

A Representation for the Modules of a Graph and Applications

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Abstract

We describe a simple representation for the modules of a graph G . We show that the modules of G are in one-to-one correspondence with the ideals of certain posets. These posets are characterized and shown to be layered posets, that is, transitive closures of bipartite tournaments. Additionally, we describe applications of the representation. Employing the above correspondence, we present methods for solving the following problems: (i) generate all modules of G , (ii) count the number of modules of G , (iii) find a maximal module satisfying some hereditary property of G and (iv) find a connected non-trivial module of G .

Keywords: *graphs, ideals, modules, posets, bipartite tournaments, algorithms*

1 Introduction

There exists a vast literature on the study of modules of a graph, since this concept was first introduced by Gallay [5]. See [2, 3, 6, 7, 8, 12, 13]. In special there are several relevant algorithmic applications of the modular decomposition of a graph [1, 9, 10, 11, 14]. This paper describes a simple representation for the modules of a graph G . We show that the modules of G are in one-to-one correspondence with the ideals of certain posets. These posets are shown to be transitive closures of bipartite tournaments.

G denotes an undirected graph, $V(G)$ and $E(G)$ the vertex and edge sets of G , respectively, with $|V(G)| = n$ and $|E(G)| = m$. For $v \in V(G)$, $N_G(v)$ denotes the set of neighbours of v in G , and $N_G[v] = N_G(v) \cup \{v\}$.

Let $\overline{N}_G(v) = V(G) \setminus N_G[v]$. A *module* of G is a non-empty subset $M \subseteq V(G)$, such that every vertex $v \in V(G) \setminus M$ is either a neighbour of all the vertices of M or of none of them. Clearly, $V(G)$ and all one vertex subsets of G are modules of G , called *trivial modules*. Finally, define a *bipartite tournament* as an orientation of a complete bipartite graph.

D denotes a directed graph, or digraph, $V(D)$ and $E(D)$ are its sets of vertices and directed edges, respectively. For $v \in V(D)$, let $N_D^+(v) = \{w \in V(D) \mid (v, w) \in E(D)\}$, $N_D^-(v) = \{w \in V(D) \mid (w, v) \in E(D)\}$, $N_D^+[v] = N_D^+(v) \cup \{v\}$ and $N_D^-[v] = N_D^-(v) \cup \{v\}$. Additionally, define $N_D(v) = N_D^+(v) \cup N_D^-(v)$ and $N_D[v] = N_D(v) \cup \{v\}$. When convenient we may drop the symbol of the graph or digraph, in the indices of these notations. If $N_D^-(v) = \emptyset$ then v is the *source* of D . For $v, w \in V(D)$, if D contains a $v - w$ path, then v is an *ancestor* of w and w is a *descendant* of v . Say that D is *strongly connected* when it contains both $v - w$ and $w - v$ paths, for every pair $v, w \in V(D)$. The *strongly connected components* of D are the maximal strongly connected subdigraphs of D . The *condensation* C of D is the digraph whose vertices correspond to the strongly connected components of D , while $w \in N_C^+(v)$ when there is an edge in D from a vertex lying in the strongly connected component corresponding to v , to a vertex in the component corresponding to w .

A digraph is *transitive* when $(v, w), (w, z) \in E(D)$ implies $(v, z) \in E(D)$, for all $v, w, z \in V(D)$. The *transitive closure* of a digraph D is the transitive spanning superdigraph of D , preserving its reachability. A *partially ordered set (poset)* is an acyclic transitive digraph. An *ideal* of a poset P is a subset $I \subseteq V(P)$, such that $x \in I$ implies $N_D^-(x) \subseteq I$, for all $x \in V(P)$.

Section 2 presents the correspondence between

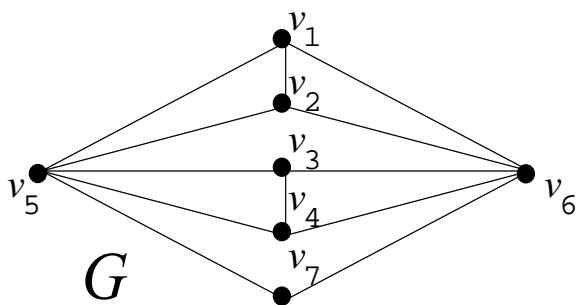


Figure 1: Graph G .

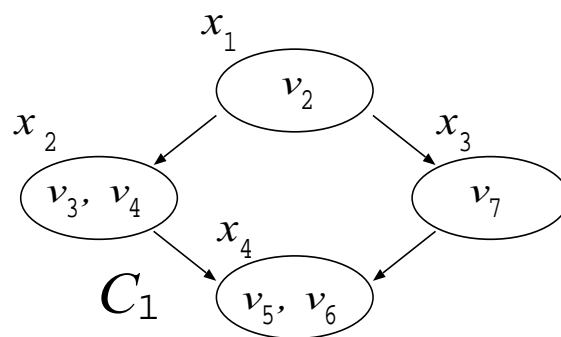


Figure 3: The condensation graph C_1 of D_1 .

