



## MATHEMATICAL SCIENCES

# On a continuous Gale-Berlekamp switching game

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**Abstract:** We propose a continuous version of the classical Gale-Berlekamp switching game. The main results of this paper concern growth estimates for the corresponding optimization problems.

**Key words:** Gale-Berlekamp switching game, unbalancing lights problem, game theory, Khinchin inequality.

## INTRODUCTION

Designed independently by Elwyn Berlekamp and David Gale in the 1960's, the Gale-Berlekamp switching game – also known as the unbalancing lights problem – represents a classic in the field of combinatorics and its applications, with deep connections to theoretical Computer Science. This single-player game consists of an  $n \times n$  square matrix of light bulbs set-up at an initial light configuration. The goal is to turn off as many lights as possible using  $n$  row and  $n$  column switches, which invert the state of each bulb in the corresponding row or column.

For an initial pattern of lights  $\Theta$ , let  $i(\Theta)$  denote the smallest final number of on-lights achievable by row and column switches starting from  $\Theta$ . The smallest possible number of remaining on-lights  $R_n$ , starting from the worst initial pattern, is then

$$R_n = \max\{i(\Theta) : \Theta \text{ is an } n \times n \text{ light pattern}\}.$$

Sometimes this optimization problem is posed as finding the maximum of the difference between the number of lights that are on and the number that are off, often denoted by  $G_n$ . Obviously both problems are equivalent as  $R_n = \frac{1}{2}(n^2 - G_n)$ .

The original problem introduced by Berlekamp asks for the exact value of  $R_{10}$  and it was proved in Carlson & Stolarski (2004) that  $R_{10} = 35$  (and thus  $G_{10} = 30$ ). Several related questions pertaining to the original problem have been investigated in depth, see e.g. Brualdi & Meyer (2015), Carlson & Stolarski (2004), Fishburn & Sloane (1989) and Schauz (2011); in particular the hardness of solving the Gale-Berlekamp switching game was studied in Roth & Viswanathan (2008).

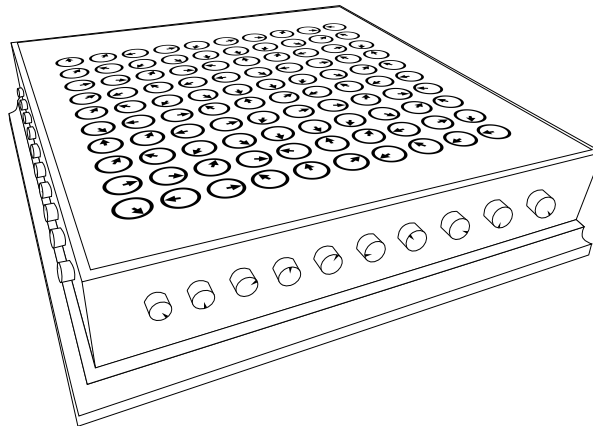
In this paper we propose a continuous version of the Gale-Berlekamp switching game. We are interested in a continuous version of the game for which vectors replace light bulbs and knobs substitute the discrete switches used to invert the state of the bulbs in the original problem. In our approach, we also allow the game-board not to be square.

To explain the new proposed game, we initially notice that by associating +1 to the on-lights and -1 to the off-lights from the array of lights  $(a_{ij})_{i,j=1}^n$  the goal of the original game can be understood mathematically as to determine

$$G_n = \min \left\{ \max_{x_i, y_j \in \{-1, 1\}} \left| \sum_{i,j=1}^n a_{ij} x_i y_j \right| : a_{ij} = -1 \text{ or } +1 \right\},$$

where  $x_i$  and  $y_j$  denote the switches of the row  $i$  and of the column  $j$ , respectively.

The new optimization problem herein proposed involves a matrix  $(a_{ij})$  with  $n_1$  rows and  $n_2$  columns whose elements are unit vectors of the plane,  $\mathbb{R}^2$ . The initial direction pattern of each  $n_1 n_2$  vectors is set up at the beginning of the game. In each row  $i$  and each column  $j$  there are knobs  $x_i$  and  $y_j$ , respectively. Rotating the knob  $x_i$  by an angle  $\theta_i$ , it rotates all vectors  $a_{ij}$  of the row  $i$  by the same angle  $\theta_i$ . Analogously, when the knob  $y_j$  is rotated by an angle  $\theta_j$ , the same happens with all the vectors  $a_{ij}$  of the column  $j$  (see Figure 1).



