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DOUBLE SAMPLING $\overline{X}$ CONTROL CHART FOR A FIRST ORDER AUTOREGRESSIVE PROCESS

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Abstract

In this paper we propose the Double Sampling $\overline{X}$ control chart for monitoring processes in which the observations follow a first order autoregressive model. We consider sampling intervals that are sufficiently long to meet the rational subgroup concept. The Double Sampling $\overline{X}$ chart is substantially more efficient than the Shewhart chart and the Variable Sample Size chart. To study the properties of these charts we derived closed-form expressions for the average run length (ARL) taking into account the within-subgroup correlation. Numerical results show that this correlation has a significant impact on the chart properties.

Keywords: autocorrelation; double sampling; statistical process control.

Resumo

Neste artigo propomos o gráfico de controle $\overline{X}$ de amostragem dupla para monitoramento de processos nos quais as observações seguem um modelo autoregressivo de primeira ordem. Nós consideramos intervalos de amostragem suficientemente longos em linha com o conceito de subgrupos racionais. O gráfico de controle $\overline{X}$ de amostragem dupla é substancialmente mais eficiente que o Gráfico de Shewhart e do que o Gráfico com Amostra de Tamanho Variável. Para estudar as propriedades destes gráficos nós derivamos expressões de forma-fechada para o Numero Médio de Amostras até o Sinal (NMA) levando em conta a correlação dentro do subgrupo. Os resultados numéricos mostram que esta correlação tem impacto significante sobre as propriedades do gráfico.

Palavras-chave: autocorrelação; amostragem dupla; controle estatístico do processo.

1. Introduction

A Shewhart control chart is a statistical tool applied to data from a process to determine if a quality characteristic has shifted from its target setting (Montgomery, 2001). This form of process monitoring is widely used in industry to distinguish common causes from special causes of variation (often responsible for process shifts) and also to identify the time of a process change in order to understand its main cause and to improve the process by preventing its future occurrence. Despite being a powerful problem-solving and process improvement tool, of inherently simple application, one of the major drawbacks of the Shewhart $\overline{X}$ chart is its slowness to detect small shifts in the process mean. For this reason, the utilization of adaptive or dynamic control charts has progressively grown in the last two decades contributing to remarkably improve the effectiveness of process monitoring.

An adaptive control chart has at least one of its parameters – sampling interval, sample size or control limits coefficient – varying in real time based on the actual values of the sample statistics which provide current information about the status of the process (Tagaras, 1998). Considerable efforts have already been undertaken in the research of the statistical properties of the adaptive Shewhart charts with Variable Sampling Interval (VSI) (Reynolds, 1996; Runger & Montgomery, 1993); with Variable Sample Size (VSS) (Prabhu, Runger, Keats & Croasdale, 1993; Costa, 1994; Costa & Rahim, 2004), and with the Variable Sampling Interval and Variable Sample Size (VSSI) (Costa, 1997, 1998, 1998a, 1999 a/b/c; Carot, Jabaloyes & Carot, 2002). Another alternative used to improve the performance of the traditional Shewhart charts is the Double Sampling (DS) procedure (Croasdale, 1974; Daudin, 1992). He & Grigoryan (2005) and Champ & Aparisi (2006) recently extended the DS procedure for multivariate processes.

The DS $\overline{X}$ chart offers faster process shift detection than the standard Shewhart chart, at the expense of a larger administrative burden. The DS $\overline{X}$ chart also outperforms the VSI $\overline{X}$ chart and performs better than the static Exponentially Weighted Moving-Average (EWMA) and Cumulative-Sum (CUSUM) control charts for large shifts; however it is not as effective in detecting small shifts.

The Average Run Length (ARL) is the usual metric to measure the performance of a control chart with fixed sampling intervals. The ARL is the expected number of samples required by the chart to signal. When there is a change in the process, it is desirable to have a low ARL so that the change will be detected quickly; when the process is in control, it is desirable to have a large ARL so that the rate of false alarms produced by the chart is low (Lu & Reynolds, 1999).

The underlying assumption in Statistical Process Control (SPC) is that the observations from the process are independent and identically distributed (i.i.d.). However, there are many practical situations in which the process data are serially correlated.

Various charting techniques to deal with autocorrelated univariate process have been proposed. In many applications the dynamics of the process induces correlation in observations closely spaced in time. If the interval between observations is short enough to produce correlation, then, one simple approach is to skip some of them, with the consequent disadvantage of disregarding the concept of rational subgroup. A second alternative would be to widen the control limits as a remedial method to deal with the autocorrelated data. Vasilopoulos & Stamboulis (1978) formulated control charts with the control limits coefficients tailored for a first or second order autoregressive model processes. A more typical approach is to fit an appropriate time-series model to the observations and charting the i.i.d. forecast errors or residuals (Alwan & Roberts, 1988; Montgomery & Mastrangelo, 1991). However, additionally to the fact that forecasts "recover" from abrupt changes and thus leave only a limited "window of opportunity" for detection (Vander Wiel, 1996), this approach is not well-suited for detecting small shifts in processes with mild positive autocorrelation. Moreover, users may have some difficulty to interpret the residuals control charts and besides, fitting and maintaining an appropriate time-series model for each required process variable can be cumbersome (Faltin, Mastrangelo, Runger & Ryan, 1997).

Thus there will be many applications for which it will be desirable, from practical considerations, to apply a standard control chart to the original correlated observations. In this case, it is important not only to adjust the chart parameters to obtain the required protection against false alarms, but also to assess the effect of the correlation on the chart's performance (Bisgaard & Kulahci, 2005). It is well-known that when the observations are independent the $\overline{X}$ chart with variable sampling interval detects process shifts faster than the $\overline{X}$ chart with fixed sampling interval. Reynolds, Arnold & Baik (1996) worked with autocorrelated observations and obtained the same conclusion. In this paper, we study the efficiency of the double sampling $\overline{X}$ chart when the observations follow a first order autoregressive model and the samples are formed according to the rational subgroup concept.

2. The process model

The primary purpose of this paper is to address the monitoring of autocorrelated processes with the observations collected in rational subgroups, as it is often the case in the application of SPC, particularly in process industries. The purpose of a sampling strategy based on rational subgroups is to minimize the chance of variability due to assignable causes within a sample and to maximize the chance of variability between samples if assignable causes are present (Montgomery, 2001). As the process autocorrelation attenuates to zero when the time between observations increases; in our model only the within-subgroup autocorrelation needs to be modelled (the correlation between subgroups is practically negligible).