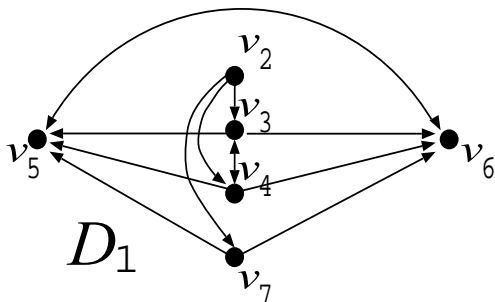


Figure 2: The modular digraph D_1 of G relative to v_1 .

modules of a graph G and ideals of certain posets, as well as a characterization of these posets.

2 The Representation

Let G be a graph and $v_i \in V(G)$. The *modular digraph* D_i of G relative to v_i , is one with vertex set $V(G) \setminus \{v_i\}$ and edge set defined as follows. For each pair of distinct vertices $v_j, v_k \in V(G)$, $i \neq j, k$,

$$(v_j, v_k) \in E(G) \Rightarrow$$

$$\begin{cases} \text{if } v_j, v_k \notin N_G(v_i) \\ \text{then } (v_j, v_k), (v_k, v_j) \in E(D_i). \end{cases} \quad (1.1)$$

$$\begin{cases} \text{if } v_j \notin N_G(v_i) \text{ and } v_k \in N_G(v_i) \\ \text{then } (v_j, v_k) \in E(D_i). \end{cases} \quad (1.2)$$

$$(v_j, v_k) \notin E(G) \Rightarrow$$

$$\begin{cases} \text{if } v_j, v_k \in N_G(v_i) \\ \text{then } (v_j, v_k), (v_k, v_j) \in E(D_i). \end{cases} \quad (2.1)$$

$$\begin{cases} \text{if } v_j \notin N_G(v_i) \text{ and } v_k \in N_G(v_i) \\ \text{then } (v_k, v_j) \in E(D_i). \end{cases} \quad (2.2)$$

D_i contains no other edges, besides those as above.

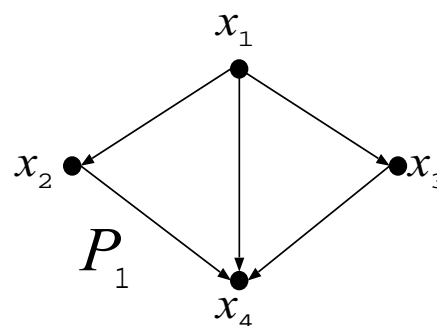


Figure 4: The modular poset P_1 of G relative to v_1 .

Further, define the *modular poset* P_i of G , relative to $v_i \in V(G)$, as the transitive closure of the condensation C_i of D_i . See Figures 1–4 for examples.

Let $v \in V(D_i)$. Denote by $S(v) \subseteq V(D_i)$ the subset of vertices belonging to the same strongly connected component of D_i , as v does. Similarly, for $x \in V(P_i)$, $S(x) \subseteq V(D_i)$ is the subset of vertices which forms the strongly connected component of D_i , corresponding to x in the condensation. Call $S(v)$ the *expansion* of v , while v is the *reduction* of $S(v)$ in D_i . On the other hand, $S(x)$ is the expansion of $x \in V(P_i)$ and x is the *reduction* of $S(x)$ in P_i .

Ehrenfeucht et al. [4] have previously employed the condensation of a digraph (different from the above) in the process of computing the modular decomposition tree of a symmetric two-structure.

The following theorem characterizes the modules of G , in terms of the ideals of modular posets.

Theorem 1 *Let G be a graph, $v_i \in V(G)$ and P_i the modular poset of G , relative to v_i . Then there exists a one-to-one correspondence between the ideals of P_i and the modules of G containing v_i .*

Proof: : Let M be a module of G containing $v_i \in V(G)$. We show that we can always choose a convenient ideal I of the modular poset P_i of G , to correspond to M . Let D_i be the modular digraph of G . We know that the vertices of P_i are in one-to-one correspondence with the strongly connected components of D_i . Let x be a vertex of P_i and $S(x)$ the expansion of x .

Proposition 2 *Either all vertices of $S(x)$ belong to M or none of them does.*

Proof: To start, note that the proposition is trivially true when $|S(x)| = 1$. When $|S(x)| > 1$, assume that it is false. In this case, $S(x)$ contains at least one vertex that belongs to M , and at least one which does not. Because the vertices of $S(x)$ form a strongly connected component of D_i , it follows that there must be an edge $(v_j, v_k) \in E(D_i)$, such that $v_j, v_k \in S(x)$, $v_j \notin M$ and $v_k \in M$. Examine the following alternatives for the pair of vertices v_j, v_k . First, assume that $(v_j, v_k) \in E(G)$. Then $v_k \in M$ and $v_j \notin M$ implies that $v_j \in N_G(v_i)$. In this case, (1.1) and (1.2) assure that $(v_j, v_k) \notin E(D_i)$. The other possibility is $(v_j, v_k) \notin E(G)$. In such situation, the implication is that $v_j \notin N_G(v_i)$. By applying this time, (2.1) and (2.2) we again conclude that $(v_j, v_k) \notin E(D_i)$, contradicting the fact that (v_j, v_k) must be an edge of D_i . Hence Proposition 2 is true.