**Figure 1.** Continuous version of the Gale-Berlekamp switching game for  $n=10$ .

The game consists of maximizing the Euclidean norm of the sum of all vectors in the final stage. More precisely, for an initial pattern  $\Theta$  of unit vectors, let  $s(\Theta)$  be the supremum of the (Euclidean) norms of the sums of all  $n_1 n_2$  vectors achievable by row and column adjusts. The extremal problem is to determine

$$G_{n_1, n_2}^{(1)} := \min \{s(\Theta) : \Theta \text{ an } n_1 \times n_2 \text{ pattern}\}.$$

Our main result estimates the asymptotic growth of  $G_{n_1, n_2}^{(1)}$ :

**Theorem 1.** For all positive integers  $n_1, n_2$ , we have

$$0.886 \leq \frac{G_{n_1, n_2}^{(1)}}{\sqrt{n_1 n_2} \max\{\sqrt{n_1}, \sqrt{n_2}\}} \leq 1. \tag{1}$$

We conclude this introduction by commenting on the ideas and techniques used to prove Theorem 1, which are of particular interest. We observe that due to the combinatorial complexity of this kind of problems, growth estimates as in Theorem 1 are often obtained by non-deterministic techniques, see for instance Alon & Spencer (1992), Araújo & Pellegrino (2019) and Bennett et al. (1975). A main

novelty proposed in this article regards a deterministic approach to estimating  $G_{n_1, n_2}^{(1)}$ , which yields improved, more precise estimates than those obtained by non-deterministic methods. We believe that the methods herein developed are likely to be applicable in an array of other problems and to exemplify the depth of these new ideas, we also prove analogues of (1) in higher-dimensional configurations.

**PROOF OF THEOREM 1**

Initially, it is more convenient to conceive the vectors in the game as complex numbers  $a_{ij}$  with modulus 1, which represent the elements of the array  $(a_{ij})_{i,j=1}^{n_1, n_2}$ . In this case, when the player rotates a knob, the action is modeled by the multiplication by unimodular complex numbers.

There is no loss of generality in supposing that  $n_1 \leq n_2$ . We start off the proof of Theorem 1 by reminding that a consequence of the Krein–Milman Theorem assures that for all  $A: \ell_\infty^{n_1} \times \ell_\infty^{n_2} \rightarrow \mathbb{C}$ , defined by

$$A(e_{j_1}, e_{j_2}) = a_{j_1 j_2},$$

where  $e_k := (0, \dots, 0, 1, 0, \dots, 0)$ , with 1 exactly at the  $k$ -th position, there holds

$$\|A\| = \sup_{\substack{|x_{j_1}^{(1)}|=|x_{j_2}^{(2)}|=1 \\ j_1, j_2=1}} \left| \sum_{j_1, j_2=1}^{n_1, n_2} a_{j_1 j_2} x_{j_1}^{(1)} x_{j_2}^{(2)} \right|,$$

i.e., the supremum norm of  $A$  is attained at the extreme points of the closed unit balls of  $\ell_\infty^{n_1}$  and  $\ell_\infty^{n_2}$ . Thus we can easily observe that

$$G_{n_1, n_2}^{(1)} = \inf \{ \|A\| : |a_{j_1 j_2}| = 1 \}$$

and our task is then to estimate the infimum of  $\|A\|$  over all bilinear forms  $A: \ell_\infty^{n_1} \times \ell_\infty^{n_2} \rightarrow \mathbb{C}$  with unimodular coefficients.

Once the problem has been described as above, the upper bound in Theorem 1 can be obtained by means of an argument from the seminal paper of Bohnenblust & Hille (1931), Theorem II, page 608. We shall explain the necessary adaptations when we deliver the proof of Theorem 2.

As for the lower estimate, we shall make use of Khinchin inequality, which we revise for the sake of completeness.

**Khinchin inequality**

To motivate, let’s state the following question: suppose that we have  $n$  real numbers  $a_1, \dots, a_n$  and a fair coin. When we flip the coin, if it comes up heads, you choose  $\beta_1 = a_1$ , and if it comes up tails, you choose  $\beta_1 = -a_1$ . When we play for the second time, if it comes up heads, you choose  $\beta_2 = \beta_1 + a_2$  and, if it comes up tails, you choose  $\beta_2 = \beta_1 - a_2$ . Repeating the process, after having flipped the coin  $k$  times we have

$$\beta_{k+1} := \beta_k + a_{k+1},$$

if it comes up heads and

$$\beta_{k+1} := \beta_k - a_{k+1},$$

if it comes up tails. After  $n$  steps, what should be the expected value of

$$|\beta_n| = \left| \sum_{k=1}^n \pm a_k \right|?$$

Khinchin’s inequality, see for instance Diestel et al. (1995), page 10, shows that the “average”

$$\frac{1}{2^n} \sum_{\varepsilon \in D_n} \left| \sum_{j=1}^n \varepsilon_j a_j \right|,$$

where  $D_n = \{-1, 1\}^n$  and  $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n)$ , behaves as the  $\ell_2$ -norm of  $(a_j)_{j=1}^n$ . More precisely, it asserts that for any  $p > 0$  there are constants  $A_p, B_p > 0$  such that

$$A_p \left( \sum_{j=1}^n |a_j|^2 \right)^{\frac{1}{2}} \leq \left( \frac{1}{2^n} \sum_{\varepsilon \in D_n} \left| \sum_{j=1}^n \varepsilon_j a_j \right|^p \right)^{\frac{1}{p}} \leq B_p \left( \sum_{j=1}^n |a_j|^2 \right)^{\frac{1}{2}}$$

for all sequences of scalars  $(a_j)_{j=1}^n$  and all positive integers  $n$ . The natural counterpart for the average  $\frac{1}{2^n} \sum_{\varepsilon \in D_n} \left| \sum_{j=1}^n \varepsilon_j a_j \right|$  in the complex framework is

$$\left( \frac{1}{2\pi} \right)^n \int_0^{2\pi} \dots \int_0^{2\pi} \left| \sum_{j=1}^n a_j e^{it_j} \right| dt_1 \dots dt_n.$$

