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The Gudermannian Growth Model: Theory, Application and Statistical Analysis

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HIGHLIGHTS

- This work introduces the Gudermannian function as a sigmoid growth model.
- Hyperbolic and trigonometric identities provide several parameterizations of this model.
- The Gudermannian growth model can be a good alternative to the classical sigmoid models.

Abstract: Processes producing sigmoid curves are common in many areas such as biology, agrarian sciences, demography and engineering. Several mathematical functions have been proposed for modeling sigmoid curves. Some models such as the logistic, Gompertz, Richards and Weibull are widely used. This work introduces the Gudermannian function as an option for modeling sigmoid growth curves. The original function was transformed and the resulting equation was called the “Gudermannian growth model.” This model was applied to four sets of experimental growth data to illustrate its practical application. The results were compared with those obtained by the logistic and Gompertz models. Since all these models are nonlinear in the parameters, the statistical properties of the least squares estimators were evaluated using measures of nonlinearity. For each experimental data set, the Akaike’s corrected information criterion was utilized to discriminate among the models. In general, the Gudermannian model fitted better to the experimental data than the logistic and Gompertz models. The results showed that the Gudermannian model can be a good alternative to the classical sigmoid models.

Keywords: Gudermannian function; sigmoid growth models; logistic model; Gompertz model; measures of nonlinearity.

INTRODUCTION

Gudermannian function

The Gudermannian function is defined for all x by:

$$y = \mathrm{gd}(x) = \int_0^x \frac{dx}{\cosh x} = 2\tan^{-1} e^x - \frac{\pi}{2} \tag{1}$$

Johann Heinrich Lambert introduced this function in the 1760s and called it the transcendent angle. In 1862, the British mathematician Arthur Cayley suggested calling it the Gudermannian function as a tribute to the German mathematician Christoph Gudermann [1, 2].

Equation (1) has alternative definitions, such as:

$$y = \mathrm{gd}(x) = \arcsin(\tanh x) \tag{2}$$

$$y = \mathrm{gd}(x) = 2\mathrm{atan}\left(\tanh\left(\frac{x}{2}\right)\right) \tag{3}$$

Several other expressions can be obtained from the properties of trigonometric and hyperbolic functions [3].

The Gudermannian function is a sigmoid function. Although sigmoid functions have a wide range of applications, including population dynamics, artificial neural networks, control systems, and probability theory [4, 5, 6], the Gudermannian function is used primarily to describe the Mercator projection. The Mercator projection is the basis for many maps covering both individual countries and the whole world, except the polar regions [7]. Many major online street mapping services (e.g., Google Maps) use a variant of the Mercator projection for their map images [8]. Other applications of the Gudermannian function include the angle of the parallelism function in hyperbolic geometry [9], a non-periodic solution of the inverted pendulum [10], and a moving mirror solution of the dynamical Casimir effect [11].

This study used the Gudermannian function in modeling sigmoid growth curves. At present, there is no work available in the literature in which this function has been used for this purpose, giving this study an unprecedented character. Before being used as a growth model, the Gudermannian function had to be transformed and the resulting equation came to be called the "Gudermannian growth model." This model was fitted to four sets of experimental data and the results were compared with two classical growth models, namely the Gompertz and logistic models. As all of these models are nonlinear in the parameters, measures of nonlinearity were employed to validate the statistical properties of the least squares estimators. The measures used were the bias measure of Box [12], the intrinsic (IN) and the parameter-effects (PE) curvatures of Bates and Watts [13], and the Hougaard's measure of skewness [14]. For each experimental data set, the Akaike's corrected information criterion was utilized to discriminate among the models.