Throughout this paper we assume that the observations of the quality characteristic to be monitored fit to a First Order Autoregressive AR(1) model that has frequently been used in applications (Runger & Willemain, 1996; Montgomery & Mastrangelo, 1991; Wardell, Moskowitz & Plante, 1994; Timmer, Pignatiello Jr & Longnecker, 1998).

The process observations X_t can be written as:

$$X_t - \mu = \phi \, \left(X_{t-1} - \mu\right) + \varepsilon_t \ , \quad t = 1,2,...T \tag{1}$$

where μ is the process mean, ϕ is the autoregressive coefficient and $\varepsilon_t \sim N\ \left(0,\sigma_\varepsilon\right)$ is the i.i.d. white noise. We additionally consider that the AR(1) model is accurate. The numerical results presented in this paper are for the case of positive autocorrelation, from low to moderately high levels, believed to be prevalent in control charts applications.

3. The Double Sampling (DS) $\bar{X}$ Chart for autocorrelated process

The DS procedure assumes that one master sample (n) is formed by two consecutive subsamples ($n=n_1+n_2$) taken without any intervening time, therefore coming from the same probability distribution. The DS procedure follows:

<u>First stage</u>: take a subsample of size n_1 and compute the mean $\bar{X}_1$. Let $Z_1 = \left(\bar{X}_1 - \mu_0\right)/\sigma_{\bar{X}_1}$ where $\bar{X}_1 \sim N\left(\mu_0, \sigma_{\bar{X}_1}\right)$, μ_0 is the in-control process mean and $\sigma_{\bar{X}_1}$ is the subsample standard deviation. Alwan & Roberts (1988) give the variance of a sample mean for an AR(1) process and we employ his expression to calculate the standard deviation when the subsample size is n_1:

$$\sigma_{\bar{X}_1} = \frac{\sigma_X}{\sqrt{n_1}}\sqrt{1+\frac{2}{n_1}\left\{\frac{\phi^{n_1+1} - n_1\phi^2 + (n_1-1)\ \phi}{(\phi-1)^2}\right\}} = \frac{\sigma_X}{\sqrt{n_1}\Psi} \tag{2}$$

where σ_X is the process standard deviation given by:

$$\sigma_X = \sqrt{\sigma_\varepsilon^2/\left(1-\phi^2\right)} \tag{3}$$

and $\Psi^{-1} = \sqrt{1+\frac{2}{n_1}\left\{\frac{\phi^{n_1+1} - n_1\phi^2 + (n_1-1)\ \phi}{(\phi-1)^2}\right\}}$. Observe that $\Psi = 1$ when $n_1 = 1$ or when the process observations are independent.

At this stage there are three possibilities:

- If $|Z_1| \le L_1$, conclude that the process mean is in control and the DS procedure finishes at this stage.
- If $|Z_1| > L$, being $L > L_1 > 0$, conclude that the process mean is off-target and the DS procedure finishes at this stage.
- If $L_1 < |Z_1| < L$, the DS procedure goes to the second stage.

<u>Second stage</u>: take a subsample of size n_2 and compute the mean $\bar{X}_2$. Let $Z_2 = \left(\bar{Y} - \mu_0\right)/\sigma_{\bar{Y}}$ where $\bar{Y}$ is the master sample mean given by $\bar{Y} = (n_1\bar{X}_1 + n_2\bar{X}_2)/n$ and $\sigma_{\bar{Y}}$ is its standard deviation.

At this stage there are only two possibilities:

- If $|Z_2| \le L_2$, conclude that the process mean is on-target.
- If $|Z_2| > L_2$, conclude that the process is out of control.

Figure 1 is a graphical representation of this procedure.

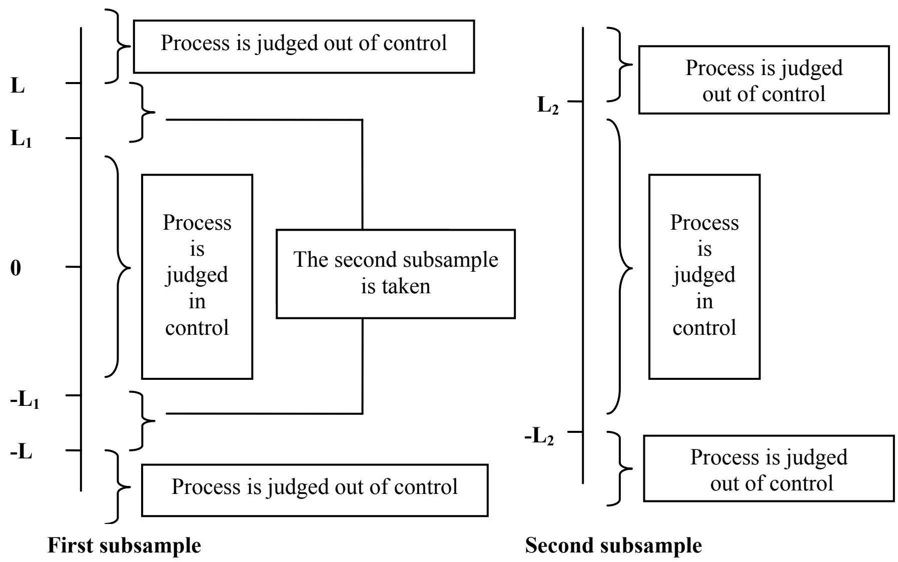

Figure 1 – Graphical view of the DS $\overline{X}$ chart.

4. Properties of the DS $\overline{X}$ control chart

We define the following intervals and unions of intervals in the first stage:

$I_1 = \left[\mu_0 - L_1 \sigma_{\overline{X}_1};\ \ \mu_0 + L_1 \sigma_{\overline{X}_1} \right]$,

$I_2 = \left[\left(\mu_0 - L \sigma_{\overline{X}_1};\ \mu_0 - L_1 \sigma_{\overline{X}_1} \right) \ \cup\ \left(\mu_0 + L_1 \sigma_{\overline{X}_1};\ \mu_0 + L \sigma_{\overline{X}_1} \right) \right]$,

$I_3 = \left[\left(-\infty;\ \ \mu_0 - L \sigma_{\overline{X}_1} \right) \ \cup\ \left(\mu_0 + L \sigma_{\overline{X}_1};\ \ +\infty \right) \right]$,

and the interval in the second stage is:

$I_4 = \left[\mu_0 - L_2 \sigma_{\overline{Y}};\ \ \mu_0 + L_2 \sigma_{\overline{Y}} \right]$

The average sample size per sampling (ASS) is:

$$\overline{n} = n_1 + n_2 \Pr\left[\overline{X}_1 \in I_2 \right] \tag{4}$$

where the probability of taking the second subsample is:

$$\Pr\left[\overline{X}_1 \in I_2 \right] = \Phi\left(L - \delta\sqrt{n_1}\Psi \right) - \Phi\left(L_1 - \delta\sqrt{n_1}\Psi \right) + \Phi\left(-L_1 - \delta\sqrt{n_1}\Psi \right) - \Phi\left(-L - \delta\sqrt{n_1}\Psi \right) \tag{5}$$

where $\Phi = (\bullet)$ is the standard Normal distribution function, $\delta = (\mu_1 - \mu_0)/\sigma_X$ and μ_1 is the off-target process mean. When the process is in-control, $\delta = 0$.

