic forms of surfaces, regarded as a G-space and endowed with a natural invariant metric, the ratio of umbilic points with a negative index to those with a positive index is evaluated in terms of the asymmetry of the metric, defined here. This work contains also a section with references pertinent to umbilic points and one with a discussion of a connection of the above defined ratio with that reported in 1977 by Berry and Hannay in the domain of Statistical Physics.

Key words: Umbilic point, principal curvature lines.

1 UMBILIC POINTS, INVARIANT METRICS AND VOLUME RATIOS

At an umbilic point p of an oriented C^3 surface $\mathcal{S}$ embedded in an oriented Euclidean 3-space $\mathbb{R}^3$ the principal curvatures coincide. In a neighborhood of such point, $\mathcal{S}$ can be written in a Monge chart as the graph $z = h(x, y)$ of a function of the form

$$h(x, y) = \frac{k}{2}(x^2 + y^2) + \frac{1}{6}(ax^3 + 3bx^2y + 3b'xy^2 + a'y^3) + o((x^2 + y^2)^{3/2}). \qquad (1)$$

The frame $(x, y; z)$ is positive and adapted to $\mathcal{S}$ at p. This means that the plane orthonormal frame (x, y) is attached to the tangent plane, positively oriented, and the z-axis is along the unit positive normal to $\mathcal{S}$ at p.

Any other such presentation as the graph $Z = H(X, Y)$ of a function

$$H(X, Y) = \frac{K}{2}(X^2 + Y^2) + \frac{1}{6}(AX^3 + 3BX^2Y + 3B'XY^2 + A'Y^3) + o((X^2 + Y^2)^{3/2}$$

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differs by a rotation

$$x = \cos\theta X - \sin\theta Y,\ \ y = \sin\theta X + \cos\theta Y,\ \ z = Z,$$

linking the positively oriented frames $(X, Y; Z)$ and $(x, y; z)$, adapted to the surface at p.

The functions are related by $H(X, Y) = h(x, y)$; substitution leads to

$$\begin{aligned}
K &= k\\
A &= a\cos^3\theta + 3b\cos^2\theta\sin\theta + 3b'\sin^2\theta\cos\theta + a'\sin^3\theta\\
B &= -a\sin\theta\cos^2\theta + b\cos\theta(3\cos^2\theta - 2) + b'\sin\theta(2 - 3\sin^2\theta) + a'\cos\theta\sin^2\theta\\
B' &= a\sin^2\theta\cos\theta + b\sin\theta(3\sin^2\theta - 2) + b'\cos\theta(3\cos^2\theta - 2) + a'\cos^2\theta\sin\theta\\
A' &= -a\sin^3\theta + 3b\sin^2\theta\cos\theta - 3b'\cos^2\theta\sin\theta + a'\cos^3\theta
\end{aligned} \tag{2}$$

Thus the group $\mathbb{O}(2)$ of rotations in the plane acts linearly, to the right, on the four dimensional space of real cubic forms

$$\frac{1}{6}(ax^3 + 3bx^2y + 3b'xy^2 + a'y^3),$$

identified with line vectors $u = (a, b, b', a')$ in $\mathbb{R}^4$.

Denote by $\Omega(\theta)$ the matrix of the linear transformation in $\mathbb{R}^4$, corresponding to the frame rotation by an angle θ. That is $U = u\Omega(\theta)$, with $U = (A, B, B', A')$ and $u = (a, b, b', a')$.

From equation 2, we get

$$\Omega(\theta) = \begin{pmatrix}
\cos^3\theta & -\sin\theta\cos^2\theta & \sin^2\theta\cos\theta & -\sin^3\theta\\
3\cos^2\theta\sin\theta & \cos\theta(3\cos^2\theta - 2) & -\sin\theta(2 - 3\sin^2\theta) & 3\cos\theta\sin^2\theta\\
3\sin^2\theta\cos\theta & -\sin\theta(3\sin^2\theta - 2) & \cos\theta(3\cos^2\theta - 2) & -3\cos^2\theta\sin\theta\\
\sin^3\theta & \sin^2\theta\cos\theta & \cos^2\theta\sin\theta & \cos^3\theta
\end{pmatrix} \tag{3}$$

The space $\mathbb{U}^4$ of umbilic intrinsic cubic forms of surfaces is defined as the G-Space $\mathbb{R}^4$, endowed with the above action of the group $G = \mathbb{O}(2)$.