The Gudermannian Growth Model

The Gudermannian function is a sigmoid function, having the limits $\pm\pi/2$ as $x \to \pm\infty$ and a rotational symmetry about the inflection point (0,0). As seen in Figure 1, the function in its original form is not suitable for fitting sigmoid growth curves. Modifications are necessary to allow different values of the lower and upper asymptotes, in addition to changes in the inflection point coordinates. An appropriate way to represent growth curves can be obtained through transformations (translations and dilations) of the parent function $y = \mathrm{gd}(x)$:

$$y = \frac{2}{\pi} a.\mathrm{gd}(b.x - c) + d \tag{4}$$

Parameters a and b are responsible for vertical and horizontal dilations, and parameters c and d are responsible for horizontal and vertical translations, respectively. The factor $2/\pi$ was chosen by the author so that the lower and upper asymptotes are equal to $(d - a)$ and $(d + a)$ instead of $\left(d - \frac{\pi}{2}a\right)$ and $\left(d + \frac{\pi}{2}a\right)$, respectively. Equation 4 is the generic form of a four-parameter growth model based on the Gudermannian function, here called the "Gudermannian growth model."

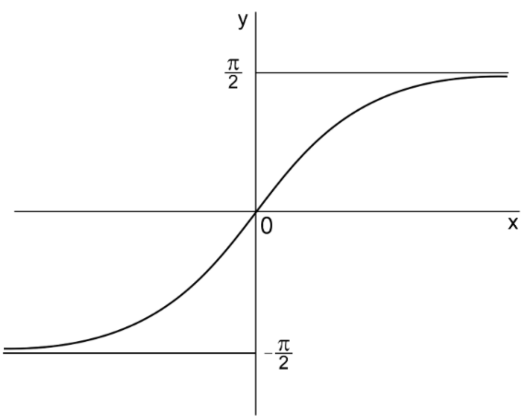

Figure 1. Graph of the Gudermannian function.

Interpreting the model: Graphical analysis

Parameters *a*, *b*, *c* and *d* of Equation 4 have no physical meaning when analyzed in isolation. However, they are useful in graphical interpretation of the model. Figure 2 shows the graph of the generic Gudermannian growth model as described in Equation 4. The upper $(\mathrm{d}+\mathrm{a})$ and lower $(\mathrm{d}-\mathrm{a})$ asymptotes are the limits of the function y as x approaches plus or minus infinity. The physical meaning of these asymptotes is obvious. For example, if the curve in Figure 2 represents a population as a function of time, (d-a) is the initial value of that population, and (d+a) is the maximum population size or the carrying capacity.

The abscissa c/b of the inflection point (IP) is determined by equating the second-order derivative of y to zero. The second-order derivative of y represents the growth acceleration, which has positive values before and negative values after the IP. In its positive phase, the growth acceleration increases to a point of maximum, called point of maximum acceleration (PMA), decreasing until presenting in its negative phase, a point of minimum, called point of maximum deceleration (PMD) [15]. The abscissas of PMA and PMD are calculated by equating the third-order derivative of y to zero and correspond respectively to $\frac{c-\operatorname{atanh}\left(\frac{\sqrt{2}}{2}\right)}{b}$ and $\frac{c+\operatorname{atanh}\left(\frac{\sqrt{2}}{2}\right)}{b}$, or simply $\frac{c-0.88}{b}$ and $\frac{c+0.88}{b}$.

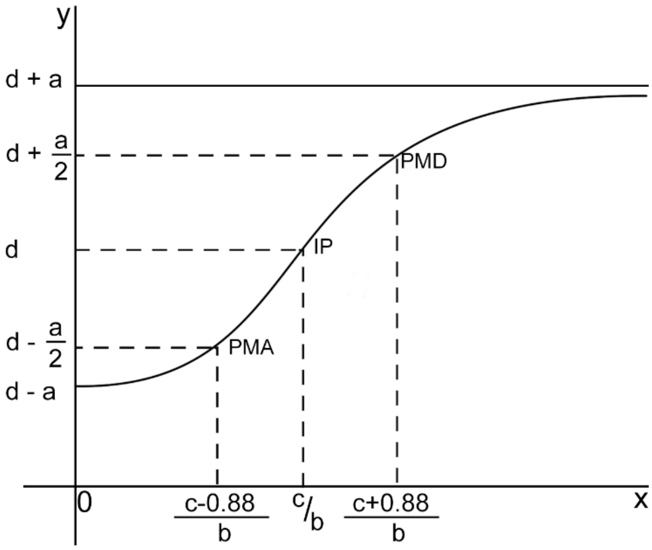

Figure 2. Graph of the Gudermannian growth model. PMA: point of maximum acceleration; IP: inflection point; PMD: point of maximum deceleration.