It is well known that in this new context we also have a Khinchin-type inequality, called Khinchin inequality for Steinhaus variables, which asserts that there exist constants  $\widetilde{A}_p$  and  $\widetilde{B}_p$  such that

$$\widetilde{A}_p \left( \sum_{j=1}^n |a_j|^2 \right)^{\frac{1}{2}} \leq \left( \left( \frac{1}{2\pi} \right)^n \int_0^{2\pi} \dots \int_0^{2\pi} \left| \sum_{j=1}^n a_j e^{it_j} \right|^p dt_1 \dots dt_n \right)^{\frac{1}{p}} \leq \widetilde{B}_p \left( \sum_{j=1}^n |a_j|^2 \right)^{\frac{1}{2}} \tag{2}$$

for every positive integer  $n$  and all scalars  $a_1, \dots, a_n$ .

Back to the proof of Theorem 1, for the purpose of establishing a lower estimate for the growth of  $G_{n_1, n_2}^{(1)}$ , we are interested in the case  $p = 1$  and only in the left hand side of (2). In König (2014) it is proven that  $\widetilde{A}_1 = \sqrt{\pi}/2$ . For a bilinear form  $A : \ell_\infty^{n_1} \times \ell_\infty^{n_2} \rightarrow \mathbb{C}$  given by

$$A(e_{j_1}, e_{j_2}) = a_{j_1 j_2}$$

with

$$|a_{j_1 j_2}| = 1,$$

we have

$$\begin{aligned} & \left( \sum_{j_1=1}^{n_1} |A(e_{j_1}, e_{j_2})|^2 \right)^{1/2} \\ & \leq \left( \frac{2}{\sqrt{\pi}} \right) \left( \frac{1}{2\pi} \right)^{n_1} \int_0^{2\pi} \dots \int_0^{2\pi} \left| \sum_{j_1=1}^{n_1} A(e_{j_1}, e_{j_2}) e^{it_{j_1}} \right| dt_1 \dots dt_{n_1} \\ & = \left( \frac{2}{\sqrt{\pi}} \right) \left( \frac{1}{2\pi} \right)^{n_1} \int_0^{2\pi} \dots \int_0^{2\pi} \left| A \left( \sum_{j_1=1}^{n_1} e^{it_{j_1}} e_{j_1}, e_{j_2} \right) \right| dt_1 \dots dt_{n_1}. \end{aligned}$$

Since

$$\int_0^{2\pi} \dots \int_0^{2\pi} \sum_{j_2=1}^{n_2} \left| A \left( \sum_{j_1=1}^{n_1} e^{it_{j_1}} e_{j_1}, e_{j_2} \right) \right| dt_1 \dots dt_{n_1}$$

$$\leq (2\pi)^{n_1} \max_{t_1, \dots, t_{n_1} \in [0, 2\pi]} \sum_{j_2=1}^{n_2} \left| A \left( \sum_{j_1=1}^{n_1} e^{it_{j_1}} e_{j_1}, e_{j_2} \right) \right|,$$

denoting the topological dual of  $\ell_\infty^n$  by  $(\ell_\infty^n)^*$  and its closed unit ball by  $B_{(\ell_\infty^n)^*}$ , we have

$$\sum_{j_2=1}^{n_2} \left( \sum_{j_1=1}^{n_1} |A(e_{j_1}, e_{j_2})|^2 \right)^{1/2}$$

$$\leq \left( \frac{2}{\sqrt{\pi}} \right) \left( \frac{1}{2\pi} \right)^{n_1} \int_0^{2\pi} \dots \int_0^{2\pi} \sum_{j_2=1}^{n_2} \left| A \left( \sum_{j_1=1}^{n_1} e^{it_{j_1}} e_{j_1}, e_{j_2} \right) \right| dt_1 \dots dt_{n_1}$$

$$\leq \left( \frac{2}{\sqrt{\pi}} \right) \max_{t_1, \dots, t_{n_1} \in [0, 2\pi]} \sum_{j_2=1}^{n_2} \left| A \left( \sum_{j_1=1}^{n_1} e^{it_{j_1}} e_{j_1}, e_{j_2} \right) \right|$$

