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Tori embedded in $\mathbb{S}^3$ with dense asymptotic lines

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ABSTRACT

In this paper are given examples of tori T^2 embedded in $\mathbb{S}^3$ with all their asymptotic lines dense.

Key words: asymptotic lines, recurrence, Clifford torus, variational equation.

1 INTRODUCTION

Let $\alpha : \mathbb{M} \to \mathbb{S}^3$ be an immersion of class C^r, $r \geq 3$, of a smooth, compact and oriented two-dimensional manifold $\mathbb{M}$ into the three dimensional sphere $\mathbb{S}^3$ endowed with the canonical inner product $\langle \cdot , \cdot \rangle$ of $\mathbb{R}^4$.

The *Fundamental Forms* of α at a point p of $\mathbb{M}$ are the symmetric bilinear forms on $\mathbb{T}_p\mathbb{M}$ defined as follows (Spivak 1999):

$$I_\alpha(p; v, w) = \langle D\alpha(p; v), D\alpha(p; w)\rangle,$$
$$II_\alpha(p; v, w) = \langle -DN_\alpha(p; v), D\alpha(p; w)\rangle.$$

Here, N_α is the positive unit normal of the immersion α and $\langle N_\alpha, \alpha\rangle = 0$.

Through every point p of the *hyperbolic region* $\mathbb{H}_\alpha$ of the immersion α, characterized by the condition that the extrinsic Gaussian Curvature $\mathcal{K}_{\text{ext}} = \det(DN_\alpha)$ is negative, pass two transverse asymptotic lines of α, tangent to the two asymptotic directions through p. Assuming $r \geq 3$ this follows from the usual existence and uniqueness theorems on Ordinary Differential Equations. In fact, on $\mathbb{H}_\alpha$ the local line fields are defined by the kernels $\mathcal{L}_{\alpha,1}$, $\mathcal{L}_{\alpha,2}$ of the smooth one-forms $\omega_{\alpha,1}$, $\omega_{\alpha,2}$ which locally split II_α as the product of $\omega_{\alpha,1}$ and $\omega_{\alpha,2}$.

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The forms $\omega_{\alpha,i}$ are locally defined up to a non vanishing factor and a permutation of their indices. Therefore, their kernels and integral foliations are locally well defined only up to a permutation of their indices.

Under the orientability hypothesis imposed on $\mathbb{M}$, it is possible to globalize, to the whole $\mathbb{H}_\alpha$, the definition of the line fields $\mathcal{L}_{\alpha,1}$, $\mathcal{L}_{\alpha,2}$ and of the choice of an ordering between them, as established in (Garcia and Sotomayor 1997) and (Garcia et al. 1999).

These two line fields, called the *asymptotic line fields* of α, are of class C^{r-2} on $\mathbb{H}_\alpha$; they are distinctly defined together with the ordering between them given by the subindexes $\{1, 2\}$ which define their *orientation ordering*: *"1"* for the *first asymptotic line field* $\mathcal{L}_{\alpha,1}$, *"2"* for the *second asymptotic line field* $\mathcal{L}_{\alpha,2}$.

The *asymptotic foliations* of α are the integral foliations $\mathcal{A}_{\alpha,1}$ of $\mathcal{L}_{\alpha,1}$ and $\mathcal{A}_{\alpha,2}$ of $\mathcal{L}_{\alpha,2}$; they fill out the hyperbolic region $\mathbb{H}_\alpha$.

In a local chart (u, v) the asymptotic directions of an immersion α are defined by the implicit differential equation

$$II = edu^2 + 2fdudv + gdv^2 = 0.$$

In $\mathbb{S}^3$, with the second fundamental form relative to the normal vector $N = \alpha \wedge \alpha_u \wedge \alpha_v$, it follows that:

$$e = \frac{\det\left[\alpha, \alpha_u, \alpha_v, \alpha_{uu}\right]}{\sqrt{EG - F^2}}, \quad f = \frac{\det\left[\alpha, \alpha_u, \alpha_v, \alpha_{uv}\right]}{\sqrt{EG - F^2}}, \quad g = \frac{\det\left[\alpha, \alpha_u, \alpha_v, \alpha_{vv}\right]}{\sqrt{EG - F^2}}.$$

There is a considerable difference between the cases of surfaces in the Euclidean and in the Spherical spaces. In $\mathbb{R}^3$ the asymptotic lines are never globally defined for immersions of compact, oriented surfaces. This is due to the fact that in these surfaces there are always elliptic points, at which $\mathcal{K}_{\text{ext}} > 0$ (Spivak 1999, Vol. III, chapter 2, pg. 64).