Measures of nonlinearity

The Gudermannian growth model is a sigmoid model and sigmoid models are nonlinear in the parameters. A regression model is called nonlinear in the parameters if the derivatives of the model with respect to the model parameters depend on one or more parameters. Since most asymptotic inferences for nonlinear regression models are based on an analogy with linear models and as these inferences are approximate, some procedures known as measures of nonlinearity have been proposed to assess whether these approximations are valid [16]. Thus, measures of nonlinearity are expressions used to evaluate the adequacy of the linear approximation of a nonlinear model and its effects on the inferences. The closer the linear behavior of a nonlinear model is, the more accurate the asymptotic results and consequently, the more reliable the inferences are. There are some nonlinear regression models whose estimators come close to being unbiased, normally distributed, and minimum variance estimators. Such models have been termed close-to-linear models. Models not possessing these properties may be termed far-from-linear [17].

The most used measures of nonlinearity are the curvature measures of Bates and Watts, the bias measure of Box, and Hougaard's measure of skewness. These measures were used in this work and are briefly summarized as follows. For further details, see the original works.

Curvature measures

Bates and Watts divide the concept of nonlinearity into two parts: intrinsic nonlinearity (IN) and parameter-effects nonlinearity (PE). The intrinsic curvature measures the curvature of the expectation surface at the parameter estimates, and the parameter-effects curvature measures the lack of parallelism of parallel parameter lines that are projected onto the expectation surface. As a result, the PE curvature can be reduced by reparameterization of the model, whereas intrinsic curvature is an inherent property of the model that cannot be affected by reparameterization [13, 18].

The statistical significance of IN and PE are evaluated by comparing these values with $1/\sqrt{\mathrm{F}}$, where $\mathrm{F} = \mathrm{F}(\alpha, n-p, p)$ is the inverse of the Fisher´s probability distribution obtained at the significance level *α*, *p* is the number of parameters, and *n* is the number of observations. The value $1/\sqrt{\mathrm{F}}$ may be regarded as the radius of the curvature of the $100.(1-\alpha)\%$ confidence region. If IN and PE are less than $1/\sqrt{\mathrm{F}}$, the model is considered close-to-linear. When IN $<1/\sqrt{\mathrm{F}}$ but PE $>1/\sqrt{\mathrm{F}}$, the model is considered far-from-linear. However, the PE curvature can be reduced by the reparameterization of the model, making it a close-to-linear model. If the intrinsic curvature IN exceeds the critical value $1/\sqrt{\mathrm{F}}$, it indicates an inherently far-from-linear model that cannot be made a close-to-linear model by reparameterization [18].

Bias and skewness

The global nonlinearity measures of Bates and Watts do not differentiate the parameters based on their contributions to the overall curvature. Manifestations of a nonlinear behavior include significant bias and skewness. Hence, it is essential to estimate at least these basic statistical properties of the parameter estimates to identify the reparameterization parameters of interest [18]. The bias and skewness of the parameter estimates of a nonlinear regression model can be estimated using the bias measure of Box and Hougaard's measure of skewness.

The Box's bias represents the discrepancy between the estimates of the parameters and the true values. According to Ratkowsky [19], the parameter estimate behavior is considered to be significantly nonlinear if the percentage bias is greater than 1% in the absolute value. In this case, a reparameterization of the model is necessary.

Skewness is a measure of lack of symmetry. Hougaard's measure of skewness can be employed to assess whether a parameter is close to linear or whether it contains considerable nonlinearity because of the close link between the extent of nonlinear behavior of an estimator and the extent of nonnormality in the sampling distribution of this estimator. According to Ratkowsky [19], the estimator of the parameter is very close to linear if $|g1i| < 0.1$, and the estimator is reasonably close to linear if $0.1 < |g1i| < 0.25$. For $|g1i| > 0.25$, the skewness is very apparent and $|g1i| > 1$ indicates a considerable nonlinear behavior.