The efficiency of the control chart is generally measured by the ARL, which is the average number of samples until an out-of-control signal, expressed as:

$$ARL = 1/(1-P) \tag{6}$$

where $P = P_1+P_2$ and P_i (i=1,2) is the probability of deciding that the process is in control at stage i of the sampling procedure. The properties of a DS chart depend on the parameters n_1, n_2, L, L_1 and L_2. In a preliminary investigation we found that large values of L leads to the best chart performance, that is, when the false alarm risk during the first sampling stage is practically zero. Daudin (1992) recommends to adopt L=4 or L=5 and following his suggestion we set L=5. In this case, considering expression (4) one has that $P_1 = \left(\bar{n}-n_1\right)/n_2$ and from expression (6), when the process is in control, it follows that:

$$P_2 = 1 - P_1 - (1/ARL_0) \tag{7}$$

When the process is in control $ARL=ARL_0$.

The expressions (A11) and (A12) with $\delta = 0$, see the appendix, are used to obtain the values of L_1 and L_2. According to expression (A11), P_1 is an increasing function of L_1 so, by grid search, it is always possible to find the value of L_1 that matches both sides in the expression (A11) for any given P_1. According to expression (A12), P_2 is also an increasing function of L_2 and again in this case, by grid search, it is always possible to find the value of L_2 that matches both sides in the expression (7) for any given values of P_1 and ARL_0.

The VSS $\overline{X}$ chart and the $\overline{X}$ chart, for the autocorrelated process model under study, are introduced next in order to be compared with the DS procedure.

5. The competing control charts

When the VSS $\overline{X}$ chart is in use, one takes random samples from the process at a fixed unit of time. The process starts in a state of statistical control with mean $\mu = \mu_0$ and standard deviation σ, and at some random time in the future the mean shifts from μ_0 to $\mu_0 \pm \delta\sigma_X$, where $\delta > 0$. The sample means are plotted on the $\overline{X}$ control chart with warning limits $\mu_0 \pm k_1\sigma_{\overline{X}}$ and action limits $\mu_0 \pm k\sigma_{\overline{X}}$, where $0 < k_1 < k$ and $\sigma_{\overline{X}}$ is the standard deviation of the sample means. The size of each sample depends on what is observed in the preceding sample. If the $\overline{X}$ value falls in the chart's central region, then the size of the next sample, denoted by n_1, should be small. Alternatively, if the $\overline{X}$ point falls in the chart's warning region, then the next sample size, denoted by n_2, should be large. The properties of the VSS $\overline{X}$ control chart for the model considered in this study were determined using the expressions given in Costa (1994), however, considering that the standard deviation of the sample means and the standard deviation of the process are given by expressions (2) and (3), respectively.

The design of the $\overline{X}$ chart for autocorrelated processes follows the concepts of the classical methodology applied to independent data, with centreline at μ_0 and control limits (CL) set at $\mu_0 \pm k\sigma_{\overline{X}}$; however, taking the autocorrelation into account to determine the sample standard deviation, that is, we employed expression (2) only replacing n_1 by $\bar{n}$. To make a fair comparison between the $\bar{X}$ chart with single sampling and with double sampling, we designed the single sampling scheme with samples of size $\bar{n}$.

6. The performance of the control charts

The DS $\bar{X}$ chart control limits coefficients L_1 and L_2 for $\bar{n} = \{ 3 \text{ and } 5 \}$ and $\phi = \{ 0.00,\ 0.25,\ 0.50,\ 0.75 \}$ are in Table 1. The ARL and the probability of taking the second subsample are given in Tables 2 through 4. The ARL for the $\bar{X}$ chart and for the VSS $\bar{X}$ chart are in Tables 3 and 4.

Table 1 – Control Limits coefficients L_1 and L_2 for the Double Sampling $\bar{X}$ chart.

n_1	$\bar{n}$	n_2	L_1	L_2			
				$\phi = 0.00$	$\phi = 0.25$	$\phi = 0.50$	$\phi = 0.75$
1	3	4	0.6745	2.9360	2.9513	2.9779	2.9985
		8	1.1503	2.7521	2.7707	2.8230	2.9338
		12	1.3830	2.6056	2.6140	2.6648	2.8078
	5	12	0.9674	2.7696	2.7814	2.8140	2.9045
		16	1.1503	2.6714	2.6810	2.7138	2.8157
		20	1.2815	2.5870	2.5966	2.6278	2.7292
2	3	4	1.1503	2.9323	2.9507	2.9770	2.9984
		8	1.5341	2.7269	2.7496	2.8029	2.9176
		12	1.7317	2.5400	2.5604	2.6138	2.7623
	5	12	1.1503	2.7906	2.8029	2.8339	2.9115
		16	1.3180	2.6878	2.6984	2.7299	2.8240
		20	1.4395	2.5962	2.6061	2.6371	2.7323

The ARL of all the control charts deteriorates when the correlation is positive, particularly in the range of low mean shifts. For example, for the DS $\bar{X}$ chart, when $n_1 = 1$, $n_2 = 8$, $\bar{n} = 3$ and $\delta = 0.50$ the ARL increases from 19.8 for $\phi = 0$ (independent case) to 53.5 for $\phi = 0.5$, (see Table 2).

Table 2 – ARL and probability of taking the second subsample (shown between brackets) for the DS $\bar{X}$ chart (n_1=1).