The proof of the theorem proceeds by assigning a label 0 or 1 to each vertex x of P_i , as follows. Examine the set $S(x)$. By Proposition 2, either all vertices of $S(x)$ belong to M , or none of them does. In the former alternative, assign the label 0 to x . In the latter, assign 1. Let I be the subset of vertices of P_i having label 0. In other words, $M = \{v_i\} \cup_{x \in I} S(x)$. We show that the subset I is an ideal of P_i . If possible choose two vertices x, y of P_i , such that x is labelled 1, y has label 0 and $(x, y) \in E(P_i)$. Since P_i is the transitive closure of the condensation of D_i , the latter digraph must contain an edge $(v_j, v_k) \in E(D_i)$, from some vertex $v_j \in S(x)$ to $v_k \in S(y)$. Because of the values of the labels of x and y , it follows that $v_j \notin M$ and $v_k \in M$. Examine the following alternatives for the pair v_j, v_k . When $(v_j, v_k) \in E(G)$, the conditions $v_j \notin M$ and $v_k \in M$ imply $v_j \in N_G(v_i)$. By applying (1.1) and (1.2), we conclude that $(v_j, v_k) \notin E(D_i)$, a contradiction. When $(v_j, v_k) \notin E(G)$ it follows that $v_j \notin N_G(v_i)$. The same contradiction $(v_j, v_k) \notin E(D_i)$ arises by (2.1) and (2.2). The conclusion is that no such edge $(v_j, v_k) \in E(D_i)$ may exist, and consequently the above choice of vertices x, y of P_i is not

possible. That is, no vertex labelled 1 in P_i has any descendant labelled 0. Consequently, I is in fact an ideal of P_i .

Conversely, let I be an ideal of P_i . We show the existence of a module M of G containing v_i , that can be made to correspond to I . Let $x \in V(P_i)$ and $S(x)$ the expansion of x . Let $M = \{v_i\} \cup_{x \in I} S(x)$. We will conclude that M is a module of G . If $I = \emptyset$ or $I = V(P_i)$ then M is a trivial module of G and the argument is complete. Otherwise, there exists $v_j \in V(G) \setminus M$, with $j \neq i$. If for any such v_j , $M \subseteq N_G(v_j)$ or $M \subseteq \overline{N}_G(v_j)$, then M is indeed a module of G , again completing the argument. Otherwise, there is $v_j \in V(G) \setminus M$ such that $M \not\subseteq N_G(v_j)$ and $M \not\subseteq \overline{N}_G(v_j)$. Then we can choose $v_k, v_l \in M$, such that $v_k \notin N_G(v_j)$ and $v_l \in N_G(v_j)$. First, we examine if i can be equal to k or l . Suppose $i = k$. Then $v_j \notin N_G(v_i)$. Since $(v_j, v_l) \in E(G)$, by (1.1) and (1.2) we conclude that $(v_j, v_l) \in E(D_i)$. Denote by $x' \in V(P_i)$ and $x'' \in V(P_i)$ the reductions in P_i of $S(v_j)$ and $S(v_l)$, respectively. Because $v_j \notin M$ and $v_l \in M$, we conclude that $x' \notin I$ and $x'' \in I$. However $(v_j, v_l) \in E(D_i)$. Hence I can not be an ideal of P_i , a contradiction. The second alternative, $i = l$ is similar and can not occur too. Consequently, $i \neq k, l$.

The proof proceeds by examining all possibilities of containments in $N_G(v_i)$ and $\overline{N}_G(v_i)$ of the vertices v_j, v_k, v_l .

Case 1 $v_j, v_k \in N_G(v_i)$

By (2.1) it follows that $(v_j, v_k), (v_k, v_j) \in E(D_i)$. Then v_j and v_k belong to a same strongly connected component of D_i . In this case, $v_j, v_k \in S(x)$, for some vertex x of P_i . By the construction of M , if $x \in I$ then $v_j, v_k \in M$. The latter contradicts $v_j \notin M$. The opposite case, $x \notin I$, implies $v_j, v_k \notin M$, contradicting $v_k \in M$. Hence Case 1 can not occur.

Case 2 $v_j \in N_G(v_i)$ and $v_k \in \overline{N}_G(v_i)$.

By (2.2), it follows that $(v_j, v_k) \in E(D_i)$. If v_j and v_k belong to the same strongly connected component of D_i , a similar argument as the Case 1, leads to a contradiction. When v_j and v_k belong to distinct components, let x and x' represent the reductions in P_i of $S(v_j)$ and $S(v_k)$, respectively. That is, $v_j \in S(x)$ and $v_k \in S(x')$. Consequently, $x \notin I$ and $x' \in I$. However, $(v_j, v_k) \in E(D_i)$ implies that x is an ancestor of x' in P_i . The latter contradicts I to be an ideal of P_i .