MATERIAL AND METHODS

Models

Using Equations 1–3 as parent functions and applying the same transformations that gave rise to Equation 4, three parameterizations of the Gudermannian growth model are obtained:

$$\mathrm{y} = \frac{2}{\pi}\mathrm{a}.\left(2\mathrm{atan}\left(\exp(\mathrm{b.x}-\mathrm{c})\right)-\frac{\pi}{2}\right)+\mathrm{d} \tag{5}$$

$$\mathrm{y} = \frac{2}{\pi}\mathrm{a}.\left(\arcsin(\tanh(\mathrm{b.x}-\mathrm{c}))\right)+\mathrm{d} \tag{6}$$

$$\mathrm{y} = \frac{4}{\pi}\mathrm{a}.\left(\mathrm{atan}\left(\tanh\left(\frac{\mathrm{b.x}-\mathrm{c}}{2}\right)\right)\right)+\mathrm{d} \tag{7}$$

These parameterizations were fitted to experimental growth data and the fits were compared with those of the logistic and Gompertz models. For a fair comparison, three parameterizations of the logistic (Equations 8-10) and Gompertz models (Equations 11-13) were used.

$$\mathrm{y} = \delta + \frac{\alpha}{1+\exp(\beta-\gamma\mathrm{X})} \tag{8}$$

$$y = \delta + \frac{\alpha}{1 + \exp(-\gamma(X - \beta))} \tag{9}$$

$$y = \delta + \frac{\alpha}{1 + \exp(\beta(1 - \gamma X))} \tag{10}$$

$$y = \delta + \alpha \exp(-\exp(\beta - \gamma X)) \tag{11}$$

$$y = \delta + \alpha \exp(-\exp(\exp(\beta) - \gamma X) \tag{12}$$

$$y = \delta + \alpha \exp(-\exp(\beta - \exp(\gamma)X)) \tag{13}$$

These classical sigmoid models were not selected at random. The logistic model was chosen because it has a rotational symmetry about the IP, as well as the Gudermannian model. The Gompertz model was included as an alternative to the logistic model because it has an asymmetric IP and a shorter period of fast growth.

Experimental data

The models were fitted to four sets of experimental growth data collected from the literature: data set 1 - water content versus distance from the growth tip in bean root cells [14]; data set 2 - average height of *Casuarina equisetifolia versus* time [20]; data set 3 - cell mass concentration of *Lactobacillus helveticus versus* time [21]; and data set 4 - cumulative number of polio cases diagnosed on a monthly basis in the United States in 1949 [22]. These data were chosen as they are typical examples of sigmoid growth curves in different areas.

Nonlinear regression and measures of nonlinearity

The parameter estimates of the models and their respective measures of nonlinearity were obtained using the NLIN procedure of the SAS software. Details about the development, procedure, and equations for determining the bias measure of Box [12], the curvature measures of nonlinearity of Bates and Watts [13], and Hougaard's measure of skewness [14] are found in the original works.

Akaike's corrected Information Criterion

For each experimental data set, the Akaike's corrected information criterion (AICc) was utilized to compare the curve-fitting effectiveness of the different models.

$$\text{AICc} = \text{N}.\ln\left(\frac{\text{SS}}{\text{N}}\right) + 2\text{K}\left(\frac{\text{K}+1}{\text{N}-\text{K}-1}\right) \tag{14}$$

Where N is the sample size, SS is the sum of the square of the vertical distances of the points from the curve, and K is the number of model parameters plus one. Lower AICc values indicate a better-fitted model. Details about this method can be found in Akaike [23], Burnham and Anderson [24], and others.

RESULTS AND DISCUSSION

The Gudermannian, logistic, and Gompertz models (Equations 5–13) were fitted to the four experimental data sets. Tables 1, 2, 3, and 4 show the least-squares parameter estimates with the respective values for the measures of nonlinearity for each model.

The models (Equations 5–13) presented IN < $1/\sqrt{F}$ for all data sets. Therefore, for each fit, the solution locus can be considered sufficiently linear within a confidence interval of approximately 95%. On the other hand, some models exhibited high PE curvature (PE < $1/\sqrt{F}$) to certain data sets. High PE curvature indicates that at least one parameter in the model deviates from the linear behavior, and Hougaard's skewness and Box's bias indicate which parameter or parameters are responsible. Parameter estimates that present a percentage bias greater than 1% in absolute value are considered to be significantly nonlinear. Similarly, a value of the standardized Hougaard's skewness measure greater than 0.25 in absolute value indicates a nonlinear behavior [14]. The PE curvature can be reduced by a suitable reparameterization of the model. In this way, a far-from-linear model can become a close-to-linear and thus make statistical inferences valid. However, the aim of this work is to test the models as they are presented and not to search for suitable reparameterizations.