			δ								
$\bar{n}$	n_2	ϕ	0.00	0.25	0.50	0.75	1.00	1.25	1.50	1.75	2.00
			(0.50)	(0.51)	(0.55)	(0.61)	(0.67)	(0.74)	(0.81)	(0.87)	(0.91)
		0.00	370.4	136.5	35.1	11.5	4.9	2.6	1.7	1.3	1.2
	4	0.25	370.4	174.2	54.6	19.7	8.5	4.3	2.6	1.8	1.4
		0.50	370.4	212.8	81.3	33.0	15.0	7.7	4.4	2.8	2.0
		0.75	370.4	249.8	116.0	53.4	26.4	14.1	8.2	5.1	3.4
			(0.25)	(0.26)	(0.31)	(0.37)	(0.46)	(0.55)	(0.64)	(0.73)	(0.80)
		0.00	370.4	94.1	19.8	6.5	3.1	2.0	1.6	1.4	1.2
3	8	0.25	370.4	129.5	32.5	10.8	4.7	2.7	1.8	1.4	1.3
		0.50	370.4	172.0	53.5	19.3	8.4	4.3	2.6	1.8	1.4
		0.75	370.4	224.3	91.0	38.3	17.9	9.3	5.3	3.4	2.3
			(0.17)	(0.18)	(0.22)	(0.28)	(0.36)	(0.45)	(0.55)	(0.64)	(0.73)
		0.00	370.4	74.8	15.1	5.5	3.1	2.2	1.8	1.6	1.4
	12	0.25	370.4	106.6	24.4	8.3	4.0	2.5	1.9	1.6	1.4
		0.50	370.4	147.6	40.9	14.2	6.3	3.4	2.2	1.7	1.4
		0.75	370.4	203.9	74.6	29.5	13.3	6.9	4.0	2.6	1.9
			(0.33)	(0.35)	(0.39)	(0.46)	(0.54)	(0.62)	(0.71)	(0.79)	(0.86)
		0.00	370.4	65.9	11.9	4.0	2.2	1.6	1.4	1.3	1.2
	12	0.25	370.4	98.4	20.9	6.7	3.1	1.9	1.5	1.3	1.2
		0.50	370.4	141.8	37.7	12.7	5.4	2.9	1.9	1.4	1.2
		0.75	370.4	202.2	73.2	28.7	12.9	6.6	3.8	2.5	1.8
			(0.25)	(0.26)	(0.31)	(0.37)	(0.46)	(0.55)	(0.64)	(0.73)	(0.80)
		0.00	370.4	54.0	9.7	3.7	2.3	1.8	1.6	1.4	1.1
5	16	0.25	370.5	82.9	16.6	5.5	2.8	1.9	1.6	1.4	1.2
		0.50	370.4	123.5	30.1	9.9	4.4	2.5	1.8	1.4	1.3
		0.75	370.4	184.8	61.2	22.9	10.0	5.1	3.0	2.1	1.6
			(0.20)	(0.21)	(0.25)	(0.32)	(0.40)	(0.49)	(0.59)	(0.68)	(0.75)
		0.00	370.4	46.5	8.6	3.7	2.5	2.0	1.7	1.5	1.2
	20	0.25	370.4	72.5	14.2	5.0	2.8	2.1	1.7	1.5	1.3
		0.50	370.4	110.6	25.6	8.5	3.9	2.4	1.8	1.5	1.3
		0.75	370.4	171.0	53.2	19.2	8.4	4.4	2.7	1.9	1.5

Larger sample sizes contribute to improve the statistical power of all the control charts under study, for all the autocorrelation levels and mean shifts considered. For example, for the variable sample size $\bar{X}$ chart, assuming $n_1 = 2$, $n_2 = 12$, $\delta = 0.50$ and $\phi = 0.50$, the ARL is reduced from 83.7 when $\bar{n} = 3$ to 54.9 when $\bar{n} = 5$, (see Tables 3 and 4).

Table 3 – ARL for the Shewhart chart (n=3), VSS chart and DS $\bar{X}$ chart (n_1=2, $\bar{n}$=3) (probability of taking the second subsample when using the DS chart shown between brackets).

ϕ	n_2	Chart	δ = 0.00	0.25	0.50	0.75	1.00	1.25	1.50	1.75	2.00
0.00	4	VSS	370.4	148.0	34.1	9.8	4.1	2.4	1.7	1.5	1.3
		DS	370.4	119.0	27.7	8.8	3.7	2.1	1.4	1.2	1.1
			(0.25)	(0.28)	(0.36)	(0.48)	(0.61)	(0.73)	(0.83)	(0.91)	(0.95)
	8	VSS	370.4	156.2	28.9	7.0	3.3	2.3	1.9	1.7	1.5
		DS	370.4	85.9	17.3	5.6	2.7	1.8	1.4	1.2	1.1
			(0.13)	(0.15)	(0.22)	(0.32)	(0.45)	(0.59)	(0.72)	(0.83)	(0.90)
	12	VSS	370.4	154.0	24.4	6.1	3.4	2.5	2.1	1.8	1.6
		DS	370.4	71.2	14.1	5.1	2.8	2.0	1.5	1.3	1.2
			(0.08)	(0.10)	(0.16)	(0.25)	(0.38)	(0.51)	(0.65)	(0.77)	(0.86)
		Shewhart	370.4	184.3	60.7	22.5	9.8	5.0	2.9	2.0	1.5
0.25	4	VSS	370.4	187.8	57.1	18.5	7.4	3.8	2.5	1.8	1.5
		DS	370.4	158.3	45.6	15.7	6.7	3.4	2.1	1.5	1.2
			(0.25)	(0.27)	(0.34)	(0.44)	(0.55)	(0.67)	(0.77)	(0.86)	(0.92)
	8	VSS	370.4	197.2	54.0	14.3	5.4	3.1	2.3	1.9	1.7
		DS	370.4	121.7	29.4	9.7	4.3	2.4	1.7	1.4	1.2
			(0.13)	(0.14)	(0.20)	(0.29)	(0.40)	(0.52)	(0.64)	(0.75)	(0.84)
	12	VSS	370.4	197.1	48.8	11.8	4.8	3.0	2.4	2.0	1.8
		DS	370.4	103.4	23.4	8.0	3.8	2.4	1.8	1.5	1.3
			(0.08)	(0.10)	(0.14)	(0.22)	(0.32)	(0.44)	(0.57)	(0.69)	(0.79)
		Shewhart	370.4	215.6	83.5	34.1	15.6	8	4.6	2.9	2.1
0.50	4	VSS	370.4	224.5	87.0	33.5	14.3	7.1	4.1	2.7	2.0
		DS	370.4	200.6	71.9	28.0	12.5	6.4	3.7	2.4	1.7
			(0.25)	(0.27)	(0.33)	(0.41)	(0.51)	(0.62)	(0.72)	(0.81)	(0.88)
	8	VSS	370.4	231.8	87.2	29.6	11.1	5.3	3.2	2.4	1.9
		DS	370.4	165.4	49.8	17.7	7.7	4.0	2.5	1.7	1.4
			(0.13)	(0.14)	(0.19)	(0.26)	(0.36)	(0.47)	(0.58)	(0.69)	(0.78)
	12	VSS	370.4	232.3	83.7	25.8	9.2	4.6	3.0	2.3	2.0
		DS	370.4	145.0	39.9	13.9	6.2	3.4	2.3	1.7	1.4
			(0.08)	(0.10)	(0.13)	(0.20)	(0.28)	(0.39)	(0.50)	(0.61)	(0.72)
		Shewhart	370.4	242.2	108.0	48.4	23.5	12.4	7.2	4.5	3.0
0.75	4	VSS	370.4	256.1	121.2	55.2	26.6	13.8	7.9	4.9	3.3
		DS	370.4	243.3	109.1	49.0	23.8	12.6	7.3	4.6	3.1
			(0.25)	(0.27)	(0.32)	(0.39)	(0.48)	(0.58)	(0.68)	(0.77)	(0.84)
	8	VSS	370.4	260.0	123.4	54.3	24.4	11.9	6.5	4.1	2.9
		DS	370.4	219.6	87.0	36.1	16.7	8.6	5.0	3.2	2.2
			(0.13)	(0.14)	(0.18)	(0.24)	(0.33)	(0.42)	(0.53)	(0.63)	(0.73)
	12	VSS	370.4	260.4	122.4	52.1	22.3	10.4	5.6	3.6	2.6
		DS	370.4	201.8	73.1	28.8	13.0	6.7	3.9	2.6	1.9
			(0.08)	(0.09)	(0.13)	(0.18)	(0.26)	(0.35)	(0.45)	(0.56)	(0.66)
		Shewhart	370.4	263.7	132.1	64.2	32.9	18.1	10.6	6.6	4.4