Case 3 $v_j \in \overline{N}_G(v_i)$ and $v_l \in N_G(v_i)$

By (1.2) we conclude that $(v_j, v_l) \in E(D_i)$. A contradiction arises by applying an argument similar as in Case 2, with v_l replacing v_k .

Case 4 $v_j, v_l \in \overline{N}_G(v_i)$

Applying (1.1) we conclude that $(v_j, v_l), (v_l, v_j) \in E(D_i)$. Hence v_j and v_l belong to a same strongly connected component of D_i . Similarly as in Case 1, we conclude that the present situation can not occur too.

The above four cases cover all eight possibilities for containment in $N(v_i)$ of v_j, v_k, v_l . Since none of them can occur, we conclude that there is no triple $v_j, v_k, v_l \in V(G)$, with $v_j \notin M$ and $v_k, v_l \in M$, satisfying $v_k \notin N_G(v_j)$ and $v_l \in N_G(v_j)$. Therefore, M is indeed a module of G .

Each of the two above described correspondences is the inverse of the other. This completes the proof of Theorem 1. \square

For example, in the graph G of Figure 1, the correspondences between modules M_i containing the vertex v_1 and ideals I_i of the modular poset P_1 of G relative to v_1 of Figure 4 are as follows:

$$\begin{aligned} M_0 &= \{v_1\} \text{ corresponds to } I_0 = \{\emptyset\}, \\ M_1 &= \{v_1, v_2\} \text{ corresponds to } I_1 = \{x_1\}, \\ M_2 &= \{v_1, v_2, v_3, v_4\} \text{ corresponds to } I_2 = \{x_1, x_2\}, \\ M_3 &= \{v_1, v_2, v_7\} \text{ corresponds to } I_3 = \{x_1, x_3\}, \\ M_4 &= \{v_1, v_2, v_3, v_4, v_7\} \text{ corresponds to } I_4 = \{x_1, x_2, x_3\}, \\ M_5 &= \{v_1, v_2, v_3, v_4, v_7, v_5, v_6\} \text{ corresponds to } I_5 = \{x_1, x_2, x_3, x_4\}. \end{aligned}$$

Theorem 1 has shown a correspondence between the modules of G containing v_i and ideals of the corresponding modular poset of G . In the sequel, the interest is to characterize these posets. We use more notation.

Let D be an acyclic digraph. A *layer decomposition* of D is a sequence L_1, \dots, L_t of subsets $L_k \subseteq V(D)$, such that $\bigcup_{1 \leq k \leq t} L_k = V(D)$ and L_k is the set of sources of the digraph $D \setminus L_k^-$, where $L_1^- = \emptyset$ and $L_k^- = \bigcup_{1 \leq l < k} L_l$, for $k > 1$.

Each L_k is called a *layer* of D . Clearly the layer decomposition of D is unique. Say that the digraph D is *layered*, when its layer decomposition L_1, \dots, L_t is such that $(v_j, v_l) \in E(D)$, for any $v_j \in L_k$ and $v_l \in L_{k+1}$, $1 \leq k < t$.

Theorem 3 *The following affirmatives are equivalent:*

- (i) P_i is the modular poset of some graph G , relative to $v_i \in V(G)$,

- (ii) P_i is a layered poset,

- (iii) P_i is the transitive closure of an acyclic bipartite tournament.

Proof: (i) \Rightarrow (ii): Let G be a graph, $v_i \in V(G)$, D_i the modular digraph of G , relative to v_i , C_i the condensation of D_i , and P_i the corresponding modular poset of G . We have to show that P_i is layered. We can classify the vertices of P_i (and C_i) into three types as follows. Let $x \in V(P_i)$ and $S(x) \subseteq V(D_i)$ the expansion of x . Then

$$x \text{ is of type } \begin{cases} 0 & \text{when } S(x) \subseteq N_G(v_i) \\ 1 & \text{when } S(x) \subseteq \overline{N}_G(v_i) \\ 2 & \text{otherwise} \end{cases}$$

Similarly, say that a vertex $v_j \in S(x)$ is a *type l* vertex of D_i , when x is a type l vertex of P_i , $0 \leq l \leq 2$. Denote by T_l the set of type l vertices of P_i . The following proposition is useful.

Proposition 4 *Let x be a vertex of C_i . Then*

$$x \text{ is of type } \begin{cases} 0 & \Rightarrow N_{C_i}(x) = T_1 \cup T_2 \\ 1 & \Rightarrow N_{C_i}(x) = T_0 \cup T_2 \\ 2 & \Rightarrow N_{C_i}[x] = V(C_i) \end{cases}$$

Proof: To prove the above fact, we can assume that P_i has at least two vertices, otherwise the proposition is trivial. Let x, y be distinct vertices of P_i . Consider the following alternatives.

Case 1: x and y are type 0.