The study of asymptotic lines on surfaces $\mathbb{M}$ of $\mathbb{R}^3$ and $\mathbb{S}^3$ is a classical subject of Differential Geometry. See (do Carmo 1976, chapter 3), (Darboux 1896, chapter II), (Spivak 1999, vol. IV, chapter 7, Part F) and (Struik 1988, chapter 2).

In (Garcia and Sotomayor 1997) and (Garcia et al. 1999) ideas coming from the Qualitative Theory of Differential Equations and Dynamical Systems such as Structural Stability and Recurrence were introduced into the subject of Asymptotic Lines. Other differential equations of Classical Geometry have been considered in (Gutierrez and Sotomayor 1991, 1998); a recent survey can be found in (Garcia and Sotomayor 2008).

The interest on the study of foliations with dense leaves goes back to Poincaré, Birkhoff, Denjoy, Peixoto, among others.

In $\mathbb{S}^3$ the asymptotic lines can be globally defined, an example is the Clifford torus, $\mathcal{C} = \mathbb{S}^1(r) \times \mathbb{S}^1(r) \subset \mathbb{S}^3$, where $\mathbb{S}^1(r) = \{(x, y) \in \mathbb{R}^2 : \ x^2 + y^2 = r^2\}$ and $r = \sqrt{2}/2$. In $\mathcal{C}$ all asymptotic lines are closed curves, in fact, Villarceau circles. (See Villarceau 1848) and illustration in Figure 1.

An asymptotic line γ is called *recurrent* if it is contained in the hyperbolic region and $\gamma \subseteq L(\gamma)$, where $L(\gamma) = \alpha(\gamma) \cup \omega(\gamma)$ is the limit set of γ, and it is called *dense* if $L(\gamma) = \mathbb{M}$.

In this paper is given an example of an embedded torus (deformation of the Clifford torus) with both asymptotic foliations having all their leaves dense.

2 PRELIMINARY CALCULATIONS

In this section will be obtained the variational equations of a quadratic differential equation to be applied in the analysis in Section 3.

PROPOSITION 1. *Consider a one parameter family of quadratic differential equations of the form*

$$\begin{aligned} &a(u,v,\epsilon)dv^2 + 2b(u,v,\epsilon)du dv + c(u,v,\epsilon)du^2 = 0,\\ &a(u,v,0) = c(u,v,0) = 0, \quad b(u,v,0) = 1. \end{aligned} \tag{1}$$

Let $v(u, v_0, \epsilon)$ *be a solution of* (1) *with* $v(u, v_0, 0) = v_0$ *and* $u(u_0, v, \epsilon)$ *solution of* (1) *with* $u(u_0, v, 0) = u_0$. *Then the following variational equations holds:*

$$\begin{aligned} &c_\epsilon \ + 2v_{\epsilon u} = 0, \qquad a_\epsilon \ + 2u_{\epsilon v} = 0,\\ &c_{\epsilon\epsilon} + 2c_{v\epsilon}v_\epsilon - 2b_\epsilon c_\epsilon + 2v_{u\epsilon\epsilon} = 0,\\ &a_{\epsilon\epsilon} + 2a_{u\epsilon}u_\epsilon - 2b_\epsilon a_\epsilon + 2u_{v\epsilon\epsilon} = 0. \end{aligned} \tag{2}$$

PROOF. Differentiation with respect to ϵ of (1) written as

$$a(u,v,\epsilon)\left(\frac{dv}{du}\right)^2 + 2b(u,v,\epsilon)\frac{dv}{du} + c(u,v,\epsilon) = 0, \quad v(u,v_0,0) = v_0,$$

taking into account that

$$a_v = \frac{\partial a}{\partial v}, \quad a_\epsilon = \frac{\partial a}{\partial \epsilon}, \quad a_{\epsilon u} = a_{u\epsilon} = \frac{\partial^2 a}{\partial \epsilon \partial u} = \frac{\partial^2 a}{\partial u \partial \epsilon},$$

leads to:

$$(a_\epsilon + a_v v_\epsilon)\left(\frac{dv}{du}\right)^2 + 2a\frac{dv}{du}v_{\epsilon u} + 2\big(b_\epsilon + b_v v_\epsilon\big)\frac{dv}{du} + 2bv_{\epsilon u} + c_\epsilon + c_v v_\epsilon = 0. \tag{3}$$

Analogous notation for $b = b(u, v(u, v_0, \epsilon), \epsilon)$, $c = c(u, v(u, v_0, \epsilon), \epsilon)$ and for the solution $v(u, v_0, \epsilon)$.