Table 4 – ARL for the Shewhart chart (n=5), VSS chart and DS $\overline{X}$ chart (n_1=2, $\bar{n}$=5) (probability of taking the second subsample when using the DS chart shown between brackets).

ϕ	n_2	Chart	δ 0.00	0.25	0.50	0.75	1.00	1.25	1.50	1.75	2.00
0.00	12	VSS	370.4	97.7	12.5	3.8	2.3	1.9	1.7	1.5	1.4
		DS	370.4	59.5	10.2	3.4	1.8	1.4	1.2	1.1	1.0
			(0.25)	(0.28)	(0.36)	(0.48)	(0.61)	(0.73)	(0.83)	(0.91)	(0.95)
	16	VSS	370.4	92.0	10.3	3.5	2.4	2.0	1.8	1.6	1.5
		DS	370.4	49.4	8.4	3.1	1.9	1.5	1.3	1.1	1.1
			(0.19)	(0.21)	(0.29)	(0.41)	(0.54)	(0.67)	(0.79)	(0.88)	(0.93)
	20	VSS	370.4	86.6	9.0	3.6	2.6	2.1	1.9	1.7	1.5
		DS	370.4	43.0	7.6	3.1	2.0	1.6	1.3	1.2	1.1
			(0.15)	(0.18)	(0.25)	(0.36)	(0.49)	(0.63)	(0.75)	(0.85)	(0.92)
		Shewhart	370.4	133.1	33.4	10.8	4.5	2.4	1.6	1.2	1.1
0.25	12	VSS	370.4	145.1	26.9	6.9	3.2	2.2	1.8	1.6	1.5
		DS	370.4	91.5	18.6	5.8	2.7	1.7	1.3	1.2	1.1
			(0.25)	(0.27)	(0.34)	(0.44)	(0.55)	(0.67)	(0.77)	(0.86)	(0.92)
	16	VSS	370.4	141.6	22.7	5.8	3	2.3	1.9	1.7	1.6
		DS	370.4	77.8	15	4.9	2.5	1.7	1.4	1.2	1.1
			(0.19)	(0.21)	(0.27)	(0.37)	(0.48)	(0.61)	(0.72)	(0.81)	(0.89)
	20	VSS	370.4	137.8	19.8	5.3	3.1	2.4	2	1.8	1.6
		DS	370.4	68.6	13	4.5	2.5	1.8	1.5	1.3	1.2
			(0.15)	(0.17)	(0.23)	(0.32)	(0.43)	(0.56)	(0.68)	(0.78)	(0.86)
		Shewhart	370.4	173.0	53.7	19.3	8.2	4.2	2.5	1.7	1.3
0.50	12	VSS	370.4	194.5	54.9	15.6	6.0	3.3	2.3	1.9	1.6
		DS	370.4	135.4	34.7	11.5	4.9	2.7	1.8	1.4	1.2
			(0.25)	(0.27)	(0.33)	(0.41)	(0.51)	(0.62)	(0.72)	(0.81)	(0.88)
	16	VSS	370.4	193.1	49.8	12.9	5.1	3.0	2.3	1.9	1.7
		DS	370.4	118.6	28.1	9.2	4.1	2.3	1.7	1.4	1.2
			(0.19)	(0.21)	(0.26)	(0.34)	(0.44)	(0.55)	(0.66)	(0.76)	(0.84)
	20	VSS	370.4	191.2	45.6	11.2	4.6	2.9	2.3	2.0	1.8
		DS	370.4	106.8	24.1	8.0	3.7	2.3	1.7	1.4	1.2
			(0.15)	(0.17)	(0.22)	(0.29)	(0.39)	(0.50)	(0.62)	(0.72)	(0.81)
		Shewhart	370.4	212.7	81.2	32.9	15.0	7.7	4.4	2.8	2.0
0.75	12	VSS	370.4	242.1	100.8	39.1	16.2	7.8	4.4	3	2.2
		DS	370.4	197.8	70	27.1	12	6.1	3.6	2.3	1.7
			(0.25)	(0.27)	(0.32)	(0.39)	(0.48)	(0.58)	(0.68)	(0.77)	(0.84)
	16	VSS	370.4	241.8	97.9	35.8	14	6.6	3.9	2.7	2.1
		DS	370.4	181.3	59.1	21.9	9.6	4.9	2.9	2	1.5
			(0.19)	(0.20)	(0.25)	(0.32)	(0.41)	(0.51)	(0.61)	(0.71)	(0.79)
	20	VSS	370.4	241.2	95.1	32.9	12.3	5.8	3.5	2.5	2.1
		DS	370.4	168.4	51.5	18.5	8.1	4.2	2.6	1.8	1.4
			(0.15)	(0.16)	(0.21)	(0.27)	(0.36)	(0.46)	(0.57)	(0.67)	(0.76)
		Shewhart	370.4	249.8	115.9	53.4	26.4	14.1	8.2	5.1	3.4