$$\leq \left( \frac{2}{\sqrt{\pi}} \right) \|A\| \sup_{\varphi \in B_{(\ell_\infty^n)^*}} \sum_{j_2=1}^{n_2} |\varphi(e_{j_2})|$$

$$= \left( \frac{2}{\sqrt{\pi}} \right) \|A\|,$$

where in the last equality we have used the isometric isomorphism

$$\ell_1^n \quad \longrightarrow \quad (\ell_\infty^n)^*$$

$$(a_j)_{j=1}^n \quad \longmapsto \quad \varphi,$$

with  $\varphi : \ell_\infty^n \rightarrow \mathbb{C}$  defined by

$$\varphi \left( (x_j)_{j=1}^n \right) = \sum_{j=1}^n a_j x_j.$$

Finally, since  $|A(e_{j_1}, e_{j_2})| = 1$ , we conclude that

$$\|A\| \geq \left( \frac{\sqrt{\pi}}{2} \right) n_2 n_1^{\frac{1}{2}}.$$

Hence, as  $n_2 \geq n_1$ , we have

$$\left( \frac{\sqrt{\pi}}{2} \right) \leq \frac{G_{n_1 n_2}^{(1)}}{\sqrt{n_1 n_2} \max\{\sqrt{n_1}, \sqrt{n_2}\}}.$$

### THE GAME IN HIGHER DIMENSIONS

The Gale–Berlekamp switching game has a natural extension to higher dimensions. Let  $m \geq 2$  be an integer and let an  $n \times \dots \times n$  array  $(a_{j_1 \dots j_m})$  of lights be given, each either on ( $a_{j_1 \dots j_m} = 1$ ) or off ( $a_{j_1 \dots j_m} = -1$ ). Let us also suppose that for each  $k = 1, \dots, m$  and each  $j_k = 1, \dots, n$  there is a switch  $x_{j_k}^{(k)}$  so that if the switch is pulled ( $x_{j_k}^{(k)} = -1$ ) all of the corresponding lights  $a_{j_1 \dots j_m}$  (with  $j_k$  fixed) are

switched: on to off or off to on. The goal is to maximize the difference between the number of lights that are on and the number of lights that are off. As in the two-dimensional case, maximizing the difference between the number of on-lights and off-lights is equivalent to estimating

$$\max_{x_{j_1}^{(1)}, \dots, x_{j_m}^{(m)} \in \{-1, 1\}} \left| \sum_{j_1, \dots, j_m=1}^n a_{j_1 \dots j_m} x_{j_1}^{(1)} \dots x_{j_m}^{(m)} \right|$$

and the extremal problem consists of estimating

$$S_n = \min \left\{ \max_{x_{j_1}^{(1)}, \dots, x_{j_m}^{(m)} \in \{-1, 1\}} \left| \sum_{j_1, \dots, j_m=1}^n a_{j_1 \dots j_m} x_{j_1}^{(1)} \dots x_{j_m}^{(m)} \right| : a_{j_1 \dots j_m} = 1 \text{ or } -1 \right\},$$

As in the bilinear case,

$$S_n = \min \|A : \ell_\infty^n \times \dots \times \ell_\infty^n \rightarrow \mathbb{R}\|,$$

with

$$A(x^{(1)}, \dots, x^{(m)}) = \sum_{j_1, \dots, j_m=1}^n a_{j_1 \dots j_m} x_{j_1}^{(1)} \dots x_{j_m}^{(m)}.$$

The anisotropic case allows to consider  $n_1 \times \dots \times n_m$  arrays, not necessarily square arrays and, in this case, we write

$$S_{n_1, \dots, n_m} = \min \left\{ \max_{x_{j_1}^{(1)}, \dots, x_{j_m}^{(m)} \in \{-1, 1\}} \left| \sum_{j_1, \dots, j_m=1}^{n_1, \dots, n_m} a_{j_1 \dots j_m} x_{j_1}^{(1)} \dots x_{j_m}^{(m)} \right| : a_{j_1 \dots j_m} = 1 \text{ or } -1 \right\}.$$

From a recent result of Albuquerque & Rezende (2019), we can easily obtain

$$\frac{1}{m(\sqrt{2})^{m-1}} \leq \frac{S_{n_1, \dots, n_m}}{\sqrt{n_1 \dots n_m} \max\{\sqrt{n_1}, \dots, \sqrt{n_m}\}} \leq 8m\sqrt{m!} \sqrt{\log(1 + 4m)}.$$