The quadratic form

$$T(u) = ab' + a'b - b^2 - (b')^2 \tag{4}$$

is invariant under $G = \mathbb{O}(2)$. That is $T(U) = T(u)$ if $U = u\Omega(\theta)$. Thus it is defined on $\mathbb{U}^4$. It characterizes the transversal umbilic points, as those with $T(u) \neq 0$.

It is well known that the Index $I(u)$ of a transversal umbilic u is $I(u) = \frac{1}{2}sign(T(u))$. See (Bruce and Fidal 1989) and (Gutierrez and Sotomayor 1982, 1983) where the identification of $T \neq 0$ with the transversality to the manifold of umbilic 2-jets is made.

The index of an isolated umbilic counts the number of turns made by a principal direction at a point of the surface that makes a small circuit around the umbilic, (Spivak 1980) and (Smyth and Xavier 1992).

According to (Spivak 1980) and (Struik 1988), the differential equation of principal lines around p in this chart is defined as a variety in the projective bundle. In the chart $(x, y, [dx : dy])$, the variety is given by the equation:

$$P(x, y; [dx : dy]) = Ldy^2 + Mdxdy + Ndx^2 = 0, \tag{5}$$

where the functions L, M and N are:

$$\begin{aligned} L = h_x h_y h_{yy} - (1 + h_y^2)h_{xy} &= -bx - b'y + h.o.t \\ M = (1 + h_x^2)h_{yy} - (1 + h_y^2)h_{xx} &= (b' - a)x + (a' - b)y + h.o.t \\ N = (1 + h_x^2)h_{xy} - h_x h_y h_{xx} &= bx + b'y + h.o.t \end{aligned}$$

and therefore $T(u) = \frac{\partial(N,M)}{\partial(x,y)}|_{(0,0)}$.

THEOREM 1. (Gutierrez and Sotomayor 1982, 1991, Bruce and Fidal 1989). *Let p be an umbilic point and consider the Monge chart as in equation (1). The transversality condition $T \neq 0$ holds if and only if the surface $P = 0$ in equation 5 is regular along the projective line $x = y = 0$ covers a punctured neighborhood of p. Then it defines a cylinder whose projection, with the projective line removed, covers twice a punctured neighborhood of p, one for each of the two open cylinders –one for each direction field– resulting from the removal of the projective line. The covering is orientation preserving or reversing according to $T > 0$ or $T < 0$. See Figure 1 for an illustration of one of the cylinders and its projection.*

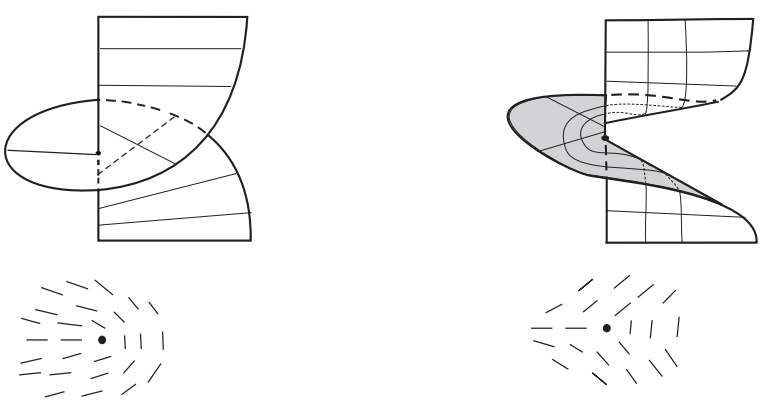

Fig. 1 – Index of Transversal Umbilic Points: left positive, right negative.

A *metric* in $\mathbb{U}^4$ is a positive definite quadratic form such as $q(u) = uQu^*$, Q being a positive definite real 4×4 symmetric matrix, invariant under the action of $\mathbb{O}(2)$. That is $Q = \Omega(\theta)Q\Omega(\theta)^*$, for all θ.

PROPOSITION 1. *Any metric $q(u) = uQu^*$, invariant under the action of $\mathbb{O}(2)$, is given by Q of the*

form:

$$Q = \begin{pmatrix} \frac{2}{3}\alpha + \frac{1}{3}\beta & 0 & \alpha & 0 \\ 0 & \beta & 0 & \alpha \\ \alpha & 0 & \beta & 0 \\ 0 & \alpha & 0 & \frac{2}{3}\alpha + \frac{1}{3}\beta \end{pmatrix} \tag{6}$$

where $\beta > 0$, *and* $\beta(\frac{2}{3}\alpha + \frac{1}{3}\beta) - \alpha^2 > 0$, *which gives the positivity of* Q.