Regardless the magnitude of the process shift, the effect of setting (n_1) equal to one or two for the DS $\bar{X}$ chart is practically negligible on the efficiency. As a general rule, larger subsamples in the second stage (n_2) contribute to reduce the ARL of the DS $\bar{X}$ chart in the interval $0.25 < \delta < 1.50$. For example, considering $\phi = 0.25$; $n_1 = 2$, $\bar{n} = 3$ and $\delta = 0.50$ the ARL varies from 45.6 when $n_2 = 4$ to 23.4 when $n_2 = 12$, (see Table 3). Regarding the VSS $\bar{X}$ control chart, larger samples in the second stage (n_2) improve the efficiency of the control chart depending on the values of ϕ, δ and $\bar{n}$. For example, assuming $\delta = 0.50$, $n_1 = 2$, $\bar{n} = 5$ and increasing n_2 from 12 to 20 reduces the ARL from 26.9 to 19.8 (26.4% reduction) when $\phi = 0.25$; from 54.9 to 45.6 (16.9% reduction) when $\phi = 0.50$ and from 100.8 to 95.1 (only 5.7% reduction) when $\phi = 0.75$. The adoption of larger values of n_2 for moderately autocorrelated processes ($\phi = 0.50$) improves the control chart efficiency in the interval $0.75 < \delta < 1.75$ for $\bar{n} = 3$ and in the interval $0.50 < \delta < 1.25$ for $\bar{n} = 5$.

Assuming a correlation level of $\phi = 0.5$, $\bar{n} = 5$ and $0.50 \le \delta \le 2.00$ we compare the ARL of the DS control chart (n_1=2, n_2=20) with the ARL of the Shewhart chart ($\bar{n} = 5$) and the ARL of the VSS chart (n_1=2, n_2=20), (see Figure 2).

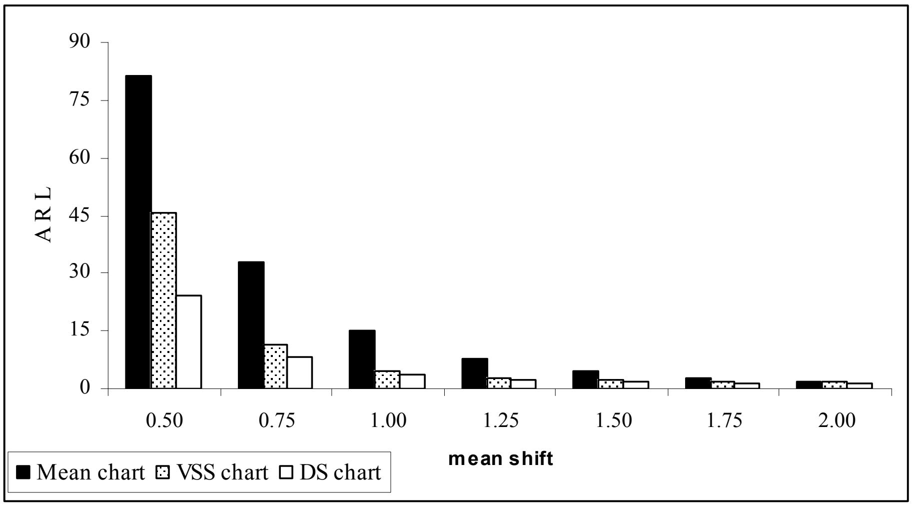

Figure 2 – Comparison of the ARL of the Mean Chart (n=5), the VSS chart ($\bar{n}$ =5, n_1=2, n_2=20) and the DS control chart ($\bar{n}$ =5, n_1=2, n_2=20) with correlation ϕ =0.5.

7. A simulation example

The purpose of the following example is to show the effect of the autocorrelation on the ability of the Shewhart and the DS $\overline{X}$ control charts in detecting process location shifts. To this end, we considered the AR(1) model, with $\phi = 0.847$ and normally distributed i.i.d. residuals with mean zero and $\sigma_\varepsilon = 3.867$, presented in the example 1 of Montgomery & Mastrangelo (1991), resulting from the monitoring of a critical parameter in a chemical process.

Assuming initially a process in-control with $\mu_0 = 84.60$ and samples of size n=3, taken according to the rational subgroup concept, we plot the mean values on a $\overline{X}$ chart, with the process standard deviation estimated by $\overline{R}/d_2$, where $\overline{R}$ is the sample range mean of 50 samples. The control limits of CL= 84.60±4.95 were determined by $CL = \overline{\overline{X}} \pm \left(3\overline{R}/d_2\sqrt{n}\right)$ (Montgomery, 2001) with $\overline{R} = 4.84$ and $d_2 = 1.693$, ignoring the serial correlation, (see Figure 3). The occurrence of many sample points beyond the control limits (samples 1, 20, 23 to 27, 29, 36, 37, 40, 41, 43, 47 and 48) indicates very tight control limits and high probability of false alarms due to the design of the control chart without recognizing the serial dependency within the subgroups.

Afterwards, using the same data set and with $\sigma_{\overline{X}}$ determined employing the expression (2) with n_1 replaced by n, new control limits calculated by ($CL = \overline{\overline{X}} \pm 3\sigma_{\overline{X}} = 84.60 \pm 20.34$) were set to account for correlation. Additionally, the last 20 samples were simulated again, now with $\mu_1 = 1.5\sigma_X$. In this case, the control chart signalled for the first time at sample 43 (run length = 13), (see Figure 4).

Finally, we investigated the process monitoring strategy using the DS $\overline{X}$ chart. Subsamples with size n_1=2 and n_2=4 were adopted and we set $\overline{n} = 3$, aiming a fair comparison between the DS and the Shewhart control charts. With coefficients L_1 = 1.1503 and L_2 = 3.0000, the control limits were set at $\overline{\overline{X}} \pm 8.041$ and $\overline{\overline{X}} \pm 18.824$, for the first and second stages, respectively CL1 and CL2. The action limits of the first stage, with coefficient L=5.00, are not shown on the chart. The values of $\overline{X}_1$ and $\overline{Y}$ are plotted on the chart according to the left hand vertical scale and the right hand vertical scale, respectively. As before, the last 20 master samples were simulated with $\mu_1 = 1.5\sigma_X$. According to the left hand vertical scale, an $\overline{X}_1$ value beyond the control limit CL1 triggers the inspection of the whole sample. In this case, the $\overline{Y}$ value is computed. According to the right hand vertical scale, a signal is given by an $\overline{Y}$ value beyond the control limit CL2. The control chart now signals the disturbance at sample 39 (run length = 9), (see Figure 5), faster than in the previous simulation with the $\overline{X}$ chart.

