Center of mass of a system of variable mass particles

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We study the dynamics of the center of mass (CM) of a set of particles when the number of particles in the
system varies during a short interval of time, or when the value of the mass of one of the particles depends on
time. We show that in both situations, while the population or mass of the system varies, the linear momentum of
the CM differs from the total linear momentum of the system. During this time interval, the dynamics of the
CM is driven by the external forces that act on the system plus transient forces that depend on the position and
velocity of the CM. This dynamics is applied to study the time evolution of the vector position of the CM of a
system of free particles moving on an air rail in which the mass of one the particles varies in a short time interval.
We want to show it to students that with the knowledge and tools they have at an elementary Mechanics course
they can formulate and answer questions which are not in the textbooks.
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one-dimensional motion.

1. Introduction

The concept of the center of mass (CM) of a system with fixed number $N \geq 2$ of particles with constant masses is commonly approached in introductory courses of Mechanics [1–6]. To the CM of the system one associates a fictitious particle, with mass chosen to be equal to the sum of the masses of the particles in the set. The dynamics of the CM is driven by the external forces that act on the N particles. The kinematics of the fictitious particle located at the CM of the system, for instance, its position, velocity and acceleration vectors, are obtained from the respective physical quantities of the particle that belong to the system.

Although the position of the CM of this type of set of particles may not coincide with the position of any of the physical particles of the set, the CM of the system has interesting properties. For instance, the total weight forces that act on the particles of a rigid body has the same effect on the motion of the body as it would if the resultant of those forces were applied on the CM. An important physical quantity associated to the CM is its linear momentum, that is equal to the total linear momentum of the system of particles. The equality of these two vectors are very important in the study of particle collision, and their conservation is helpful for the study of the elementary Particle Physics inside atoms, nuclei and nucleons.

Understanding the dynamics of systems with variable mass is of great relevance, for instance, to the study of ship and rocket motions [7–10]. In the present paper we characterize the CM of a particle system in two different scenarios: a) when the number of particles in the system varies by one unit, and this transition happens in a very short time, but all particle masses are constant; b) when the number of particles in the system is fixed, but the mass of one of them varies in time. We study the dynamics of the CM of a system of particles in both scenarios and determine the form of Newton's Second Law for the dynamics of the CM during the whole movement, including the transition time interval of each scenario. Our aim is that of relating, in each of those systems, the linear momentum of the CM to the total linear momentum of system at any time t. We point out that outside the transition time interval, we have the same conditions in the system as is usually described in the literature and the known results are recovered from ours. It is certainly interesting to ask questions that go beyond what is traditionally presented in textbooks which may possibly lead to new knowledge, regardless of how experimentally feasible those possibilities might currently be.

In section 2 we describe the kinematics of a set of N fixed particles that is split in two subsystems: S_1 and S_2. The masses of all particles are constant. A given particle, the L-th particle, migrates from S_1 to S_2 during a short transition time Δt. In subsection 2A we relate the linear momentum of the CM of subsystems S_1, and

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S_2 to the total linear momentum of their respective particles, for $t \in (-\infty, \infty)$. In subsection 2B we derive the dynamics of the CM of S_1, including the transition time interval when the L-th particle moves from S_1 to S_2. In section 3 we deal with a single system with N particles in which one of those particles has time-dependent mass. The treatment is analogous to that of section 2, but the resulting equation of motion for the CM differs slightly. In section 4 we show the behavior of a simple example of a unidimensional system consisting of three free particles sliding on an air rail, in which one of them has variable mass. In section 5 we present our conclusions. In Appendix A we calculate at any time t the position of the CM of a system with N particles, $N \geq 2$, in any scenario, when the number of particles can vary with time and the value of their masses can depend on time. In Appendix B we present a particular function that has the same properties as the transition function $f(t)$.

2. The case of two subsystems with varying number of particles

2.1. Kinematics

Consider a classical system S of N particles with constant masses $\{m_1, m_2, \ldots, m_N\}$. Its *center of mass* (CM) has a position vector given by

$$\vec{R}_{CM}(t) = \frac{m_1\,\vec{r}_1(t) + m_2\,\vec{r}_2(t) + \cdots + m_N\,\vec{r}_N(t)}{m_1 + m_2 + \cdots + m_N}, \quad (1)$$

in which m_i and $\vec{r}_i(t)$ are the mass and the position of the i-th particle at time t, respectively, for $i \in \{1, 2, \ldots, N\}$. A fictitious particle is assumed to coincide with the position of the CM of the system of the physical particles. The mass of this fictitious particle is chosen to be equal to the value of the total mass of the N particles. We assume an inertial frame of reference, in which all measurements are made. In a [possibly non-inertial] frame of reference solidary to the CM, the system total momentum is zero [1]. The kinematics of the center of mass (CM) stems from the kinematics of the particles that compose the system.