As can be seen in Tables 1 and 2, the Gudermannian model described by Equation 5 presented high PE curvature for data sets 1 and 2. For data set 1, this was due to the estimates of a, b, and c, whose high skewness values suggest a nonlinear behavior. As for data set 2, the nonlinear behavior can be attributed to the estimate of a, which has high skewness. For data sets 3 and 4 (Tables 3 and 4, respectively), all parameter estimates of this parameterization are unbiased and not skewed, giving this model a close-to-linear behavior and ensuring the statistical validity of the parameters estimated by the method of least squares. In relation to the other parameterizations of the Gudermanian model, Equation 6 presented a close-to-linear behavior for all data sets (Tables 1–4) and Equation 7 presented a nonlinear behavior only for data set 2 (Table 2), probably due to the high skewness of the estimate of b.

Table 1. Statistical results of the least-squares estimation using data set 1: water content *versus* distance from the growth tip in bean root cells.

Model	IN	PE	Parameter	Estimate	Skewness	% Bias
Gudermannian Equation (5)	0.13	0.89	a	10.34	0.33	0.28
			b	0.56	0.29	0.82
			c	3.65	0.27	0.82
			d	11.06	-0.09	-0.06
Gudermannian Equation (6)	0.13	0.46	a	10.34	-0.23	-0.28
			b	0.56	0.11	-0.10
			c	3.65	0.13	-0.05
			d	11.06	0.09	0.06
Gudermannian Equation (7)	0.13	0.40	a	10.34	0.02	-0.02
			b	0.56	0.03	0.67
			c	3.65	0.03	0.70
			d	11.06	-0.09	0.00
Logistic Equation (8)	0.17	0.79	α	20.40	0.29	0.26
			β	4.54	0.25	0.72
			γ	0.70	0.28	0.72
			δ	0.88	-0.28	-3.89
Logistic Equation (9)	0.17	0.77	α	20.40	0.29	0.26
			β	6.50	-0.07	-0.05
			γ	0.70	0.28	0.72
			δ	0.88	-0.28	-3.89
Logistic Equation (10)	0.17	0.83	α	20.40	0.29	0.26
			β	4.53	0.25	0.72
			γ	0.15	0.21	0.10
			δ	0.88	-0.28	-3.89
Gompertz Equation (11)	0.29	0.81	α	20.21	0.24	0.27
			β	2.78	0.35	0.96
			γ	0.48	0.33	0.91
			δ	1.65	-0.12	-1.42
Gompertz Equation (12)	0.29	1.63	α	20.20	0.24	0.27
			β	1.02	0.00	0.29
			γ	0.48	0.33	0.91
			δ	1.65	-0.12	-1.42
Gompertz Equation (13)	0.29	1.34	α	20.20	0.24	0.27
			β	2.78	0.35	0.96
			γ	-0.73	0.00	-0.42
			δ	1.65	-0.12	-1.42

$1/\sqrt{F}$ = 0.55

The logistic model (Equations 8, 9, and 10) presented a close-to-linear behavior for data sets 3 and 4 (Tables 3 and 4) and a far-from-linear behavior for data sets 1 and 2. For data set 1, the significant parameter-effect curvature in Equations 8 and 9 can be attributed to the estimates of α, γ, and δ. The estimates of α and γ are skewed, while the estimate δ is skewed and biased. The high PE curvature of Equation 10 is probably due to the high skewness values of α and δ.

For data set 2, the high skewness values of α and δ estimates (Equations 8, 9, and 10) are responsible for the nonlinear behavior.

Table 2. Statistical results of the least-squares estimation using data set 2: average height of *Casuarina equisetifolia versus* time.