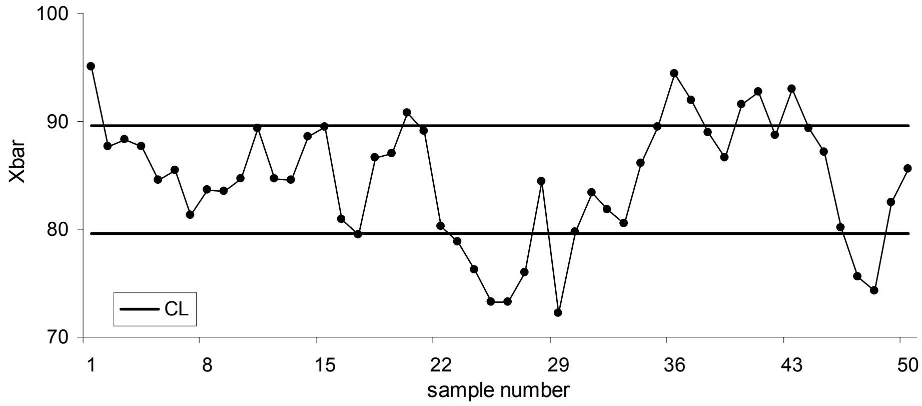

Figure 3 – Shewhart Control Chart (n=3) with unrecognized correlation (potential false alarms at samples #1, 20, 23 to 27, 29, 36, 37, 40, 41, 43, 47 and 48).

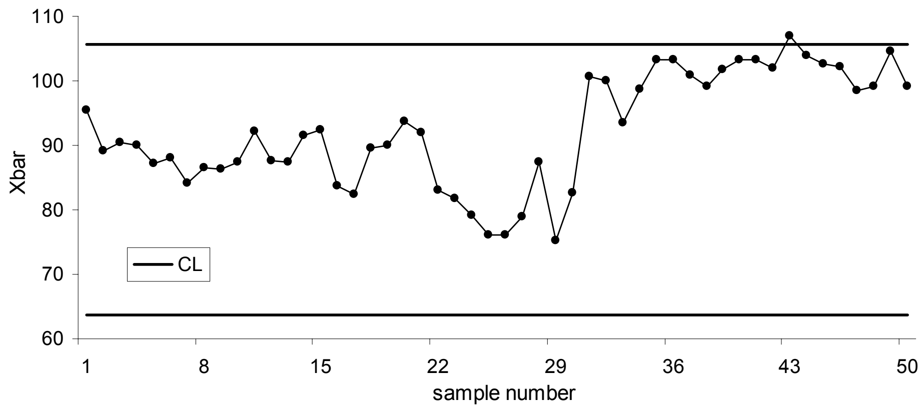

Figure 4 – Shewhart Control Chart (n=3) modified to account for the correlation.

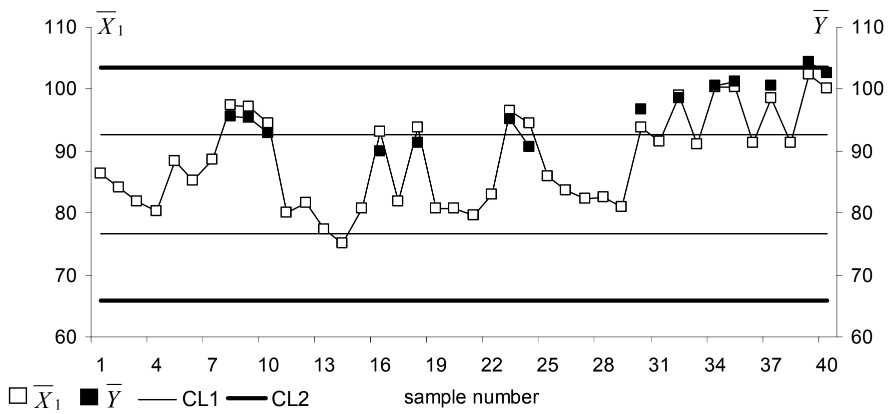

Figure 5 – Double Sampling Control Chart (n_1=2, n_2=4 and $\overline{n}$ =3) for the autocorrelated process.

8. Conclusion

Most standard control charting schemes for Statistical Process Control are based on the assumption that measurements of the product quality variable are independent within and between the subgroups. However, positive autocorrelation at low lags is commonplace in chemical processes (Montgomery, 2001), where, given the advances in sensor technologies, observations are closely spaced in time. In the present paper, we assume that successive observations collected from the process fit to a First Order Autoregressive model, and furthermore, that such observations are grouped following the rational subgroup concept (each sample consists of units that were consecutively produced and the subgroups are far spaced in time to maximize the chance of detecting difference between samples, if an assignable cause is present). This sampling strategy results in within-subgroup correlation and significant effects on the performance of control charts if the traditional control chart methodology is applied. Aiming to partially recover the chart efficiency lost due to the autocorrelation, we propose a Double Sampling $\overline{X}$ control chart for autocorrelated processes, as an extension of a previously introduced methodology (Daudin, 1992). We set up all the charts for the same false alarm rate and the same Average Sample Size per sampling when the process mean is on target, we derived suitable control limits and measured the process monitoring efficiency in fair comparison with competing schemes. The major findings from the present research are the following:

- In all cases, the positive correlation has a detrimental effect on the control chart's ability to detect process mean shifts;
- When the correlation within the subgroup ranges from low to moderate, the Average Run Length of the Double Sampling $\overline{X}$ chart is substantially better than the Average Run Length of both, the Shewhart and the Variable Sample Size $\overline{X}$ control charts, regardless the magnitude of the process shift;
- For processes with moderate correlation level, larger values of n_2 improve the mean shift detection ability of the Double Sampling scheme.