2.1.1. Subsystems, permanent particles and the migrating particle

Let S be decomposed into two subsystems S_1 and S_2 so that that the first $(L-1)$ particles, in which L is an integer and $1 < L < N$, *always* belong to S_1 — let us call them the *permanent* particles of S_1. In the same token, the last $(N-L)$ particles are the permanent particles of the subsystem S_2. For the remaining L-th particle — let it be called the *migrating* particle — suppose that there is some *transition* time interval $T = [t_1, t_2]$ so that such particle belong to S_1 for all $t < t_1$ and belong to S_2 for all $t > t_2$. During that interval, which can be written as

$$T = \left[\bar{t} - \frac{\Delta t}{2}, \quad \bar{t} + \frac{\Delta t}{2} \right] \quad (2)$$

in terms of its amplitude

$$\Delta t = (t_2 - t_1) > 0 \quad (3)$$

and its mid-point

$$\bar{t} = \frac{t_1 + t_2}{2}, \quad (4)$$

the migrating L-th particle crosses the boundary from S_1 to S_2 (say, due to the particle and/or the boundary motions).

In order to have non-singular kinematic functions defined and the total number of physical particles in the set S equal to N at any time t, consider some smooth real *transition function* $f(t)$ that has the following properties:

$$f(t) = \begin{cases} 0, & \text{for } t < -\dfrac{\Delta t}{2} \\ \dfrac{1}{2} + h(t) \in [0,1], & \text{for } t \in \left[-\dfrac{\Delta t}{2}, \dfrac{\Delta t}{2} \right], \\ 1, & \text{for } t > \dfrac{\Delta t}{2}, \end{cases} \quad (5)$$

so that $h(t)$ is an odd smooth function, $h(-t) = -h(t)$. Consequently,

$$f(-t) = \begin{cases} 1, & \text{for } t < -\dfrac{\Delta t}{2} \\ \dfrac{1}{2} - h(t) \in [0,1], & \text{for } t \in \left[-\dfrac{\Delta t}{2}, \dfrac{\Delta t}{2} \right], \\ 0, & \text{for } t > \dfrac{\Delta t}{2}, \end{cases} \quad (6)$$

and

$$f(t) + f(-t) = 1 \quad (7)$$

for all t. The value of the time interval Δt determines the rapidity of the transition of the L-th particle from the subsystem S_1 to the other subsystem (S_2). In Appendix B we present an explicit example of a function that satisfies conditions (5). With $f(t)$ defined as such, the functions

$$N_1(t) \equiv L - f(t - \bar{t}) \quad (8)$$

and

$$\begin{aligned} N_2(t) &= N - N_1(t) = N - L + f(t - \bar{t}) \\ &= N - L + 1 - f(\bar{t} - t), \end{aligned} \quad (9)$$

can be interpreted (at least for $t \notin T$) as the number of particles in subsystems S_1 and S_2, respectively. Before the transition, which occurs while $t \in \left[\bar{t} - \dfrac{\Delta t}{2}, \bar{t} + \dfrac{\Delta t}{2} \right]$, where Δt and $\bar{t}$ are given by (3) and (4), respectively, we have $N_1 = L$ and $N_2 = N - L$; after the transition, we have $N_1 = L - 1$ and $N_2 = N - L + 1$.

For our present purposes, we do not need to specify $f(x)$ any further. For a particular realization of the function $f(t)$, the reader is referred to Appendix B.

Our aim is that of obtaining Newton's Second Law for the CM of a subsystem like S_1 or S_2, with varying number of particles, within a larger system with total constant number of particles. All particles have constant masses. In what follows, we will obtain expressions for the kinematics of the fictitious particles whose locations coincide with the CM of those subsystems, and relate them to the kinematics of the physical particles that belong to these subsystems, that is, S_1 and/or S_2.

2.1.2. The masses and centers of mass of the subsystems S_1 and S_2

Analogous to (1), the position of the CM of S_1 is written as

$$\vec{R}_{CM}^{(1)}(t) = \frac{\sum_{i=1}^{L-1} m_i\,\vec{r}_i(t) + m_L f(\bar{t}-t)\,\vec{r}_L(t)}{\sum_{i=1}^{L-1} m_i + m_L f(\bar{t}-t)}, \quad (10)$$

whereas the position of the CM of S_2, $\vec{R}_{CM}^{(2)}(t)$ is

$$\vec{R}_{CM}^{(2)}(t) = \frac{m_L f(t-\bar{t})\vec{r}_L(t) + \sum_{i=L+1}^{N} m_i\,\vec{r}_i(t)}{m_L f(t-\bar{t}) + \sum_{i=L+1}^{N} m_i}. \quad (11)$$

The masses of the subsystems S_1 and S_2 at any time t are

$$M_1(t) = m_1 + m_2 + \cdots + m_{L-1} + m_L f(\bar{t}-t) \quad (12)$$

and

$$M_2(t) = m_L f(t-\bar{t}) + m_{L+1} + \cdots + m_{N-1} + m_N, \quad (13)$$

respectively.