Model	IN	PE	Parameter	Estimate	Skewness	% Bias
Gudermannian Equation (5)	0.07	1.37	a	1.84	0.46	0.36
			b	0.42	0.07	0.22
			c	2.20	0.05	0.21
			d	2.38	-0.23	-0.07
Gudermannian Equation (6)	0.07	0.35	a	1.84	-0.06	-0.36
			b	0.42	0.06	0.50
			c	2.20	0.99	0.61
			d	2.38	0.23	0.07
Gudermannian Equation (7)	0.07	0.88	a	1.84	0.07	0.04
			b	0.42	0.28	0.44
			c	2.20	-0.06	0.47
			d	2.38	0.23	-0.02
Logistic Equation (8)	0.08	1.28	α	3.53	0.48	0.38
			β	2.89	0.01	0.14
			γ	0.55	0.05	0.17
			δ	0.62	-0.48	-1.41
Logistic Equation (9)	0.08	1.22	α	3.53	0.48	0.38
			β	5.25	-0.20	-0.10
			γ	0.55	0.05	0.17
			δ	0.62	-0.48	-1.41
Logistic Equation (10)	0.08	1.40	α	3.53	0.48	0.38
			β	2.89	0.01	0.14
			γ	0.19	0.34	0.15
			δ	0.62	-0.48	-1.41
Gompertz Equation (11)	0.07	0.70	α	3.36	0.30	0.15
			β	1.91	0.01	0.06
			γ	0.40	0.03	0.07
			δ	0.93	-0.26	-0.26
Gompertz Equation (12)	0.07	1.09	α	3.36	0.30	0.15
			β	0.65	-0.15	-0.13
			γ	0.40	0.03	0.07
			δ	0.93	-0.26	-0.26
Gompertz Equation (13)	0.07	0.82	α	3.36	0.30	0.15
			β	1.91	0.01	0.06
			γ	-0.91	-0.12	0.06
			δ	0.93	-0.26	-0.26

$1/\sqrt{F} = 0.49$

The three parameterizations of the Gompertz model (Equations 11, 12 and 13) presented a far-from-linear behavior for data sets 1 and 2 (Tables 1 and 2). For data set 1, the nonlinear behavior can be attributed to the estimates of β, γ, and δ. The estimates of β are skewed in Equations 11 and 13. The estimate of γ is skewed in Equation 12. In the three parameterizations (Equations 11, 12 and 13), the estimate of δ is slightly biased. For data set 2, the high skewness values of α and δ estimates (Equations 11, 12, and 13) are responsible for the nonlinear behavior. For data set 3, unlike Equations 12 and 13, Equation 11 presented close-to-linear behavior. The estimates of γ in Equation 12 and β in Equation 13 are skewed and are probably responsible for the nonlinear behavior of these two parameterizations. All parameterizations of this model (Equations 11, 12, and 13) presented close-to-linear behavior for data set 4.

Table 3. Statistical results of the least-squares estimation using data set 3: cell mass concentration of *Lactobacillus helveticus versus* time.

Model	IN	PE	Parameter	Estimate	Skewness	% Bias
Gudermannian Equation (5)	0.01	0.03	a	3.09	0.10	0.04
			b	0.31	0.17	0.22
			c	5.27	0.17	0.22
			d	3.16	0.00	0.04
Gudermannian Equation (6)	0.01	0.07	a	3.09	-0.10	-0.04
			b	0.31	-0.03	-0.09
			c	5.27	-0.03	-0.10
			d	3.16	0.00	0.00
Gudermannian Equation (7)	0.01	0.03	a	3.09	0.00	0.00
			b	0.31	0.12	0.13
			c	5.27	0.12	0.13
			d	3.16	0.00	0.00
Logistic Equation (8)	0.07	0.25	α	6.12	0.07	0.03
			β	6.53	0.15	0.15
			γ	0.39	0.15	0.15
			δ	0.10	-0.05	-0.82
Logistic Equation (9)	0.07	0.25	α	6.12	0.08	0.03
			β	16.86	0.00	0.00
			γ	0.39	0.15	0.15
			δ	0.10	-0.05	-0.82
Logistic Equation (10)	0.07	0.25	α	6.12	0.07	0.01
			β	6.53	0.15	0.15
			γ	0.06	0.04	0.00
			δ	0.10	-0.05	-0.82
Gompertz Equation (11)	0.22	0.06	α	-6.13	-0.04	0.01
			β	-4.81	-0.09	0.06
			γ	-0.26	-0.10	0.07
			δ	6.09	0.01	0.00
Gompertz Equation (12)	0.22	1.52	α	6.17	0.16	0.12
			β	1.38	0.04	0.16
			γ	0.26	0.27	0.57
			δ	0.21	-0.03	-0.91
Gompertz Equation (13)	0.22	1.31	α	6.17	0.16	0.12
			β	3.98	0.29	0.60
			γ	-1.35	0.02	-0.15
			δ	0.21	-0.03	-0.91