Then $S(x), S(y) \subseteq N_G(v_i)$. Let $v_j \in S(x)$ and $v_k \in S(y)$. Examining (1.1) - (1.2) and (2.1) - (2.2), since $v_j, v_k \in N_G(v_i)$ we conclude that (2.1) gives $(v_j, v_k), (v_k, v_j) \in E(D_i)$. This implies that v_j, v_k belong to the same strongly connected component of D_i , contradicting x, y to be distinct. Hence alternative (2.1) does not occur. Consequently, $v_k \notin N_{D_i}(v_j)$. Since v_j and v_k are arbitrary vertices of $S(x)$ and $S(y)$, respectively, we conclude that there are no edges in C_i between a vertex of $S(x)$ and another of $S(y)$. Consequently, $y \notin N_{C_i}(x)$.

Case 2: x is type 0 and y is type 1.

Then $S(x) \subseteq N_G(v_i)$ and $S(y) \subseteq \overline{N}_G(v_i)$. Again, let $v_j \in S(x)$ and $v_k \in S(y)$. If $(v_j, v_k) \in E(G)$, applying (1.2) implies that $(v_k, v_j) \in E(D_i)$. When $(v_j, v_k) \notin E(G)$, (2.2) leads to $(v_j, v_k) \in E(D_i)$. The conclusion is that $v_k \in N_{D_i}(v_j)$, meaning that $y \in N_{C_i}(x)$.

Case 3: x is type 0 and y is type 2.

We know that $S(x) \subseteq N_G(v_i)$, while $S(y) \not\subseteq N_G(v_i)$ and $S(y) \not\subseteq \overline{N}_G(v_i)$. Then we can choose $v_j \in S(x)$ and $v_k \in S(y)$, such that $v_j \in N_G(v_i)$ and $v_k \in \overline{N}_G(v_i)$. Similarly as in Case 2, (1.2) and (2.2) imply that either $(v_k, v_j) \in E(D_i)$ or $(v_j, v_k) \in E(D_i)$. Consequently, $v_k \in N_{D_i}(v_j)$, that is $y \in N_{C_i}(x)$.

Case 4: x and y are type 1.

Similarly, as in Case 1, we obtain $y \notin N_{C_i}(x)$.

Case 5: x is type 1 and y type 2.

Analogous to Case 3, and we conclude that $y \in N_{C_i}(x)$.

Case 6: x and y are type 2.

That is, $S(x), S(y) \not\subseteq N_G(v_i)$ and $S(x), S(y) \subseteq \overline{N}_G(v_i)$. This means that the following choice of vertices is possible. Let $v_j \in S(x)$, and $v_k \in S(y)$ satisfying $v_j \in N_G(v_i)$ and $v_k \in \overline{N}_G(v_i)$. This situation is again similar to Case 2, implying that $y \in N_{C_i}(x)$.

As the last step of the proof, look at all the above cases. Let $x \in V(P_i)$ be a type 0 vertex. Applying Cases 1, 2 and 3, we conclude that $N_{C_i}(x) = T_1 \cup T_2$. If x is of type 1, then Cases 2, 4 and 5 lead to $N_{C_i}(x) = T_0 \cup T_2$. The last alternative is that x is type 2, which implies $N_{C_i}[x] = V(C_i)$, by Cases 3, 5 and 6. Proposition 4 is proved.

In the sequel, let L_1, \dots, L_t be a layer decomposition L of C_i . Examine the types of vertices in C_i belonging to a same layer or to consecutive layers in L . Let x, y be distinct vertices of C_i . Suppose that x, y belong to a same layer of L . Then they can not be adjacent in C_i . By Proposition 4, it follows that if x is of type 0, so is y ; when x is type of 1, so is y ; and if x is of type 2 implies that y can not exist. Consequently, there can be no layer formed by vertices of distinct types. Moreover, any type 2 vertex is the sole vertex in its layer. Then we can say that layer L_q is of type l , when L_q contains a type l vertex, $0 \leq l \leq 2$. We study the alternatives when x, y belong to consecutive layers of L . Suppose that the theorem is false, that is, P_i is not a layered poset. Therefore, C_i is not layered too. Then there exist $x \in L_q$ and $y \in L_{q+1}$, such that $(x, y) \notin E(C_i)$. Because L is a layered decomposition, we know that $(y, x) \notin E(C_i)$. Then $y \notin N_{C_i}(x)$. Consequently, neither layer L_q nor L_{q+1} can be of type 2. Since L is a layer decomposition, there are $x' \in L_q$

and $y' \in L_{q+1}$, such that $(x', y') \in E(C_i)$. By Proposition 4, the latter implies that L_q and L_{q+1} can not be both of a same type. Consequently, L_q is of type 0 and L_{q+1} is of type 1, or vice-versa. Again, by Proposition 4, this implies that $(x, y) \in E(C_i)$, contradicting our inicial assumption. Therefore, C_i is layered and consequently so is P_i .

(ii) \Rightarrow (iii): Let P_i be a layered poset. The proof is to construct a bipartite tournament B_i , such that P_i is the transitive closure of B_i . Let L_1, \dots, L_t be the layer decomposition L of P_i . Denote by B_i the subdigraph obtained from P_i by removing the edges of P_i between vertices belonging to layers L_q, L_p , such that q and p are both odd or both even. The vertices of B_i can be partitioned into two subsets L', L'' , the first formed by the odd layers and the second by the even layers of L .