The mass of the fictitious particle at the CM of the subsystem S_1, $\vec{R}_{CM}^{(1)}(t)$, is chosen to be equal to $M_1(t)$, and the mass of the fictitious particle at the CM of the subsystem S_2 is also chosen to be equal to the sum of masses of the physical particles that belong to S_2 at every time t.[1]

The total mass M of the system S is constant,

$$\begin{aligned} M &= M_1(t) + M_2(t) \\ &= \sum_{i=1}^{L-1} m_i + m_L \underbrace{\left[f(\bar{t}-t) + f(t-\bar{t}) \right]}_{=1} + \sum_{j=L+1}^{N} m_j \\ &= \sum_{i=1}^{N} m_i. \end{aligned} \quad (14)$$

One can rewrite (10) and (11) as

$$M_1(t)\,\vec{R}_{CM}^{(1)}(t) = \sum_{i=1}^{L-1} m_i\,\vec{r}_i(t) + m_L f(\bar{t}-t)\,\vec{r}_L(t) \quad (15)$$

and

$$M_2(t)\,\vec{R}_{CM}^{(2)}(t) = m_L f(t-\bar{t})\vec{r}_L(t) + \sum_{i=L+1}^{N} m_i\,\vec{r}_i(t) \quad (16)$$

and then obtain

$$M_1(t)\,\vec{R}_{CM}^{(1)}(t) + M_2(t)\,\vec{R}_{CM}^{(2)}(t) = \sum_{i=1}^{L-1} m_i\,\vec{r}_i(t)$$

$$+ m_L \underbrace{\left[f(\bar{t}-t) + f(t-\bar{t}) \right]}_{=1} \vec{r}_L + \sum_{i=L+1}^{N} m_i\,\vec{r}_i(t)$$

$$= \sum_{i=1}^{N} m_i\,\vec{r}_i(t) \quad (17)$$

which, according to (1) and (14), equals to

$$M_1(t)\,\vec{R}_{CM}^{(1)}(t) + M_2(t)\,\vec{R}_{CM}^{(2)}(t) = M\vec{R}_{CM}(t). \quad (18)$$

The velocities of each CM of the total system S and the subsystems S_1 and S_2 are, respectively,

$$\vec{V}_{CM}(t) = \frac{d\vec{R}_{CM}(t)}{dt}, \quad (19)$$

$$\vec{V}_{CM}^{(1)}(t) = \frac{d\vec{R}_{CM}^{(1)}(t)}{dt}, \quad (20)$$

$$\vec{V}_{CM}^{(2)}(t) = \frac{d\vec{R}_{CM}^{(2)}(t)}{dt}. \quad (21)$$

One should readily notice that

$$\vec{V}_{CM}^{(1)} \neq \frac{1}{M(t)} \left[\sum_{i=1}^{L-1} m_i\vec{v}_i(t) + \underbrace{m_L(t)}_{m_L f(\bar{t}-t)}\,\vec{v}_L(t) \right]. \quad (22)$$

In fact, by differentiating (10), one obtains[2]

$$\vec{V}_{CM}^{(1)} = \frac{1}{M(t)} \left[\sum_{i=1}^{L-1} m_i\vec{v}_i(t) + m_L\,f(\bar{t}-t)\vec{v}_L(t) + \right.$$

$$+ m_L\,\frac{df(\bar{t}-t)}{dt}\vec{r}_L(t) \right] - \frac{1}{M(t)^2}\frac{dM(t)}{dt}\left[\sum_{i=1}^{L-1} m_i\vec{r}_i(t) \right.$$

$$+ m_L\,f(\bar{t}-t)\vec{r}_L(t). \right]. \quad (23)$$

In order to relate the linear momentum of the CM of the system S and those of the subsystems S_1 and S_2, we differentiate (18), obtaining

$$\frac{dM_1(t)}{dt}\,\vec{R}_{CM}^{(1)}(t) + M_1(t)\,\vec{V}_{CM}^{(1)}(t)$$

$$+ \frac{dM_2(t)}{dt}\,\vec{R}_{CM}^{(2)}(t) + M_2(t)\,\vec{V}_{CM}^{(2)}(t) = M\vec{V}_{CM}(t). \quad (24)$$

[1]If the system is under a constant gravitational acceleration $\vec{g}$, the total weight forces of the particles that constitute the subsystems S_1 and S_2 can be considered to be applied in their centers of mass, located at $\vec{R}_{CM}^{(1)}(t)$ and $\vec{R}_{CM}^{(2)}(t)$, respectively.

[2]During the transition time interval T, the expression of $V_{CM}^{(i)}$, $i \in \{1,2\}$, is different from that of a set particles with constant mass.

As

$$\frac{df(\bar{t}-t)}{dt} = \frac{d}{dt}\left(1 - f(t-\bar{t})\right) = -\frac{df(t-\bar{t})}{dt}, \quad (25)$$

from (12), (13) and (7), one obtains

$$\frac{dM_1(t)}{dt} = m_L \frac{df(\bar{t}-t)}{dt} = -m_L \frac{df(t-\bar{t})}{dt}, \quad (26)$$

$$\frac{dM_2(t)}{dt} = m_L \frac{df(t-\bar{t})}{dt}. \quad (27)$$

2.1.3. The relation between the linear momentum of the CM of the system S and the CM's of the subsystems S_1 and S_2

The relation (24) can be written as

$$M_1(t)\,\vec{V}_{CM}^{(1)}(t) + M_2(t)\,\vec{V}_{CM}^{(2)}(t)$$
$$= M\vec{V}_{CM}(t) + m_L \frac{df(t-\bar{t})}{dt}\left(\vec{R}_{CM}^{(1)}(t) - \vec{R}_{CM}^{(2)}(t)\right), (28)$$