Evaluation of equation (3) at $\epsilon = 0$ results in:

$$c_\epsilon + 2v_{\epsilon u} = 0.$$

Differentiating twice the equation (1) and evaluating at $\epsilon = 0$ leads to:

$$\begin{aligned} &c_{\epsilon\epsilon} + 2c_{v\epsilon}v_\epsilon + 4b_\epsilon v_{\epsilon u} + 2bv_{u\epsilon\epsilon} = 0,\\ &c_{\epsilon\epsilon} + 2c_{v\epsilon}v_\epsilon - 2b_\epsilon c_\epsilon + 2v_{u\epsilon\epsilon} = 0. \end{aligned} \tag{4}$$

Similar calculation gives the variational equations for u_ϵ and $u_{\epsilon\epsilon}$. This ends the proof. ☐

3 DOUBLE RECURRENCE FOR ASYMPTOTIC LINES

Consider the Clifford torus $\mathcal{C} = \mathbb{S}^1\left(\frac{1}{\sqrt{2}}\right) \times \mathbb{S}^1\left(\frac{1}{\sqrt{2}}\right) \subset \mathbb{S}^3$ parametrized by:

$$C(u, v) = \frac{\sqrt{2}}{2}\big(\cos(-u+v), \sin(-u+v), \cos(u+v), \sin(u+v)\big), \tag{5}$$

where C is defined in the square $Q = \{(u, v) : \quad 0 \le u \le 2\pi, \quad 0 \le v \le 2\pi\}$.

PROPOSITION 2. *The asymptotic lines on the Clifford torus in the coordinates given by equation* (5) *are given by* $dudv = 0$, *that is, the asymptotic lines are the coordinate curves (Villarceau circles). See Figure* 1.

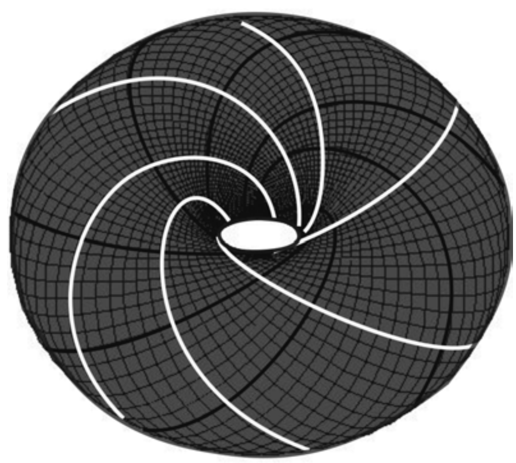

Fig. 1 – Torus and Villarceau circles.

PROOF. The coefficients of the first fundamental form $I = Edu^2 + 2Fdudv + Gdv^2$ and the second fundamental form $II = edu^2+2fdudv+gdv^2$ of C with respect to the normal vector field $N = C\wedge C_u\wedge C_v$ are given by:

$$\begin{aligned} E(u, v) &= 1, \qquad e(u, v) = 0,\\ F(u, v) &= 0, \qquad f(u, v) = 1,\\ G(u, v) &= 1, \qquad g(u, v) = 0. \end{aligned}$$

Therefore the asymptotic lines are defined by $dudv = 0$ and so they are the coordinate curves. Figure 1 is the image of the Clifford torus by a stereographic projection of $\mathbb{S}^3$ to $\mathbb{R}^3$. □

THEOREM 1. *There are embeddings* $\alpha : \mathbb{T}^2 \to \mathbb{S}^3$ *such that all leaves of both asymptotic foliations,* $\mathcal{A}_{\alpha,1}$ *and* $\mathcal{A}_{\alpha,2}$, *are dense in* $\mathbb{T}$. *See Figure* 2.