Following the notation of Araújo and Pellegrino (2019), let  $m \geq 2$  be an integer and  $(a_{i_1 \dots i_m})$  be an  $n \times \dots \times n$  array of complex scalars such that  $|a_{i_1 \dots i_m}| = 1$ . For  $p \in (1, \infty]$ , let

$$g_{m,n}^{\mathbb{C}}(p) = \max \left| \sum_{i_1, \dots, i_m=1}^n a_{i_1 \dots i_m} x_{i_1}^{(1)} \dots x_{i_m}^{(m)} \right|,$$

where the maximum is evaluated over all  $x_j^{(j)} \in \mathbb{C}$  such that  $\|(x_j^{(j)})_{j=1}^n\|_p = 1$  for all  $j$ . It is not difficult to prove that

$$g_{m,n}^{\mathbb{C}}(p) = \|A : \ell_p^n \times \dots \times \ell_p^n \rightarrow \mathbb{C}\|,$$

with

$$A(x^{(1)}, \dots, x^{(m)}) = \sum_{i_1, \dots, i_m=1}^n a_{i_1 \dots i_m} x_{i_1}^{(1)} \dots x_{i_m}^{(m)}.$$

Denoting

$$G_{m,n}(p) = \min g_{m,n}^{\mathbb{C}}(p), \tag{3}$$

where minimum is evaluated over all unimodular  $m$ -linear forms  $A : \ell_p^n \times \dots \times \ell_p^n \rightarrow \mathbb{C}$ , the best information we can collect (combining results from Araújo & Pellegrino (2019) and Pellegrino et al. (2020)) is the following:

$$\left| \begin{array}{l} \frac{1}{1.3m^{0.365}} \leq \frac{G_{m,n}(p)}{n^{\frac{mp+p-2m}{2p}}} \leq 8\sqrt{m!}\sqrt{\log(1+4m)} \text{ for } p \in [2, \infty) \\ 1 \leq \frac{G_{m,n}(p)}{n^{1-\frac{1}{p}}} \leq C_{m,p} \text{ for } p \in (1, 2], \end{array} \right.$$

where  $C_{m,p}$  is obtained by interpolation (via the Riesz–Thorin Theorem) of the constant 1 (the constant when  $p = 1$ ) and  $8\sqrt{m!}\sqrt{\log(1+4m)}$  (the constant when  $p = 2$ ).

The above solution rests in a non-deterministic tool. We shall show in what follows that for  $p = \infty$  we can find deterministic solutions with better constants.

We begin with a matrix  $(a_{j_1 \dots j_m})_{j_1, \dots, j_m=1}^{n_1, \dots, n_m}$  whose elements are unit vectors in the Euclidean space  $\mathbb{R}^2$ . The initial direction pattern of each  $n_1 \dots n_m$  vectors is set up at the beginning of the game. For each  $k \in \{1, \dots, m\}$ , we have  $n_k$  control knobs  $x_1^{(k)}, \dots, x_{n_k}^{(k)}$ . When the knob  $x_{j_k}^{(k)}$  is rotated by an angle  $\theta_{j_k}^{(k)}$ , the same happens with all the vectors  $a_{j_1 \dots j_m}$  with  $j_k$  fixed. Defining  $\Theta$  and  $s(\Theta)$  as in the two-dimensional case, the extremal problem is to determine

$$G_{n_1 \dots n_m}^{(1)} := \min\{s(\Theta) : \Theta \text{ an } n_1 \times \dots \times n_m \text{ pattern}\}.$$

It is worth mentioning that, as a consequence of the Krein–Milman Theorem, we know that  $G_{n_1 \dots n_m}^{(1)}$  coincides with  $G_{m,n}(\infty)$ , as defined in (3). We prove the following:

**Theorem 2.** *For all positive integers  $m \geq 2, n_1, \dots, n_m \geq 1$  we have*

$$(0.886)^{m-1} \simeq \left(\frac{\sqrt{\pi}}{2}\right)^{m-1} \leq \frac{G_{n_1 \dots n_m}^{(1)}}{\sqrt{n_1 \dots n_m} \max\{\sqrt{n_1}, \dots, \sqrt{n_m}\}} \leq 1.$$

Moreover, the universal upper bound 1 cannot be improved.

The proof that, in general, the upper bound 1 cannot be improved is trivial – just consider  $n_2 = \dots = n_m = 1$  and note that in this case

$$G_{n_1 \dots n_m}^{(1)} = n_1 = \sqrt{n_1 \dots n_m} \max\{\sqrt{n_1}, \dots, \sqrt{n_m}\}.$$

The case  $n_1 = \dots = n_m$  was investigated in Araújo & Pellegrino (2019), but the techniques used by the authors do not provide good estimates for the upper constants: for instance, if we follow the arguments from Araújo & Pellegrino (2019) we just obtain  $8\sqrt{m!}\sqrt{\log(1+4m)}$ , due to Kahane–Salem–Zygmund inequality, instead of the universal sharp constant 1.

**Proof of Theorem 2**

Our task is to estimate  $\inf\{\|A\| : |a_{j_1 \dots j_m}| = 1\}$ , where the infimum runs over all  $m$ -linear forms  $A : \ell_\infty^{n_1} \times \dots \times \ell_\infty^{n_m} \rightarrow \mathbb{C}$  with unimodular coefficients.