or

$$\vec{P}_{CM}^{(1)}(t) + \vec{P}_{CM}^{(2)}(t) = \vec{P}_{CM}(t) +$$
$$+ m_L \frac{df(t-\bar{t})}{dt}\left(\vec{R}_{CM}^{(1)}(t) - \vec{R}_{CM}^{(2)}(t)\right), \quad (29)$$

in which

$$\vec{P}_{CM}^{(1)}(t) \equiv M_1(t)\,\vec{V}_{CM}^{(1)}(t), \quad (30)$$

$$\vec{P}_{CM}^{(2)}(t) \equiv M_2(t)\vec{V}_{CM}^{(2)}(t) \quad (31)$$

are the linear momenta of the CM of S_1 and S_2, respectively, whereas

$$\vec{P}_{CM}(t) \equiv M\vec{V}_{CM}(t), \quad (32)$$

in which M is the total mass (14) of the CM of the N physical particles and $\vec{P}_{CM}(t)$ is the linear momentum of the CM of the whole system S. Observe that the sum of $\vec{P}_{CM}^{(1)}(t)$ and $\vec{P}_{CM}^{(2)}(t)$ does *not* add up to $\vec{P}_{CM}(t)$ during the transition interval (2). The linear momenta $\vec{P}_{CM}^{(1)}(t)$ and $\vec{P}_{CM}^{(2)}(t)$ recover the usual definition of the momenta of each subsystem for times t outside the transition interval T in which the L-th particle migrates from S_1 to S_2, as discussed in the next subsection.

The linear momentum of the i-th particle is

$$\vec{p}_i(t) = m_i\,\vec{v}_i(t) = m_i\,\frac{d\vec{r}_i(t)}{dt}, \quad (33)$$

for $i \in \{1, 2, \ldots, N\}$. The total linear momentum $\vec{P}(t)$ of the whole system, sum of all individual linear momenta, corresponds, as expected, to the linear momentum of (the fictitious particle at) the CM of the system S:

$$\vec{P}(t) = \sum_{i=1}^{N} \vec{p}_i(t) = \sum_{i=1}^{N} m_i\,\frac{d\vec{r}_i(t)}{dt}$$
$$= \frac{d}{dt}\sum_{i=1}^{N} m_i\,\vec{r}_i(t) = \frac{d}{dt}\left(M\vec{R}_{CM}(t)\right)$$
$$= M\,\vec{V}_{CM}(t) \quad (34)$$

which, according to (32), leads to

$$\vec{P}(t) = \vec{P}_{CM}(t), \quad (35)$$

showing that the linear momentum of the CM of S is equal to the total liner momentum of the set of the N physical particles.

2.1.4. The linear momentum of a subsystem is NOT ALWAYS equal to the total linear momenta of its physical particles

A relation analogous to that of (35) *does not hold* for all t, however, for the subsystems S_1 and S_2. Let us examine, for instance, the linear momentum of the CM of S_1, defined in (30), and compare it to the total momentum $\vec{P}^{(1)}(t)$ of subsystem S_1, which reads

$$\vec{P}^{(1)}(t) = \vec{p}_1(t) + \vec{p}_2(t) + \cdots + \vec{p}_{L-1}(t) + f(\bar{t}-t)\vec{p}_L(t)$$
$$= \sum_{k=1}^{L-1} \vec{p}_k(t) + f(\bar{t}-t)\vec{p}_L(t). \quad (36)$$

Eq.(10) is used to relate the vector $\vec{P}_{CM}^{(1)}(t)$ and the information about the physical particles that belong to the subsystem S_1 at each moment,

$$\vec{P}_{CM}^{(1)}(t) = M_1(t)\,\vec{V}_{CM}^{(1)}(t) = M_1(t)\frac{d\vec{R}_{CM}^{(1)}(t)}{dt}$$
$$= M_1(t)\frac{d}{dt}\left\{\frac{1}{M_1(t)}\left[m_1\vec{r}_1(t) + m_2\vec{r}_2(t) + \cdots + m_{L-1}\vec{r}_{L-1}(t) + m_L\,f(\bar{t}-t)\,\vec{r}_L(t)\right]\right\}. \quad (37)$$

So,

$$
\begin{aligned}
\vec{P}_{CM}^{(1)}(t) &= M_1(t) \left\{ -\frac{1}{M_1(t)^2} \frac{dM_1(t)}{dt} \left[\sum_{i=1}^{L-1} m_i \vec{r}_i(t) + m_L \, f(\bar{t}-t) \, \vec{r}_L(t) \right] \right. \\
&\quad \left. +\frac{1}{M_1(t)} \left[\sum_{i=1}^{L-1} m_i \frac{d\vec{r}_i(t)}{dt} + m_L \left(\frac{df(\bar{t}-t)}{dt} \, \vec{r}_L(t) + f(\bar{t}-t) \frac{d\vec{r}_L(t)}{dt} \right) \right] \right\}.
\end{aligned} \tag{38}
$$