$1/\sqrt{F} = 0.56$

Table 4. Statistical results of the least-squares estimation using data set 4: cumulative number of polio cases diagnosed on a monthly basis in the United States in 1949.

Model	IN	PE	Parameter	Estimate	Skewness	% Bias
Gudermannian Equation (5)	0.05	0.23	a	20792.4	0.10	0.05
			b	1.07	0.23	0.34
			c	7.50	0.24	0.34
			d	21335.4	0.05	0.02
Gudermannian Equation (6)	0.05	0.14	a	20792.4	-0.10	-0.05
			b	1.07	-0.08	-0.18
			c	7.50	-0.09	-0.19
			d	21335.4	-0.05	-0.02
Gudermannian Equation (7)	0.05	0.07	a	20792.4	0.00	-0.01
			b	1.07	0.14	0.18
			c	7.50	0.14	0.18
			d	21335.4	0.00	0.00
Logistic Equation (8)	0.10	0.32	α	-41291.1	-0.09	0.05
			β	-9.20	-0.22	0.30
			γ	-1.31	-0.22	0.30
			δ	41965.9	0.09	0.04
Logistic Equation (9)	0.10	0.22	α	41291.1	0.09	0.05
			β	7.01	0.03	0.01
			γ	1.31	0.22	0.30
			δ	674.8	-0.03	-0.75
Logistic Equation (10)	0.10	0.21	α	41291.1	0.09	0.05
			β	9.20	0.22	0.30
			γ	0.14	0.02	0.00
			δ	674.8	-0.03	-0.75
Gompertz Equation (11)	0.03	0.05	α	42202.1	0.11	0.04
			β	5.73	0.16	0.18
			γ	0.87	0.15	0.17
			δ	1159.9	0.00	-0.20
Gompertz Equation (12)	0.03	0.43	α	42201.1	0.11	0.04
			β	1.75	0.01	0.04
			γ	0.87	0.15	0.17
			δ	1159.9	0.00	-0.20
Gompertz Equation (13)	0.03	0.48	α	42201.1	0.11	0.04
			β	5.73	0.15	0.18
			γ	-0.14	0.00	-0.39
			δ	1159.9	0.00	-0.20

$1/\sqrt{F}$ = 0.51

The Akaike's corrected information criterion (AICc) was used to identify the model that best fitted each of the experimental data sets. According to this criterion, for a given dataset, the model with the smallest AICc is the “best” model, the model with the second smallest AICc is the “second best” model, and so on. It is important to highlight that for each data set, the AICc scores were calculated only for models that presented a close-to-linear behavior, as it would not make sense to include models whose statistical inferences may be invalid. Table 5 shows the results.

Table 5. AICc values of Gudermannian, logistic, and Gompertz models†.

	AICc			
Model	**Data set 1**	**Data set 2**	**Data set 3**	**Data set 4**
Gudermannian (Equation 5)	-	-	-63.76	173.45
Gudermannian (Equation 6)	0.37	-50.07	-63.76	173.45
Gudermannian (Equation 7)	0.37	-	-63.76	173.45
Logistic (Equation 8)	-	-	-67.70	173.55
Logistic (Equation 9)	-	-	-67.70	173.55
Logistic (Equation 10)	-	-	-67.70	173.55
Gompertz (Equation 11)	-	-	-81.38	189.97
Gompertz (Equation 12)	-	-	-	189.97
Gompertz (Equation 13)	-	-	-	189.97

† Calculated only for data sets in which the model exhibited close-to-linear behavior.