By construction, B_i has no edges between vertices of the same subset L' or L'' . On the other hand, since P_i is a layered poset, $x \in L'$ and $y \in L''$ imply that x and y are adjacent in P_i , and therefore adjacent in B_i . That is, B_i is a bipartite tournament. Since P_i is acyclic, so is B_i . In addition, all the edges of $E(P_i) \setminus E(B_i)$ are necessarily implied by transitivity. Consequently, P_i is the transitive closure of B_i . That is, (ii) \Rightarrow (iii).

(iii) \Rightarrow (i): By hypothesis, P_i is the transitive closure of an acyclic bipartite tournament B_i . We have to prove that P_i is the modular poset of some graph G , relative to a vertex $v_i \in V(G)$. With this purpose, we construct an undirect graph G , having B_i as its modular digraph, relative to v_i . Define $V(G) = V(B_i) \cup \{v_i\}$, where $v_i \notin V(B_i)$. The edges of G are defined in the sequel. Let $V_0 \cup V_1 = V(B_i)$ be a bipartition of the vertices of B_i . The vertices of G adjacent to v_i are defined as being all of V_0 and none of V_1 . The set V_0 is a clique of G , while V_1 is an independent set of G . It remains to define the edges (v_j, v_k) of G , such that $v_j \in V_0$ and $v_k \in V_1$. For each pair v_j, v_k , where $v_j \in V_0$ and $v_k \in V_1$, we know that B_i contains exactly one of the directed edges (v_j, v_k) or (v_k, v_j) . The decision of whether or not (v_j, v_k) will be included as an undirected edge of G depends on its direction in B_i , as follows. Include (v_j, v_k) in $E(G)$ if and only if its direction in B_i is from v_k to v_j . The construction of G is complete.

The next task is to show that B_i is precisely the modular digraph D_i of G relative to v_i . Clearly, $V(D_i) = V(B_i)$, as both of them are equal to $V(G) \setminus \{v_i\}$. By the construction of G , $N_G(v_i) = V_0$ and

$\overline{N}_G(v_i) = V_1$. Since V_0 is a clique in G , (1.1) and (1.2) assure that D_i has no edges between two vertices of V_0 . Similarly, from (2.1) and (2.2) we conclude that there are no edges in D_i between two vertices of V_1 . Finally, for each pair $v_j \in V_0$ and $v_k \in V_1$, (1.2) assures that when $(v_j, v_k) \in E(G)$, D_i contains the directed edge (v_k, v_j) . Similarly, from (2.2) we conclude that whenever $(v_j, v_k) \notin E(G)$, it follows $(v_j, v_k) \in E(D_i)$. This completes the description of D_i . Observe that D_i is a bipartite digraph, with bipartition $V_0 \cup V_1$. In addition, for $v_j \in V_0$ and $v_k \in V_1$, D_i contains either the edge (v_j, v_k) or (v_k, v_j) , its direction being the same as in B_i . Consequently, $D_i = B_i$. Since B_i is acyclic, so is D_i . Therefore the condensation C_i of D_i coincides with D_i . Because P_i is the transitive closure of B_i , we conclude that P_i is the transitive closure of C_i . Hence P_i is the modular poset of G , relative to v_i . This completes the proof of Theorem 3. \square

3 Applications

In the last section, it has been shown that the set of modules of a graph G , containing a vertex $v_i \in V(G)$, can be described by the modular poset of G , relative to v_i . In this section, we present applications of this representation.

3.1 Enumerating the modules

The first application is on the enumeration of the modules of a graph G . We present methods for generating and counting the modules of G . The following simple proposition is useful.

Proposition 5 *Let P be a layered poset with layers L_1, \dots, L_t . Then there exists an one-to-one correspondence between non-empty ideals I of P and non-empty subsets $L'_k \subseteq L_k$, for all $1 \leq k \leq t$. Moreover $I = L'_k \cup_{1 \leq \ell < k} L_\ell$.*

Proof: Let I be an ideal of P , and k the largest index of a layer L_k of P , such that $I \cap L_k \neq \emptyset$. Then $L_1, \dots, L_{k-1} \subseteq I$. Let $L'_k = I \setminus \bigcup_{1 \leq \ell < k} L_\ell$. Choose $L'_k \subseteq L_k$ to correspond to I . Conversely, by hypothesis $L'_k \neq \emptyset$ and $L'_k \subseteq L_k$, for some $1 \leq k \leq t$. Then $I = L'_k \cup_{1 \leq \ell < k} L_\ell$ is clearly an ideal, precisely the one corresponding to L'_k . \square

Using Theorem 1, a possible method for enumerating the modules of G is to choose an ordering v_1, \dots, v_n of the vertices of G , and iteratively enumerate the modules of G containing v_i , except those also containing any of the preceding vertices v_1, \dots, v_{i-1} . With

the purpose of applying this idea let P_i be the modular poset of G relative to v_i . Define the *modular poset P'_i relative to v_1, \dots, v_i* , as follows. If $i = 1$ then $P'_i = P_i$, and for $i > 1$ $P'_i = P_i \setminus \bigcup_{1 \leq j < i} N_{P_i}^+[x]$, where $x \in V(P_i)$ is the reduction of $S(v_j)$ in P_i . Clearly, P'_i is also a layered poset. The next proposition describes a correspondence between all modules of G and ideals of P'_i .