PROOF. Let $N(u, v) = (\alpha \wedge \alpha_u \wedge \alpha_v)/|\alpha \wedge \alpha_u \wedge \alpha_v|(u, v)$ be the unit normal vector to the Clifford torus.

We have that,

$$N(u, v) = \frac{\sqrt{2}}{2}\big(\cos(-u+v), \sin(-u+v), -\cos(u+v), -\sin(u+v)\big).$$

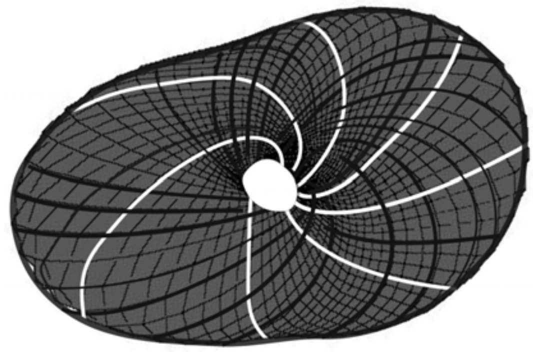

Fig. 2 – Stereographic projection of a deformation of a Clifford torus with $\epsilon = 2/3$.

Let $c(u, v) = h(u, v)N(u, v)$, h being a smooth $2\pi-$ double periodic function, and consider for $\epsilon \neq 0$ small the one parameter family of embedded torus

$$\alpha_\epsilon(u, v) = \frac{C(u, v) + \epsilon c(u, v)}{|C(u, v) + \epsilon c(u, v)|}. \tag{6}$$

Then the coefficients of the second fundamental form of α_ϵ with respect to

$$N_\epsilon = \alpha_\epsilon \wedge (\alpha_\epsilon)_u \wedge (\alpha_\epsilon)_v / |\alpha_\epsilon \wedge (\alpha_\epsilon)_u \wedge (\alpha_\epsilon)_v|,$$

after multiplication by $1/(1 + \epsilon^2 h^2)^2$, are given by:

$$\begin{aligned} e &= \epsilon h_{uu} + 2\epsilon^2 h_u h_v + \epsilon^3\big(2hh_u^2 - h^2 h_{uu}\big), \\ f &= 1 + \epsilon h_{uv} + \epsilon^2\big(h_u^2 + h_v^2 - h^2\big) + \epsilon^3\big(2hh_u h_v - h^2 h_{uv}\big) + \epsilon^4 h^4, \\ g &= \epsilon h_{vv} + 2\epsilon^2 h_u h_v + \epsilon^3\big(2hh_v^2 - h^2 h_{vv}\big). \end{aligned} \tag{7}$$

By Proposition 1 the variational equations of the implicit differential equation

$$e(u, v, \epsilon) + 2f(u, v, \epsilon)\frac{dv}{du} + g(u, v, \epsilon)\left(\frac{dv}{du}\right)^2 = 0, \tag{8}$$

with $e(u, v, 0) = g(u, v, 0) = 0$, $f(u, v, 0) = 1$ and $v(u, v_0, 0) = v_0$ are given by:

$$e_\epsilon + 2v_{\epsilon u} = 0, \quad e_{\epsilon\epsilon} + 2e_{v\epsilon}v_\epsilon - 2f_\epsilon e_\epsilon + 2v_{u\epsilon\epsilon} = 0. \tag{9}$$

In fact, differentiating equation (8) with respect to ϵ it is obtained:

$$\begin{aligned} (e_v(u, v, \epsilon)v_\epsilon + e_\epsilon(u, v, \epsilon)) + 2(f_v(u, v, \epsilon)v_\epsilon + f_\epsilon(u, v, \epsilon))v_u + 2f v_{u\epsilon}(u, v, \epsilon) \\ + (g_v(u, v, \epsilon)v_\epsilon + g_\epsilon(u, v, \epsilon))(v_u)^2 + 2g(u, v, \epsilon)v_u v_{u\epsilon} = 0. \end{aligned} \tag{10}$$

Making $\epsilon = 0$ leads to equation $e_\epsilon + 2v_{\epsilon u} = 0$.