This can be rewritten due to (26) as

$$
\begin{aligned}
\vec{P}_{CM}^{(1)}(t) &= \frac{m_L}{M_1(t)} \frac{df(t-\bar{t})}{dt} \left[m_1\vec{r}_1(t) + m_2\vec{r}_2(t) + \cdots + m_{L-1}\vec{r}_{L-1}(t) + m_L \, f(\bar{t}-t) \, \vec{r}_L(t) \right] \\
&\quad +\vec{p}_1(t) + \vec{p}_2(t) + \cdots + \vec{p}_{L-1}(t) + f(\bar{t}-t)\vec{p}_L(t) + m_L \frac{df(\bar{t}-t)}{dt} \, \vec{r}_L(t)
\end{aligned} \tag{39}
$$

We simplify (39) by using (15) and (25),

$$
\vec{P}_{CM}^{(1)}(t) = m_L \frac{df(t-\bar{t})}{dt} \vec{R}_{CM}^{(1)}(t) + \left[\sum_{i=1}^{L-1} \vec{p}_i(t) + f(\bar{t}-t)\vec{p}_L(t) \right] - m_L \frac{df(t-\bar{t})}{dt} \, \vec{r}_L(t), \tag{40}
$$

which amounts to

$$
\vec{P}_{CM}^{(1)}(t) = \vec{P}^{(1)}(t) + m_L \frac{df(t-\bar{t})}{dt} \left(\vec{R}_{CM}^{(1)}(t) - \vec{r}_L(t) \right). \tag{41}
$$

We use (30) to rewrite $V_{CM}^{(1)}$ in terms of the linear momenta of the particles that belong to the subset S_1,

$$
\vec{V}_{CM}^{(1)}(t) = \frac{\vec{p}_1(t) + \vec{p}_2(t) + \cdots + \vec{p}_{L-1}(t) + f(\bar{t}-t)\vec{p}_L(t)}{M_1(t)} - \frac{m_L}{M_1(t)} \frac{df(t-\bar{t})}{dt} \left(\vec{r}_L(t) - \vec{R}_{CM}^{(1)}(t) \right). \tag{42}
$$

Unlike the case of the whole system stated in (35), the linear momentum of the CM of the subsystem S_1, $\vec{P}_{CM}^{(1)}(t)$, is *not* equal to the total momentum $\vec{P}^{(1)}(t)$ of that subsystem during the interval of time that the function $f(t-\bar{t})$ varies between the values 0 and 1. This interval of time corresponds to the transition of the L-th particle between the two subsystems S_1 and S_2.

An analogous statement can be found for the subsystem S_2, namely,

$$
\vec{P}_{CM}^{(2)}(t) = \vec{P}^{(2)}(t) - m_L \frac{df(t-\bar{t})}{dt} \left(\vec{R}_{CM}^{(2)}(t) - \vec{r}_L(t) \right), \tag{43}
$$

relating the linear momentum of its CM, as defined in (31), and the total momentum $\vec{P}^{(2)}(t)$ of the physical particles that belong to S_2, which reads

$$
\begin{aligned}
\vec{P}^{(2)}(t) &= f(t-\bar{t})\vec{p}_L(t) + \vec{p}_{L+1}(t) + \ldots \\
&\quad +\vec{p}_{N-1}(t) + \vec{p}_N(t).
\end{aligned} \tag{44}
$$

By adding (41) and (43), one recovers (29).

2.2. Dynamics of the CM of S_1

2.2.1. Equation of motion for the CM of the subsystem S_1

The equation of motion of the i-th particle of the system S, observed from an inertial frame, can be written as

$$
\frac{d\vec{p}_i(t)}{dt} = \vec{F}_i^{(ext)}(t) + \sum_{\substack{j=1 \\ j\neq i}}^{N} \vec{F}_{j\to i}^{(int)}(t), \tag{45}
$$

in which $\vec{F}_i^{(ext)}(t)$ is the sum of the external forces acting on the i-th particle (that is, due to interactions with other bodies not belonging to the system S) and $\vec{F}_{j\to i}^{(int)}(t)$ is the internal force on the i-th particle due to its interaction with the j-th particle of S. The summation index j runs over all particles of S, excluding the i-particle itself. Newton's Third Law can be expressed as

$$
\vec{F}_{i\to j}^{(int)}(t) = -\vec{F}_{j\to i}^{(int)}(t). \tag{46}
$$

The equation of motion for the ficticious particle on the CM of subsystem S_1 is, from the time derivative of (41),

$$
\begin{aligned}
\frac{d\vec{P}_{CM}^{(1)}(t)}{dt} &= m_L \frac{d^2 f(t-\bar{t})}{dt^2} \left(\vec{R}_{CM}^{(1)}(t) - \vec{r}_L(t) \right) \\
&\quad +m_L \frac{df(t-\bar{t})}{dt} \left(\vec{V}_{CM}^{(1)}(t) - \vec{v}_L(t) \right) + \frac{d\vec{P}^{(1)}(t)}{dt} \tag{47} \\
&= \vec{F}_R^{(CM,1)}(t). \tag{48}
\end{aligned}
$$

Equality (48) is Newton's Second Law for a variable mass particle. $\vec{F}_R^{(CM,1)}(t)$ is the resulting force acting on the fictitious particle located in the CM of S_1 so that it follows the path $\vec{R}_{CM}^{(1)}(t)$ as its mass changes with time.