PROOF. The proof follows by solving the equation $Q = \Omega(\theta)Q\Omega(\theta)^*$ first for $\theta = \pi/2$ and $\theta = \pi/4$ and finally checking that it holds for all θ, the proof follows.

A criterion for the positivity of a symmetric matrix, consists of the positivity of all principal minors (Gantmacher 1960), Chap. X, page 306. Notice that for Q in (6) the positivity of the second and third principal minors imply that of the other two. □

REMARK 1. Another way to obtain the expression of Q in (6) consists in projecting the 10 dimensional space $\mathcal{M}^{10}$ of 4×4 symmetric matrices M via the averaging $\mathcal{A}$ along the orbits of $\Omega(\theta)$:

$$\mathcal{A}(M) = \frac{1}{2\pi}\int_0^{2\pi} \Omega(\theta)M\Omega(\theta)^* d\theta. \tag{7}$$

Denoting by m_{ij} the entries of the symmetric matrix M, integration in expression (7) gives:

$$\begin{aligned} \alpha &= (6m_{24} + 3m_{11} + 3m_{44} - m_{33} - m_{22} + 6m_{13})/16 \\ \beta &= (-6m_{24} + 9m_{11} + 9m_{44} + 5m_{33} + 5m_{22} - 6m_{13})/16 \end{aligned} \tag{8}$$

for the invariant symmetric matrix $\mathcal{A}(M)$. The other entries of Q in (6) are also corroborated by integration in (8).

For the identity matrix I, $\mathcal{A}(I)$ has $\alpha = 1/4,\ \beta = 7/4$.

PROPOSITION 2. *The planes* $\mathbb{U}_1 :\ a = 3b', a' = 3b$ *and* $\mathbb{U}_2 :\ a = -b', a' = -b$ *are invariant under the action of* $\mathbb{O}(2)$. *These spaces are mutually orthogonal, relative to* q.

The quadratic forms $r_1 = {r_{11}}^2 + {r_{12}}^2$ *and* $r_2 = {r_{21}}^2 + {r_{22}}^2$, *where*

$$r_{11} = (a + b')/8,\ r_{12} = -(a' + b)/8,\ r_{21} = (a - 3b')/24,\ r_{22} = (a' - 3b)/24,$$

are invariant under the action of $\mathbb{O}(2)$. *Also* r_1 *and* r_2 *vanish respectively on* $\mathbb{U}_2$ *and* $\mathbb{U}_1$.

Furthermore, the symmetric matrices R_1 *and* R_2 *which define* r_1 *and* r_2 *generate the lines* $\beta = \alpha$ *and* $\alpha = -\beta/3$, *which form the border of the admissible region* $\beta(\frac{2}{3}\alpha + \frac{1}{3}\beta) - \alpha^2 > 0$, *in Proposition 1.*

PROOF. The invariance of the planes is straightforward.

The plane $\mathbb{U}_1$ is spanned by $u_{1,1} = (3, 0, 1, 0)$ and $u_{1,2} = (0, 1, 0, 3)$; $\mathbb{U}_2$ is spanned by $u_{2,1} = (-1, 0, 1, 0)$ and $u_{2,2} = (0, 1, 0, -1)$. Scalar multiplication relative to Q of these vectors ends the proof.

The second and third items follow from a straightforward calculation. □

Although other possibilities exist, in this work the forms r_1 and r_2 will be used as a reference.

DEFINITION 1. Let $q(u) = uQu^*$ be as in Proposition 1. Write, $q_1 = r_1/m_1^2$, $q_2 = r_2/m_2^2$, for positive constants m_1 and m_2, uniquely determined by $q_i = q|_{\mathbb{U}_i}$, $i = 1, 2$.

The *asymmetry* of q is defined by the ratio $\sigma(q) = m_2/m_1$.

Clearly $\sigma(q) = m_2/m_1$ ranges over all positive reals. Its expression in terms of α, β is given in proposition 3.

THEOREM 2. *Let T be the quadratic form in equation 4, giving the index of transversal umbilic points.*

Relative to unit ball $B(1, q) = \{q(u) \leq 1\}$ of any invariant metric q in $\mathbb{U}^4$, the ratio of the volume V_- of the cone C_-, where T is negative, to that of the volume V_+ of the cone C_+, where T is positive, is given by $9(\sigma(q))^2$, where $\sigma(q)$ is the asymmetry of q, as in Definition 1 and Proposition 3.