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Appendix A: Finding P_1 and P_2 expressions

The observations X_t for a first order autoregressive process AR(1), expression (1), are given by $X_t - \mu = \phi(X_{t-1} - \mu) + \varepsilon_t$, t=1, 2, 3,...,T ; where ϕ is the autoregressive coefficient, μ is the process mean and ε_t is the white noise, i.i.d., $\varepsilon_t \sim N\ (0, \sigma_\varepsilon)$. Without loss of generality, we take $\mu = \mu_0 = 0$ and $\sigma_\varepsilon = 1$ when the process is in-control. Based on the DS procedure, a master sample of size n, comprising sequential process observations, can be partitioned in two subsamples of sizes n_1 and n_2 units with $n = n_1+n_2$. Furthermore, we assume that the first master sample is taken at any given random point of time, and the subsequent master samples are taken in fixed time intervals sufficiently long, so that the effect of the autocorrelation between them is practically negligible. The mean of the first subsample n_1 is written as:

$$\overline{X}_1 = \frac{\sum_{i=1}^{n_1} X_i}{n_1} = \frac{\sum_{t=1}^{n_1} \sum_{i=1}^{t} \phi^{t-i}\,\omega_i}{n_1} \tag{A1}$$

where: $\omega_i \begin{cases} = x_1 \text{ when } t = 1 \\ = \varepsilon_i \text{ when } t \rangle 1 \end{cases}$

Given the fixed correlation structure within the subgroup, the expression of the second subsample mean ($\overline{X}_2$) comprises two terms, $X_{i,k}$ and $X_{i,u}$, to distinguish respectively the known and unknown information (ε_t values) after the measurement of the first subsample.

$$\overline{X}_2 = \frac{\left(\sum_{i=n_1+1}^{n} X_{i,k} + \sum_{i=n_1+1}^{n} X_{i,u} \right)}{n_2} \tag{A2}$$

where:

$$X_{\mathrm{i,k}} = \sum_{t=1}^{n_1} \phi^{i-t}\omega_{\mathrm{t}} \qquad \mathrm{i} = n_1+1,.......,n \tag{A3}$$

and

$$X_{\mathrm{i,u}} = \sum_{t=n_1+1}^{i} \phi^{i-t}\omega_{\mathrm{t}} \qquad \mathrm{i} = n_1+1,.......,n \tag{A4}$$

where: $\omega_t \begin{cases} = x_1 \text{ when } t=1 \\ = \varepsilon_t \text{ when } t \rangle 1 \end{cases}$

The combined (or master) sample mean ($\overline{Y}$) is:

$$\overline{Y} = \frac{n_1}{n}\overline{X}_1 + \frac{n_2}{n}\overline{X}_2 \tag{A5}$$

thus:

$$\overline{Y} = \frac{1}{n}\left(\sum_{t=1}^{n_1}\sum_{i=1}^{t}\phi^{t-i}\omega_{\mathrm{i}} + \sum_{i=n_1+1}^{n} X_{\mathrm{i,k}}\right) + \frac{1}{n}\left(\sum_{i=n_1+1}^{n} X_{\mathrm{i,u}}\right)$$

$$\therefore \overline{Y} = \frac{1}{n}\left(\sum_{t=1}^{n_1}\sum_{i=1}^{t}\phi^{t-i}\omega_{\mathrm{i}} + \sum_{i=n_1+1}^{n}\sum_{t=1}^{n_1}\phi^{i-t}\omega_{\mathrm{t}}\right) + \frac{n_2}{n}\left(\frac{\sum_{i=n_1+1}^{n}\sum_{t=n_1+1}^{i}\phi^{i-t}\omega_{\mathrm{t}}}{n_2}\right) \tag{A6}$$

and finally:

$$\overline{Y} = \Delta + \frac{n_2}{n}\overline{X}_2^{*} \tag{A7}$$

where:

$$\Delta = \frac{1}{n}\left(\sum_{i=1}^{n_1}\sum_{t=1}^{n-i+1}\phi^{t-1}\omega_{\mathrm{i}}\right) = \frac{1}{n}\sum_{i=1}^{n_1} B_i\omega_i \tag{A8}$$

with $B_i = \sum_{t=1}^{n-i+1}\phi^{t-1}$ and $\sigma_{\Delta} = \sqrt{\frac{B_1^2}{1-\phi^2} + B_2^2 + ... + B_{n_1}^2}$. The variable $\overline{X}_2^{*}$ introduced at (A7) is given by:

$$\overline{X}_2^{*} = \frac{\sum_{t=n_1+1}^{n}\sum_{i=t}^{n}\phi^{i-t}\omega_{\mathrm{t}}}{n_2} = \frac{\sum_{j=1}^{n_2} C_j\varepsilon_{\mathrm{n_1}+j}}{n_2} \tag{A9}$$

where: $C_j = \sum_{i=n_1+j}^{n}\phi^{i-(n_1+j)}$ $\qquad j$=1,2,3,……..n_2

and the variance is:

$$\sigma^2_{\overline{X}_2^*} = \sum_{i=1}^{n_2} C_i^2 \tag{A10}$$

Based on the DS procedure, one has:

$$P_1 = \Pr\left[\overline{X}_1 \in I_1\right] = \Pr\left[-L_1\sigma_{\overline{X}_1} - \delta\sigma_X \langle \overline{X}_1 \langle L_1\sigma_{\overline{X}_1} - \delta\sigma_X\right]$$

that can be rewritten as:

$$P_1 = \Pr\left[-L_1 - \delta\sqrt{n_1}\Psi \langle Z \langle L_1 - \delta\sqrt{n_1}\Psi\right] \tag{A11}$$

where: $\delta = \dfrac{\mu_1 - \mu_0}{\sigma_X}$

The probability P_2 is:

$$P_2 = \Pr\left[\left(\overline{Y} \in I_4\right) \cap \left(\overline{X}_1 \in I_2\right)\right]$$

that can be rewritten as:

$$P_2 = \int_{-L-\delta\sqrt{n_1}\Psi}^{-L_1-\delta\sqrt{n_1}\Psi} \left\{\Phi\left(\frac{n}{n_2}\right)\left(\frac{L_2\sigma_{\overline{Y}} - \Delta - \delta\sigma_X}{\sigma_{\overline{X}_2^*}}\right) - \Phi\left(\frac{n}{n_2}\right)\left(\frac{-L_2\sigma_{\overline{Y}} - \Delta - \delta\sigma_X}{\sigma_{\overline{X}_2^*}}\right)\right\}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}\,dz +$$

$$\int_{L_1-\delta\sqrt{n_1}\Psi}^{L-\delta\sqrt{n_1}\Psi} \left\{\Phi\left(\frac{n}{n_2}\right)\left(\frac{L_2\sigma_{\overline{Y}} - \Delta - \delta\sigma_X}{\sigma_{\overline{X}_2^*}}\right) - \Phi\left(\frac{n}{n_2}\right)\left(\frac{-L_2\sigma_{\overline{Y}} - \Delta - \delta\sigma_X}{\sigma_{\overline{X}_2^*}}\right)\right\}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}\,dz \tag{A12}$$

where $\sigma_{\overline{Y}}$ is given by expression (2) replacing n_1 by n.