One is reminded that $M_1 \vec{V}_{CM}^{(1)}(t) = \vec{P}_{CM}^{(1)}(t)$ and $m_L \vec{v}_L(t) = \vec{p}_L(t)$, the linear momentum of the CM of S_1 and of the migrating L-th particle, respectively. The expression of the vector $\vec{P}^{(1)}(t)$ is given by (36). Ob-

serve that the r.h.s of this equation contains not only the expected (last) term, the time derivative of the total momentum of the physical particles that belong to the subsystem S_1, but also terms on the first and second time derivatives of the transition function $f(t - \bar{t})$. Hence, for a well defined equation of motion for the CM of this subsystem, a continuous function $f(t)$ with continuous derivatives in the variable t up to second order would suffice.

2.2.2. Rewriting $\dfrac{d\vec{P}^{(1)}(t)}{dt}$ in terms of internal and external forces applied to S_1

Let us rewrite (47) in terms of the forces acting on the particles of the system. First, let us observe that its last term is equivalent to

$$\frac{d\vec{P}^{(1)}(t)}{dt} = \sum_{k=1}^{L-1} \frac{d\vec{p}_k(t)}{dt} + \frac{df(\bar{t}-t)}{dt}\vec{p}_L(t) + f(\bar{t}-t)\frac{d\vec{p}_L(t)}{dt}, \tag{49}$$

by using (36). From the individual equations of motion (45), comes

$$\frac{d\vec{P}^{(1)}(t)}{dt} = \sum_{k=1}^{L-1}\left[\vec{F}_k^{(ext)}(t) + \sum_{\substack{j=1\\j\neq k}}^{N}\vec{F}_{j\to k}^{(int)}(t)\right] + \frac{df(\bar{t}-t)}{dt}\vec{p}_L(t) + f(\bar{t}-t)\left[\vec{F}_L^{(ext)}(t) + \sum_{\substack{j=1\\j\neq L}}^{N}\vec{F}_{j\to L}^{(int)}(t)\right]. \tag{50}$$

By grouping the terms related to external and internal forces applied to the physical particles that belong to S_1, one obtains

$$\frac{d\vec{P}^{(1)}(t)}{dt} = \left[\sum_{k=1}^{L-1}\vec{F}_k^{(ext)}(t) + f(\bar{t}-t)\vec{F}_L^{(ext)}(t)\right] + \left[\sum_{k=1}^{L-1}\sum_{\substack{j=1\\j\neq k}}^{N}\vec{F}_{j\to k}^{(int)}(t) + f(\bar{t}-t)\sum_{\substack{j=1\\j\neq L}}^{N}\vec{F}_{j\to L}^{(int)}(t)\right] - \frac{df(t-\bar{t})}{dt}\vec{p}_L(t), \tag{51}$$

in which once more we have used the result (25) on the derivative of the function $f(t)$.

We will focus our attention now on the double summation involving the internal forces. By splitting the inner summation in three parts, for $j < L$, $j = L$ and $j > L$, one obtains

$$\sum_{k=1}^{L-1}\sum_{\substack{j=1\\j\neq k}}^{N}\vec{F}_{j\to k}^{(int)}(t) = \sum_{k=1}^{L-1}\left(\sum_{\substack{j=1\\j\neq k}}^{L-1}\vec{F}_{j\to k}^{(int)}(t) + \vec{F}_{L\to k}^{(int)}(t) + \sum_{j=L+1}^{N}\vec{F}_{j\to k}^{(int)}(t)\right)$$

$$= \sum_{k=1}^{L-1}\sum_{\substack{j=1\\j\neq k}}^{L-1}\vec{F}_{j\to k}^{(int)}(t) + \sum_{k=1}^{L-1}\vec{F}_{L\to k}^{(int)}(t) + \sum_{k=1}^{L-1}\sum_{j=L+1}^{N}\vec{F}_{j\to k}^{(int)}(t). \tag{52}$$

(As $k \leq L - 1$, the constraint $j \neq k$ is redundant for $j \geq L$, so it has been kept solely for the summation in which $j \leq L - 1$.)