For data sets 1 and 2, only the Gudermannian model presented linear behavior. Therefore, in these two cases, this model is considered the "best" model, regardless of the value of AICc. For data set 3, the Gudermannian (Equations 5-7), the logistic (Equations 8-10), and the Gompertz (Equation 11) models showed close-to-linear behavior. According to the AICc values, the Gudermannian model is considered the "best" model, the logistic model is the "second best", and the Gompertz model is the "third best". For data set 4, all models (Equations 5-13) showed linear behavior. Again, the Gudermannian model showed a better fit to the data.

Figure 3 shows the fits of the models of Table 5 for each data set. The graphs demonstrate a good agreement between the observed values and those predicted by the models.

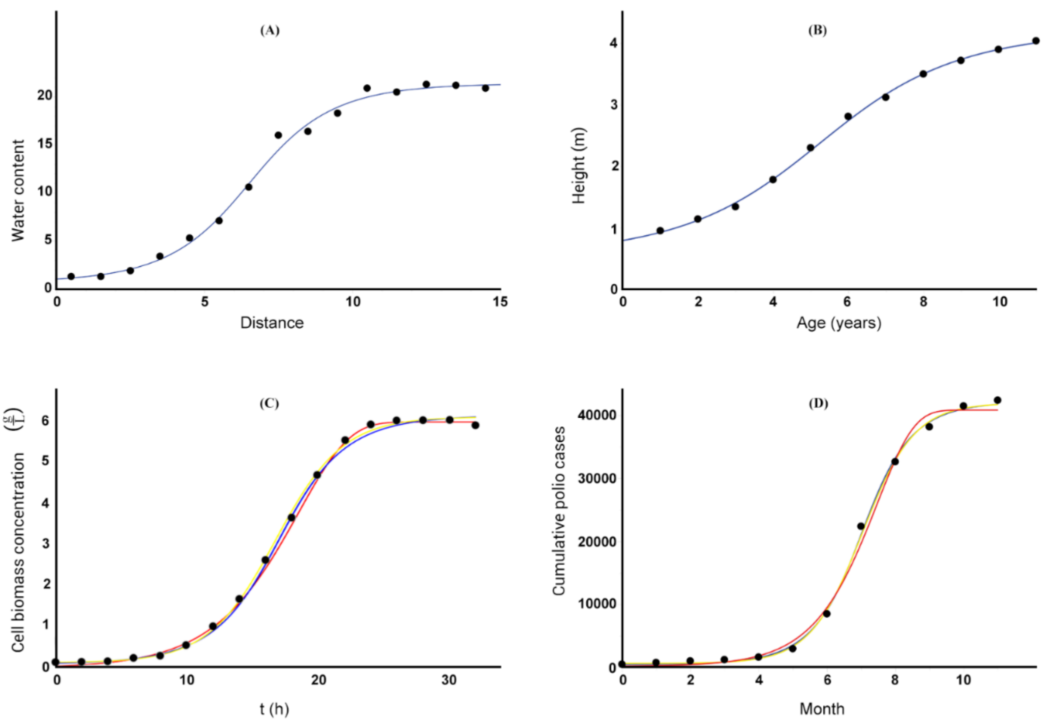

Figure 3. Growth curves fitted to each data set. **(A)**: Water content versus distance from the growth tip in bean root cells. Model: Gudermannian; **(B):** Average height of *Casuarina equisetifolia versus* time. Model: Gudermannian; **(C):** Cell mass concentration of *Lactobacillus helveticus versus* time. Models: Gudermannian (blue line), logistic (yellow line), and Gompertz (red line); **(D):** Cumulative number of polio cases diagnosed on a monthly basis in the United States in 1949. Models: Gudermannian (blue line), logistic (yellow line), and Gompertz (red line).

CONCLUSION

The Gudermannian growth model proved to be a good option for modeling sigmoid growth curves. This model fitted better than the logistic and Gompertz models to the experimental data utilized in this study. One of the parameterizations of this model presented a close-to-linear behavior for all experimental data sets.

Conflicts of Interest: The author declares no conflict of interest.

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