Proof. Direct calculation leads to

$$T = 72(-r_2 + r_1/9).$$

Therefore, in terms of $q_1,\ q_2$,

$$T = 72\left[\left(\frac{m_1}{3}\right)^2 q_1 - {m_2}^2 q_2\right].$$

The proof consists in computing the volume V_- of the solid torus cone

$$C_- : q_2 \geq \left(\frac{m_1}{3m_2}\right)^2 q_1, \quad q \leq 1$$

and divide it by the volume V_+ of the solid torus cone

$$C_+ : q_2 \leq \left(\frac{m_1}{3m_2}\right)^2 q_1, \quad q \leq 1.$$

Let $v_{i1},\ v_{i2}$ be an orthonormal basis of $\mathbb{U}_i$, $i = 1, 2$, relative to q_i, so that they form a positive orthonormal frame, relative to q, on $\mathbb{U}^4$.

In q-orthonormal coordinates (x, y, z, w) relative to the frame $v_{i1},\ v_{i2}$, $i = 1, 2$, it follows that

$$T = 72\left[(\frac{m_1}{3})^2(x^2 + y^2) - m_2^2(z^2 + w^2)\right], \quad q = (x^2 + y^2) + (z^2 + w^2). \tag{9}$$

Let $x = r\cos\theta,\ y = r\sin\theta$ and $z = R\cos\gamma,\ w = R\sin\gamma$, where $0 \leq r \leq 1,\ 0 \leq R \leq 1,\ 0 \leq \theta \leq 2\pi,\ 0 \leq \gamma \leq 2\pi$.

The element of volume dV in the metric q is given by $dxdydzdw$

Therefore, $dV = rRdrdRd\theta d\gamma$ and

$$V_1 = \int_{q\leq 1} dV = 4\pi^2 \int_{0\leq r^2+R^2\leq 1} rRdrdR.$$

Considering the change of coordinates $r = t\cos\beta$, $R = t\sin\beta$, $0 \leq \beta \leq \pi/2$, $0 \leq t \leq 1$, we obtain

$$\int_{0\leq r^2+R^2\leq 1} rRdrdR = \int_{0\leq\beta\leq\pi/2,\, 0\leq t\leq 1} t^3\sin\beta\cos\beta dtd\beta = \frac{1}{8}.$$

Therefore, as it is well known, the volume of the unit ball in a four dimensional Euclidean space is given by $V_1 = \int_{q\leq 1} dV = \pi^2/2$. See (Courant and John 1989), pg. 459.

Take $\tan\beta_0 = \frac{m_1}{3m_2}$, the volume of the solid torus cone C_+ is given by,

$$V_+ = 4\pi^2\int_0^1\int_0^{\beta_0} t^3\sin\beta\cos\beta d\beta dt = \frac{\pi^2}{2}sin^2\beta_0.$$

Analogously, the volume of the solid torus cone C_- is equal to

$$V_- = 4\pi^2\int_0^1\int_{\beta_0}^{\frac{\pi}{2}} t^3\sin\beta\cos\beta d\beta dt = \frac{\pi^2}{2}(1 - sin^2\beta_0).$$

Therefore $\frac{V_-}{V_+} = \frac{(1-\sin^2\beta_0)}{\sin^2\beta_0} = \frac{1}{\tan^2\beta_0} = 9(\frac{m_2}{m_1})^2$. □

PROPOSITION 3. *In terms of Q, as in equation 6, $\sigma(q)$ is calculated as follows:*

$$\sigma(q) = m_2/m_1 = \sqrt{\frac{3\alpha+\beta}{3(\beta-\alpha)}}. \tag{10}$$

PROOF. In fact, by the uniqueness of the simultaneous diagonalization of the quadratic forms q and T, see (Gantmacher 1960) pg. 314, equation 9 implies that the eigenvalues of matrix M_T, of T, relative to Q, the matrix of q, are $72(\frac{m_1}{3})^2$ and $-72m_2^2$.

Separate direct calculation of these relative eigenvalues, which are those of the matrix $M_T\,Q^{-1}$, gives $\frac{1}{2(3\alpha+\beta)}$ and $-\frac{3}{2(\beta-\alpha)}$.