Proposition 6 *Let G be a graph, v_1, \dots, v_i a sequence of vertices of it, and P'_i the modular poset of G relative to v_1, \dots, v_i . Then there exists a one-to-one correspondence between the ideals of P'_i and the modules of G , containing v_i and not any of the vertices v_1, \dots, v_{i-1} .*

Proof: Let M be a module of G containing v_i and not containing any v_1, \dots, v_{i-1} . By Theorem 1, P_i has an ideal I corresponding to M . Moreover, $M = \{v_i\} \cup_{x \in I} S(x)$. We show that I is also an ideal of P'_i and can be chosen as corresponding to M . If $i = 1$, this is trivial. Let $i > 1$ and consider the alternatives.

Case 1: $I \not\subseteq V(P'_i)$

Then any vertex $x \in I \setminus V(P'_i)$ is such that $x \in N_{P_i}^+[y]$, where y is the reduction in P_i of $S(v_j) \subseteq V(G)$, for some $1 \leq j < i$. Because $v_j \notin M$, it follows that $y \notin I$. This means that I is not an ideal of P_i , since $x \in I$ has an ancestor $y \notin I$ in P_i . This contradiction leads to the conclusion that this case does not occur.

Case 2: $I \subseteq V(P'_i)$

Then I is also an ideal of P'_i , since the sets of ancestors of the vertices of I are the same in P_i and P'_i .

Conversely, let I be an ideal of P'_i . Then I is also an ideal of P_i . By theorem 1, $M = \{v_i\} \cup_{x \in I} S(x)$ is a module of G containing v_i . Since $I \subseteq V(P'_i)$, I does not contain any vertex which is the reduction in P_i of $S(v_j) \subseteq V(G)$, for some $1 \leq j < i$. Consequently, $v_j \notin M$. That is, M is a module of G containing v_i and not v_1, \dots, v_{i-1} . Therefore, M is the module of G corresponding to the ideal I of P'_i . \square

The algorithm for generating all modules of a graph G follows directly from Proposition 6. Choose an arbitrary ordering v_1, \dots, v_n of the vertices of G . For $1 \leq i \leq n$ construct the modular poset P'_i , relative to v_1, \dots, v_i . Generate all ideals I of P'_i and compute the corresponding modules $\{v_i\} \cup_{x \in I} S(x)$ of G . Generating all the ideals of P'_i is equivalent to generating all the subsets of the layers of P'_i , by Theorem 3 and Proposition 5. The latter step requires constant amortized time. The overall time bound of the algorithm

is $O(n^3 + \mu)$, where μ is the total number of modules of G . The (worst-case) delay complexity is $O(n^2)$.

As for the counting problem, the number of distinct modules of G can be computed using the expression given by the proposition below. The proof of it follows directly from Theorems 1, 3 and Propositions 5 and 6.

Proposition 7 *Let G be a graph, μ the number of modules of G , and v_1, \dots, v_n an ordering of its vertices. Denote by L_{i_1}, \dots, L_{i_t} the layer decomposition of the modular poset of G , relative to v_1, \dots, v_i . Then*

$$\mu = n + \sum_{1 \leq i \leq n} \sum_{1 \leq k \leq t_i} (2^{|L_{i_k}|} - 1)$$

The corresponding counting algorithm requires $O(n^3)$ steps.

3.2 Finding special modules

Consider the problem of finding a module of a graph G satisfying a given property. Below, we describe solutions for two distinct cases. The first is to find a maximal module of G satisfying an hereditary property, while the second corresponds to finding a non-trivial connected module of a graph, if existing.

We use more notation. Let G be graph, $v_i \in V(G)$, P_i the modular poset of G , relative to v_i , and L_1, \dots, L_t the layer decomposition of P_i . Let $S(X) = \bigcup_{x \in X} S(x)$ for $X \subseteq V(P_i)$. Denote $S_0 = \{v_i\}$ and $S_k = S_{k-1} \cup S(L_k)$. Finally, $S_k^i \subseteq S_k$ represents the subset of vertices forming the connected component containing v_i of the subgraph induced in G by $S_k \subseteq V(G)$.

3.2.1 Finding modules satisfying hereditary properties

A *property π on graphs* is a collection of graphs, closed under isomorphism. When a graph G belongs to π , say that $V(G)$ *satisfies* π . When π is closed under induced subgraphs, say that it is an hereditary property.

The following proposition describes the maximal modules of a graph G containing a vertex $v_i \in V(G)$ and satisfying an hereditary property π . Examples of such properties are *planar graphs*, *chordal graphs*, *bipartite graphs*, and so on.