For the sake of mathematical simplicity only, and without any loss of generality, let us define $\vec{F}_{i\to i}^{(int)} \equiv \vec{0}$, for $i \in \{1, 2, \ldots, N\}$ — which still satisfies Newton's Third Law (46), and let us relax about the (again redundant) condition $j \neq k$. We may then write the first term of (52) as

$$\sum_{k=1}^{L-1}\sum_{\substack{j=1\\j\neq k}}^{L-1}\vec{F}_{j\to k}^{(int)}(t) = \sum_{k=1}^{L-1}\sum_{j=1}^{L-1}\vec{F}_{j\to k}^{(int)}(t) = \frac{1}{2}\left[\sum_{k=1}^{L-1}\sum_{j=1}^{L-1}\vec{F}_{j\to k}^{(int)}(t) + \sum_{k=1}^{L-1}\sum_{j=1}^{L-1}\vec{F}_{j\to k}^{(int)}(t)\right]$$

$$= \frac{1}{2}\left[\sum_{k=1}^{L-1}\sum_{j=1}^{L-1}\vec{F}_{j\to k}^{(int)}(t) + \sum_{j=1}^{L-1}\sum_{k=1}^{L-1}\vec{F}_{k\to j}^{(int)}(t)\right], \tag{53}$$

in which we simply interchanged the dummy summation indices $\{j, k\}$ without altering the summation in the second term of (53). By using Newton's Third Law (46) and then swapping the summation order, one obtains

$$\sum_{k=1}^{L-1}\sum_{\substack{j=1\\j\neq k}}^{L-1}\vec{F}_{j\to k}^{(int)}(t) = \frac{1}{2}\left[\sum_{k=1}^{L-1}\sum_{j=1}^{L-1}\vec{F}_{j\to k}^{(int)}(t) - \sum_{j=1}^{L-1}\sum_{k=1}^{L-1}\vec{F}_{j\to k}^{(int)}(t)\right] = \frac{1}{2}\left[\sum_{k=1}^{L-1}\sum_{j=1}^{L-1}\vec{F}_{j\to k}^{(int)}(t) - \sum_{k=1}^{L-1}\sum_{j=1}^{L-1}\vec{F}_{j\to k}^{(int)}(t)\right] = \vec{0}, \quad (54)$$

in which $\vec{0}$ is the null vector ($|\vec{0}| = 0$).

Hence (52) is reduced to

$$\sum_{k=1}^{L-1}\sum_{\substack{j=1\\j\neq k}}^{N}\vec{F}_{j\to k}^{(int)}(t) = \sum_{k=1}^{L-1}\vec{F}_{L\to k}^{(int)}(t) + \sum_{k=1}^{L-1}\sum_{j=L+1}^{N}\vec{F}_{j\to k}^{(int)}(t). \quad (55)$$

and consequently (51) now reads

$$\frac{d\vec{P}^{(1)}(t)}{dt} = \left[\sum_{k=1}^{L-1}\vec{F}_k^{(ext)}(t) + f(\bar{t}-t)\vec{F}_L^{(ext)}(t)\right] + \left[\sum_{k=1}^{L-1}\vec{F}_{L\to k}^{(int)}(t) + f(\bar{t}-t)\sum_{\substack{j=1\\j\neq L}}^{N}\vec{F}_{j\to L}^{(int)}(t) + \sum_{k=1}^{L-1}\sum_{j=L+1}^{N}\vec{F}_{j\to k}^{(int)}(t)\right]$$
$$- \frac{df(t-\bar{t})}{dt}\vec{p}_L(t). \quad (56)$$

But

$$\sum_{k=1}^{L-1}\vec{F}_{L\to k}^{(int)}(t) + f(\bar{t}-t)\sum_{\substack{j=1\\j\neq L}}^{N}\vec{F}_{j\to L}^{(int)}(t) = \sum_{k=1}^{L-1}\vec{F}_{L\to k}^{(int)}(t) + f(\bar{t}-t)\left[\sum_{j=1}^{L-1}\vec{F}_{j\to L}^{(int)}(t) + \sum_{j=L+1}^{N}\vec{F}_{j\to L}^{(int)}(t)\right]$$
$$= \sum_{k=1}^{L-1}\vec{F}_{L\to k}^{(int)}(t) + f(\bar{t}-t)\left[-\sum_{j=1}^{L-1}\vec{F}_{L\to j}^{(int)}(t) + \sum_{j=L+1}^{N}\vec{F}_{j\to L}^{(int)}(t)\right] = \left(1 - f(\bar{t}-t)\right)\sum_{k=1}^{L-1}\vec{F}_{L\to k}^{(int)}(t)$$
$$+ f(\bar{t}-t)\sum_{j=L+1}^{N}\vec{F}_{j\to L}^{(int)}(t) = f(t-\bar{t})\sum_{k=1}^{L-1}\vec{F}_{L\to k}^{(int)}(t) + f(\bar{t}-t)\sum_{k=L+1}^{N}\vec{F}_{k\to L}^{(int)}(t). \quad (57)$$

Thus (56) can be rewritten as

$$\frac{d\vec{P}^{(1)}(t)}{dt} = \left[\sum_{k=1}^{L-1}\vec{F}_k^{(ext)}(t) + f(\bar{t}-t)\vec{F}_L^{(ext)}(t)\right] + \left[f(t-\bar{t})\sum_{k=1}^{L-1}\vec{F}_{L\to k}^{(int)}(t) + f(\bar{t}-t)\sum_{k=L+1}^{N}\vec{F}_{k\to L}^{(int)}(t)\right.$$
$$+ \left.\sum_{k=1}^{L-1}\sum_{j=L+1}^{N}\vec{F}_{j\to k}^{(int)}(t)\right] - \frac{df(t-\bar{t})}{dt}\vec{p}_L(t). \quad (58)$$