With no loss of generality, we suppose  $n_1 \leq \dots \leq n_m$ . For the upper bound, consider, for all  $k = 1, \dots, m - 1$ , a  $n_{k+1} \times n_{k+1}$  matrix  $(a_{rs}^{(k)})$  with

$$a_{rs}^{(k)} = e^{2\pi i \frac{rs}{n_{k+1}}}.$$

A simple computation shows that

$$\begin{cases} \sum_{t=1}^{n_2} a_{rt}^{(1)} \overline{a_{st}^{(1)}} = n_2 \delta_{rs}. \\ |a_{rs}^{(1)}| = 1. \end{cases}$$

$$\vdots$$

$$\begin{cases} \sum_{t=1}^{n_m} a_{rt}^{(m-1)} \overline{a_{st}^{(m-1)}} = n_m \delta_{rs}. \\ |a_{rs}^{(m-1)}| = 1. \end{cases}$$

All the matrices are completed with zeros (if necessary) in order to get a square matrix  $n_m \times n_m$ . Define

$$A : \ell_\infty^{n_1} \times \dots \times \ell_\infty^{n_m} \rightarrow \mathbb{C}$$

by

$$A(x^{(1)}, \dots, x^{(m)}) = \sum_{i_1, \dots, i_m=1}^{n_1, \dots, n_m} a_{i_1 i_2}^{(1)} a_{i_2 i_3}^{(2)} \dots a_{i_{m-1} i_m}^{(m-1)} x_{i_1}^{(1)} \dots x_{i_m}^{(m)}$$

and note that, since  $n_1 \leq \dots \leq n_m$ , the coefficients

$$c_{i_1 \dots i_m} := a_{i_1 i_2}^{(1)} a_{i_2 i_3}^{(2)} \dots a_{i_{m-1} i_m}^{(m-1)}$$

of all monomials  $x_{i_1}^{(1)} \dots x_{i_m}^{(m)}$  with  $i_k \in \{1, \dots, n_k\}$  are unimodular. For  $x^{(1)} \in B_{\ell_\infty^{n_1}}, \dots, x^{(m)} \in B_{\ell_\infty^{n_m}}$ , consider  $y^{(1)} \in B_{\ell_\infty^{n_m}}, \dots, y^{(m)} \in B_{\ell_\infty^{n_m}}$  defined by

$$y^{(1)} = (x_1^{(1)}, \dots, x_{n_1}^{(1)}, 0, \dots, 0)$$

and so on. We have

$$\begin{aligned} & |A(x^{(1)}, \dots, x^{(m)})| \\ &= \left| \sum_{i_1, \dots, i_m=1}^{n_m} a_{i_1 i_2}^{(1)} a_{i_2 i_3}^{(2)} \dots a_{i_{m-1} i_m}^{(m-1)} y_{i_1}^{(1)} \dots y_{i_m}^{(m)} \right| \\ &\leq \sum_{i_m=1}^{n_m} \left| \sum_{i_1, \dots, i_{m-1}=1}^{n_m} a_{i_1 i_2}^{(1)} a_{i_2 i_3}^{(2)} \dots a_{i_{m-1} i_m}^{(m-1)} y_{i_1}^{(1)} \dots y_{i_{m-1}}^{(m-1)} \right| |y_{i_m}^{(m)}| \\ &\leq \left( \sum_{i_m=1}^{n_m} |y_{i_m}^{(m)}|^2 \right)^{1/2} \cdot \left( \sum_{i_m=1}^{n_m} \left| \sum_{i_1, \dots, i_{m-1}=1}^{n_m} a_{i_1 i_2}^{(1)} a_{i_2 i_3}^{(2)} \dots a_{i_{m-1} i_m}^{(m-1)} y_{i_1}^{(1)} \dots y_{i_{m-1}}^{(m-1)} \right|^2 \right)^{1/2} \\ &\leq n_m^{1/2} \left( \sum_{i_m=1}^{n_m} \sum_{\substack{i_1, \dots, i_{m-1}=1 \\ j_1, \dots, j_{m-1}=1}}^{n_m} a_{i_1 i_2}^{(1)} \overline{a_{j_1 j_2}^{(1)}} \dots a_{i_{m-1} i_m}^{(m-1)} \overline{a_{j_{m-1} j_m}^{(m-1)}} y_{i_1}^{(1)} \overline{y_{j_1}^{(1)}} \dots y_{i_{m-1}}^{(m-1)} \overline{y_{j_{m-1}}^{(m-1)}} \right)^{1/2}. \end{aligned}$$