Proposition 8 *Let G be a graph, $v_i \in V(G)$, and L_1, \dots, L_t the layer decomposition of the modular poset of G , relative to v_i . Let π be an hereditary property satisfied by $\{v_i\}$. For $M \subseteq V(G)$, M is a maximal module of G containing v_i and satisfying π if and only*

if M satisfies π and

$$\begin{aligned} &M = V(G), \text{ or} \\ &M = S \cup S(L'_k), \text{ where } L'_k \subseteq L_k, L'_k \neq L_k, \text{ for some} \\ &1 \leq k \leq t, \text{ and} \\ &M \cup \{v\} \text{ does not satisfy } \pi, \text{ for all } v \in S(L_k \setminus L'_k). \end{aligned}$$

The above proposition leads to the following algorithm for finding a maximal module of G , containing $v_i \in V(G)$ and satisfying a given hereditary property π , where $\{v_i\} \in \pi$.

In the *initial step*, given G and v_i , construct D_i and P_i , find the layer decomposition L_1, \dots, L_t of P_i , define $k = 1$, $\ell = 0$ and $M = \{v_i\}$. In the *general step*, for each $v \in L_k$, if $M \cup \{v\}$ satisfies π then include v in M , otherwise set ℓ to 1. After all vertices of L_k have been examined, if $k = t$ or $\ell = 1$, *stop*: M is the desired module. Otherwise, increase k by 1 and repeat the general step.

The complexity of the algorithm is $O(n^2 + nC_\pi)$, where C_π is the complexity of verifying whether G satisfies π .

3.2.2 Finding a connected non-trivial module

The property *connected graphs* is not hereditary. Therefore finding a connected non-trivial module of a graph can not be solved by Proposition 8. We describe below a method for finding such a module. We use an additional concept.

Let G be a connected graph, $A, B \subseteq V(G)$ and $A \cap B = \emptyset$. Say that B *separates* A in G when the vertices of A belong to more than one connected component of $G \setminus B$.

The following proposition describes the non-trivial connected modules of G .

Proposition 9 *Let G be a graph, $|V(G)| \geq 3$, $v_i \in V(G)$ and L_1, \dots, L_t the layer decomposition of the modular poset of G , relative to v_i . There exists a connected non-trivial module of G , containing v_i , if and only if*

- (a) $\{v_i\} \neq S_1^i \neq V(G)$, or
- (b) $S_{k-1} \subseteq S_k^i \neq V(G)$, for some $k > 1$, or
- (c) $S_t^i = V(G)$, $t > 1$, and $\{v\}$ does not separate S_{t-1} in G , for some $v \in S(L_t)$.

Proof: Let M be a non-trivial connected module of G , containing v_i . By Proposition 5, M is of the form $\{v_i\} \cup_{x \in L'_k} S(x)$, for some k and subset $L'_k \subseteq L_k$. That is, $S_{k-1} \subseteq M$. Since M is connected it follows that $M \subseteq S_k^i$. Consequently, $k = 1$ implies

$\{v_i\} \neq S_1^i \neq V(G)$, otherwise M is trivial. That is, (a) is valid. Suppose $1 < k < t$. Then $S_k^i \neq V(G)$ and (b) holds. Finally, let $k = t > 1$. Examine the following two alternatives. If $S_t^i \neq V(G)$ then case (b) occurs again. Consider $S_t^i = V(G)$. Since M is not trivial, there exists $B \subseteq S(L_t)$, such that $B \cap M = \emptyset$ and $B \neq \emptyset$. Because $S_{t-1} \subseteq M$ it follows that B can not separate S_{t-1} in G . Consequently, any subset $B' \subseteq B$ also does not separate S_{t-1} in G . That is, any $v \in B$ is such that $\{v\}$ does not separate S_{t-1} , meaning that condition (c) occurs. Hence M being a non-trivial connected module of G implies that at least one of the conditions (a), (b) or (c) occurs. The proof of the converse consists of exhibiting a module M having the desired properties, whenever one of the conditions (a), (b) or (c) holds.

If (a) is verified then S_1^i is a connected non-trivial module of G . So is S_k^i , when (b) occurs. Finally, suppose that (c) is true. In this case, the vertices of the connected component containing v_i , of the subgraph induced by $V(G) \setminus \{v\}$ form a non-trivial connected module of G . \square

The above proof leads directly to an algorithm for finding a connected non-trivial module of G , or reporting that one does not exist. In the worst situation, it might be necessary to check conditions (b) and (c) $O(n)$ times. Therefore the complexity of the algorithm is $O(nm)$.

4 Conclusions

We have presented a characterization of the modules of an undirected graph G , in terms of ideals of certain posets. As applications of it, we have described algorithms for (i) generating all the μ modules of G , (ii) counting the μ modules of G , (iii) finding a maximal module of G containing $v_i \in V(G)$ and satisfying an hereditary property π , and (iv) finding a non-trivial connected module of G . The complexities of the algorithms are (i) $O(n^3 + \mu)$, (ii) $O(n^3)$, (iii) $O(n^2 + nC_\pi)$, where C_π is the complexity of verifying whether G satisfies π , and (iv) $O(nm)$.

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