Equating the ratios of the eigenvalues in both calculations gives $9\left(\frac{m_2}{m_1}\right)^2 = 3\frac{3\alpha+\beta}{\beta-\alpha}$, which amounts to equation (10). □

2 AT THE CROSSROADS OF GEOMETRY AND GLOBAL ANALYSIS

The Geometric local properties of umbilic points, regarded as singularities, have been studied focusing on the three following main aspects:

i) *Topological*, related to the Index sign of the principal line fields around the umbilic.

ii) *Focal*, describing the patterns, Hyperbolic Elliptic, of normal rays envelopes. This aspect is related to Geometric Optics, Catastrophe Theory and Lagrangian Geometry. See (Thom 1972) and (Wall 1977).

iii) *Darbouxian*, which counts the number of principal lines separatrices approaching the umbilic, and, more generally, describes locally the foliations by principal lines.

These aspects and their different types are discriminated and analyzed in terms of suitable algebraic conditions in the G-space $\mathbb{U}^4$.

A coherent differential geometric and topological picture of the set morphology and inclusion relationships between the different sorts of umbilic types has been established by Porteous, see (Porteous 1994), and previous reference quoted there. See also (Zeeman 1976), for the focal aspect, and (Darboux 1896), (Gutierrez and Sotomayor 1982, 1983, 1991) and (Bruce and Fidal 1989), for the Darbouxian types.

The globalization to the whole surface of the local analysis of Darboux, in the context of Structural Stability and Genericity of principal foliations, was carried out in (Gutierrez and Sotomayor 1982, 1983, 1991).

An additional extension led Gutierrez, Garcia, Sotomayor and others, to expand the study of umbilic points and also principal foliations to surfaces and hypersurfaces in $\mathbb{R}^4$. See (Gutierrez et al. 1997), (Garcia and Sotomayor 2000) and (Garcia 2001).

Other foliations of interest in Classical Differential Geometry, such as asymptotic lines and lines of mean curvature, defined also by quadratic differential equations similar to (5), have been studied in (Garcia and Sotomayor 1997) and (Garcia et al. 1999) and (Garcia and Sotomayor 2001, 2002, 2003).

There remain deep open problems related to the structure of principal foliations around isolated umbilic points on smooth surfaces, in the non-transversal case. See (Mello and Sotomayor 1999), (Smyth and Xavier 1992) and (Ivanov 2002).

3 UMBILIC POINTS IN RANDOM SURFACES

In the domain of Statistical Physics, but still connected to Geometry and Topology, Berry and Hannay (Berry and Hannay 1977) carried out a quantitative statistical study of the proportions in which the different types of umbilic types are distributed in random surfaces, such as those modeling an ocean or a lake. An issue here is to study how the presence of umbilic points in a random surface influences the reflection on it of electromagnetic short waves. Although this work is more related to the focal interpretation of umbilic points, it considers explicitly also their Darbouxian and Index aspects.

This paper is the outcome of an initial attempt to provide a mathematical formulation and a proof, in the tradition of Geometry and Classical Analysis, that could correspond to the conclusions of (Berry and Hannay 1977), reported in the tradition of Statistical Physics.

Theorem 2 suggests a disagreement with the report of the calculations in (Berry and Hannay 1977) which claim that the statistical ratio is always 1, irrespective of the statistic anisotropy present in the evaluation. The asymmetry of the invariant metric, used to make evaluations in this work, may be considered as a geometric counterpart for the statistic anisotropy.

Considering only the local aspect of surfaces at umbilic points, this discrepancy may be due to the fact that in the calculations made in (Berry and Hannay 1977), the cubic forms are regarded

as vectors in $\mathbb{R}^4$, with a fixed frame, and not as elements of the G-space $\mathbb{U}^4$. The effect of this is that the same umbilic on a surface is counted multiple times, one for each rotated frame.

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RESUMO

No espaço das formas cúbicas de superfícies, consideradas como um $G-$espaço e munido de uma métrica invariante, é avaliado o quociente dos volume das formas cúbicas correspondentes aos umbílicos de índice negativo pelo volume daquele correspondente aos umbíicos de índice positivo. Esta avaliação é expressa em termos da assimetria da métrica, definida neste artigo. Este trabalho contém também uma seção com referências a trabalhos relacionados aos pontos umbílicos e outra com uma comparação do quociente acima citado com o obtido em 1977 por Berry e Hannay no domínio da Física Estatística.

Palavras-chave: ponto umbílico, linhas de curvaturas principais.

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