Thus

$$\begin{aligned}
 & |A(x^{(1)}, \dots, x^{(m)})| \\
 & \leq n_m^{1/2} \left( \sum_{i_m=1}^{n_m} \sum_{\substack{i_1, \dots, i_{m-1}=1 \\ j_1, \dots, j_{m-1}=1}}^{n_m} a_{i_1 i_2}^{(1)} \overline{a_{j_1 j_2}^{(1)}} \dots a_{i_{m-1} i_m}^{(m-1)} \overline{a_{j_{m-1} i_m}^{(m-1)}} y_{i_1}^{(1)} \overline{y_{j_1}^{(1)}} \dots y_{i_{m-1}}^{(m-1)} \overline{y_{j_{m-1}}^{(m-1)}} \right)^{1/2} \\
 & = n_m^{1/2} \left( \sum_{\substack{i_1, \dots, i_{m-1}=1 \\ j_1, \dots, j_{m-1}=1}}^{n_m} a_{i_1 i_2}^{(1)} \overline{a_{j_1 j_2}^{(1)}} \dots a_{i_{m-2} i_{m-1}}^{(m-2)} \overline{a_{j_{m-2} i_{m-1}}^{(m-2)}} y_{i_1}^{(1)} \overline{y_{j_1}^{(1)}} \dots y_{i_{m-1}}^{(m-1)} \overline{y_{j_{m-1}}^{(m-1)}} \sum_{i_m=1}^{n_m} a_{i_{m-1} i_m}^{(m-1)} \overline{a_{j_{m-1} i_m}^{(m-1)}} \right)^{1/2}.
 \end{aligned}$$

Since

$$\sum_{i_m=1}^{n_m} a_{i_{m-1} i_m}^{(m-1)} \overline{a_{j_{m-1} i_m}^{(m-1)}} = n_m \delta_{i_{m-1} j_{m-1}},$$

we have

$$\begin{aligned}
 & |A(x^{(1)}, \dots, x^{(m)})| \\
 & \leq n_m^{1/2} \left( \sum_{i_{m-1}=1}^{n_m} \sum_{\substack{i_1, \dots, i_{m-2}=1 \\ j_1, \dots, j_{m-2}=1}}^{n_m} a_{i_1 i_2}^{(1)} \overline{a_{j_1 j_2}^{(1)}} \dots a_{i_{m-2} i_{m-1}}^{(m-2)} \overline{a_{j_{m-2} i_{m-1}}^{(m-2)}} y_{i_1}^{(1)} \overline{y_{j_1}^{(1)}} \dots y_{i_{m-2}}^{(m-2)} \overline{y_{j_{m-2}}^{(m-2)}} y_{i_{m-1}}^{(m-1)} \overline{y_{j_{m-1}}^{(m-1)}} n_m \right)^{1/2} \\
 & = n_m \left( \sum_{i_{m-1}=1}^{n_m} \sum_{\substack{i_1, \dots, i_{m-2}=1 \\ j_1, \dots, j_{m-2}=1}}^{n_m} a_{i_1 i_2}^{(1)} \overline{a_{j_1 j_2}^{(1)}} \dots a_{i_{m-2} i_{m-1}}^{(m-2)} \overline{a_{j_{m-2} i_{m-1}}^{(m-2)}} y_{i_1}^{(1)} \overline{y_{j_1}^{(1)}} \dots y_{i_{m-2}}^{(m-2)} \overline{y_{j_{m-2}}^{(m-2)}} |y_{i_{m-1}}^{(m-1)}|^2 \right)^{1/2} \\
 & \leq n_m \left( \sum_{\substack{i_1, \dots, i_{m-2}=1 \\ j_1, \dots, j_{m-2}=1}}^{n_m} a_{i_1 i_2}^{(1)} \overline{a_{j_1 j_2}^{(1)}} \dots a_{i_{m-3} i_{m-2}}^{(m-3)} \overline{a_{j_{m-3} i_{m-2}}^{(m-3)}} y_{i_1}^{(1)} \overline{y_{j_1}^{(1)}} \dots y_{i_{m-2}}^{(m-2)} \overline{y_{j_{m-2}}^{(m-2)}} \sum_{i_{m-1}=1}^{n_m} a_{i_{m-2} i_{m-1}}^{(m-2)} \overline{a_{j_{m-2} i_{m-1}}^{(m-2)}} \right)^{1/2}.
 \end{aligned}$$

Thus

$$\begin{aligned}
 & |A(x^{(1)}, \dots, x^{(m)})| \\
 & \leq n_m \left( \sum_{\substack{i_1, \dots, i_{m-2}=1 \\ j_1, \dots, j_{m-2}=1}}^{n_m} a_{i_1 i_2}^{(1)} \overline{a_{j_1 j_2}^{(1)}} \dots a_{i_{m-3} i_{m-2}}^{(m-3)} \overline{a_{j_{m-3} i_{m-2}}^{(m-3)}} y_{i_1}^{(1)} \overline{y_{j_1}^{(1)}} \dots y_{i_{m-2}}^{(m-2)} \overline{y_{j_{m-2}}^{(m-2)}} \sum_{i_{m-1}=1}^{n_m} a_{i_{m-2} i_{m-1}}^{(m-2)} \overline{a_{j_{m-2} i_{m-1}}^{(m-2)}} \right)^{1/2}.
 \end{aligned}$$