Differentiation of equation (10) with respect to ϵ and evaluation at $\epsilon = 0$ leads to

$$e_{\epsilon\epsilon} + 2e_{v\epsilon}v_\epsilon - 2f_\epsilon e_\epsilon + 2v_{u\epsilon\epsilon} = 0.$$

Therefore, the integration of the linear differential equations (9) leads to:

$$v_\epsilon(u) = -\frac{1}{2}\int_0^u h_{uu}du, \quad v_{u\epsilon\epsilon} = \frac{1}{2}h_{uuv}\int_0^u h_{uu}du \ + \ h_{uu}h_{uv} - 2h_u h_v. \tag{11}$$

Taking $h(u, v) = \sin^2(2v - 2u)$, it results from equation (7) that:

$$e(u, v, \epsilon) = e(v, u, \epsilon) = g(u, v, \epsilon), \qquad f(u, v, \epsilon) = f(v, u, \epsilon). \tag{12}$$

In fact, from the definition of h it follows that:

$$h(u, v) = h(v, u), \qquad h_u = -h_v = -2\sin(4v - 4u),$$

$$h_{uu} = h_{vv} = 8\cos(4v - 4u), \qquad h_{uv} = -8\cos(4v - 4u).$$

So, a careful calculation shows that equation (12) follows from equation (7).

So, from equation (11), it follows that

$$\begin{aligned} v_\epsilon(u, v_0, 0) &= -\sin(4v_0) - \sin(4v_0 - 4u), \\ v_{\epsilon\epsilon}(u, v_0, 0) &= -12u - 4\sin(4u) - 4\sin(8v_0 - 4u), \\ &\quad -\frac{5}{2}\sin(8v_0) + \frac{13}{2}\sin(8v_0 - 8u). \end{aligned}$$

Therefore,

$$v_\epsilon(2\pi, v_0, 0) - v_\epsilon(0, v_0, 0) = 0, \qquad v_{\epsilon\epsilon}(2\pi, v_0, 0) - v_{\epsilon\epsilon}(0, v_0, 0) = -24\pi.$$

Consider the Poincaré map $\pi_\epsilon^1 : \{u = 0\} \to \{u = 2\pi\}$, relative to the asymptotic foliation $\mathcal{A}_{\alpha,1}$, defined by $\pi_\epsilon^1(v_0) = v(2\pi, v_0, \epsilon)$.

Therefore, $\pi_0^1 = Id$ and it has the following expansion in ϵ:

$$\pi_\epsilon^1(v_0) = v_0 + \frac{\epsilon^2}{2}v_{\epsilon\epsilon}(2\pi, v_0, 0) + O(\epsilon^3) = v_0 - 12\pi\epsilon^2 + O(\epsilon^3)$$

and so the rotation number of π_ϵ^1 changes continuously and monotonically with ϵ.

By the symmetry of the coefficients of the second fundamental form in the variables (u, v) and the fact that $e(u, v, \epsilon) = g(u, v, \epsilon)$, see equation (12), it follows that the Poincaré map $\pi_\epsilon^2 : \{v = 0\} \to \{v = 2\pi\}$, relative to the asymptotic foliation $\mathcal{A}_{\alpha,2}$, defined by $\pi_\epsilon^2(u_0) = u(u_0, 2\pi, \epsilon)$ is conjugated to π_ϵ^1 by an isometry.

Therefore we can take $\epsilon \neq 0$ small such that the rotation numbers of π_ϵ^i, $i = 1, 2$, are, modulo 2π, irrational. Therefore all orbits of π^i, $i = 1, 2$, are dense in $\mathbb{S}^1$. See (Katok and Hasselblatt 1995, chapter 12) or (Palis and Melo 1982, chapter 4). This ends the proof. □

4 CONCLUDING COMMENTS

In this paper it was shown that there exist embeddings of the torus in $\mathbb{S}^3$ with both asymptotic foliations having all their leaves dense.

The technique used here is based on the second order perturbation of differential equations.

It is worth mentioning that the consideration of only the first variational equation was was technically insufficient to achieve the results of this paper. The same can be said for the technique of local bumpy perturbations of the Clifford Torus.

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RESUMO

Neste artigo são dados exemplos de toros $\mathbb{T}^2$ mergulhados em $\mathbb{S}^3$ com todas as suas linhas assintóticas densas.

Palavras-chave: linhas assintóticas, recorrências, toro de Clifford, equação variacional.

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