Since

$$\begin{aligned} & \left( \sum_{\substack{i_1, \dots, i_{m-2}=1 \\ j_1, \dots, j_{m-2}=1}}^{n_m} a_{i_1 i_2}^{(1)} \overline{a_{j_1 j_2}^{(1)}} \dots a_{i_{m-3} i_{m-2}}^{(m-3)} \overline{a_{j_{m-3} j_{m-2}}^{(m-3)}} y_{i_1}^{(1)} \overline{y_{j_1}^{(1)}} \dots y_{i_{m-2}}^{(m-2)} \overline{y_{j_{m-2}}^{(m-2)}} \sum_{i_{m-1}=1}^{n_m} a_{i_{m-2} i_{m-1}}^{(m-2)} \overline{a_{j_{m-2} j_{m-1}}^{(m-2)}} \right)^{1/2} \\ &= n_{m-1}^{1/2} \left( \sum_{i_{m-2}=1}^{n_m} \sum_{\substack{i_1, \dots, i_{m-3}=1 \\ j_1, \dots, j_{m-3}=1}}^{n_m} a_{i_1 i_2}^{(1)} \overline{a_{j_1 j_2}^{(1)}} \dots a_{i_{m-3} i_{m-2}}^{(m-3)} \overline{a_{j_{m-3} j_{m-2}}^{(m-3)}} y_{i_1}^{(1)} \overline{y_{j_1}^{(1)}} \dots y_{i_{m-2}}^{(m-2)} \overline{y_{j_{m-2}}^{(m-2)}} \right)^{1/2} \\ &= n_{m-1}^{1/2} \left( \sum_{i_{m-2}=1}^{n_m} |y_{i_{m-2}}^{(m-2)}|^2 \sum_{\substack{i_1, \dots, i_{m-3}=1 \\ j_1, \dots, j_{m-3}=1}}^{n_m} a_{i_1 i_2}^{(1)} \overline{a_{j_1 j_2}^{(1)}} \dots a_{i_{m-3} i_{m-2}}^{(m-3)} \overline{a_{j_{m-3} j_{m-2}}^{(m-3)}} y_{i_1}^{(1)} \overline{y_{j_1}^{(1)}} \dots y_{i_{m-3}}^{(m-3)} \overline{y_{j_{m-3}}^{(m-3)}} \right)^{1/2}, \end{aligned}$$

we conclude that

$$\begin{aligned} & |A(x^{(1)}, \dots, x^{(m)})| \\ & \leq n_m n_{m-1}^{1/2} \left( \sum_{i_{m-2}=1}^{n_m} \sum_{\substack{i_1, \dots, i_{m-3}=1 \\ j_1, \dots, j_{m-3}=1}}^{n_m} a_{i_1 i_2}^{(1)} \overline{a_{j_1 j_2}^{(1)}} \dots a_{i_{m-3} i_{m-2}}^{(m-3)} \overline{a_{j_{m-3} j_{m-2}}^{(m-3)}} y_{i_1}^{(1)} \overline{y_{j_1}^{(1)}} \dots y_{i_{m-3}}^{(m-3)} \overline{y_{j_{m-3}}^{(m-3)}} \right)^{1/2} \end{aligned}$$

and repeating this procedure we finally obtain

$$\begin{aligned} |A(x^{(1)}, \dots, x^{(m)})| & \leq n_m n_{m-1}^{1/2} \dots n_2^{1/2} \left( \sum_{i_1=1}^{n_m} y_{i_1}^{(1)} \overline{y_{i_1}^{(1)}} \right)^{1/2} \\ & = n_m^{1/2} n_m^{1/2} \dots n_2^{1/2} \left( \sum_{i_1=1}^{n_1} |x_{i_1}^{(1)}|^2 \right)^{1/2} \\ & \leq n_m^{1/2} \left( n_m^{1/2} \dots n_1^{1/2} \right). \end{aligned}$$

Thus

$$\frac{G_{n_1 \dots n_m}^{(1)}}{\sqrt{n_1} \dots \sqrt{n_m} \max\{\sqrt{n_1}, \dots, \sqrt{n_m}\}} \leq 1.$$

The lower estimate is an adaptation of the bilinear case, using this well-know extension of inequality (2), in the case  $p = 1$ , to multiple sums as follows:

$$\begin{aligned} & \left( \sum_{j_1, \dots, j_m=1}^{n_1, \dots, n_m} |a_{j_1 \dots j_m}|^2 \right)^{1/2} \\ & \leq \left( \frac{2}{\sqrt{\pi}} \right)^m \left( \frac{1}{2\pi} \right)^{n_1 + \dots + n_m} \int_0^{2\pi} \dots \int_0^{2\pi} \left| \sum_{j_1, \dots, j_m=1}^{n_1, \dots, n_m} a_{j_1 \dots j_m} e^{it_{j_1}^{(1)}} \dots e^{it_{j_m}^{(m)}} \right| dt, \end{aligned}$$

where  $dt := dt_1^{(1)} \dots dt_{n_1}^{(1)} \dots dt_1^{(m)} \dots dt_{n_m}^{(m)}$ .

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