Inserting (58) into (47) yields

$$\frac{d\vec{P}_{CM}^{(1)}(t)}{dt} = m_L\frac{d^2 f(t-\bar{t})}{dt^2}\left(\vec{R}_{CM}^{(1)}(t) - \vec{r}_L(t)\right) + m_L\frac{df(t-\bar{t})}{dt}\left(\vec{V}_{CM}^{(1)}(t) - \vec{v}_L(t)\right)$$
$$+ \left[\sum_{k=1}^{L-1}\vec{F}_k^{(ext)}(t) + f(\bar{t}-t)\vec{F}_L^{(ext)}(t)\right] + \left[f(t-\bar{t})\sum_{k=1}^{L-1}\vec{F}_{L\to k}^{(int)}(t) + f(\bar{t}-t)\sum_{k=L+1}^{N}\vec{F}_{k\to L}^{(int)}(t)\right.$$
$$+ \left.\sum_{k=1}^{L-1}\sum_{j=L+1}^{N}\vec{F}_{j\to k}^{(int)}(t)\right] - \frac{df(t-\bar{t})}{dt}\vec{p}_L(t) \quad (59)$$

or

$$
\begin{aligned}
\frac{d\vec{P}_{CM}^{(1)}(t)}{dt} &= m_L \frac{d^2 f(t-\bar{t})}{dt^2}\left(\vec{R}_{CM}^{(1)}(t)-\vec{r}_L(t)\right) + m_L \frac{df(t-\bar{t})}{dt}\left(\vec{V}_{CM}^{(1)}(t)-2\,\vec{v}_L(t)\right) \\
&+ \left[\sum_{k=1}^{L-1}\vec{F}_k^{(ext)}(t)+f(\bar{t}-t)\vec{F}_L^{(ext)}(t)\right] + \left[f(t-\bar{t})\sum_{k=1}^{L-1}\vec{F}_{L\to k}^{(int)}(t)+f(\bar{t}-t)\sum_{k=L+1}^{N}\vec{F}_{k\to L}^{(int)}(t)\right. \\
&+ \left.\sum_{k=1}^{L-1}\sum_{j=L+1}^{N}\vec{F}_{j\to k}^{(int)}(t)\right] \tag{60} \\
&= \vec{F}_R^{(CM,1)}(t). \tag{61}
\end{aligned}
$$

That is the equation of motion for the CM of S_1, in terms of variables related to the center of mass itself $(\vec{R}_{CM}^{(1)},\vec{P}_{CM}^{(1)})$, to the migrating L-th particle $(\vec{r}_L,\vec{p}_L)$, to the forces acting on each particle of S_1 due to the particles in S_2 besides the external forces of S that act on the physical particles belonging to S_1 during the whole movement, that is $t \in (-\infty,+\infty)$. Notice that during the transition time in which the L-th particle moves from S_1 to S_2, its motion is affected by the internal forces acting on it, modulated by the transition function $f(t)$.

Eq. (61) relates the effective force $\vec{F}_R^{(CM,1)}(t)$ that acts on the fictitious particle located at the CM of S_1, and the forces applied to the physical particles belonging to the subset S_1. Besides the physical forces contributing to $\vec{F}_R^{(CM,1)}(t)$ we have the contribution of the migrating particle L, responsible for the time variation of the mass of the fictitious particle, $M_1(t)$.

2.2.3. The equation of motion of the CM of S_1 in terms of its acceleration, masses and external forces to S_1

Defining the acceleration of the CM of the subsystem S_1 as

$$
\vec{a}_{CM}^{(1)}(t) \equiv \frac{d\vec{V}_{CM}^{(1)}(t)}{dt}, \tag{62}
$$

and taking (26) into account, we may rewrite the l.h.s. of (60) as

$$
\frac{d\vec{P}_{CM}^{(1)}(t)}{dt}=\frac{d}{dt}\left(M_1(t)\vec{V}_{CM}^{(1)}(t)\right)=\frac{dM_1(t)}{dt}\vec{V}_{CM}^{(1)}(t)+M_1(t)\frac{d\vec{V}_{CM}^{(1)}(t)}{dt}=-m_L\frac{df(t-\bar{t})}{dt}\vec{V}_{CM}^{(1)}(t)
$$

$$
+M_1(t)\vec{a}_{CM}^{(1)}(t) \quad \Rightarrow \quad M_1(t)\,\vec{a}_{CM}^{(1)}(t)=\frac{d\vec{P}_{CM}^{(1)}(t)}{dt}+\frac{df(t-\bar{t})}{dt}\,m_L\vec{V}_{CM}^{(1)}(t). \tag{63}
$$

$$
= \vec{F}_R^{(CM,1)}(t)+\frac{df(t-\bar{t})}{dt}\cdot m_L\,\vec{V}_{CM}^{(1)}(t). \tag{64}
$$

By inserting this equality into (60) and isolating the acceleration term, we obtain the equation of motion for the CM of the subsystem S_1,

$$
\begin{aligned}
M_1(t)\vec{a}_{CM}^{(1)}(t) &= m_L\frac{d^2 f(t-\bar{t})}{dt^2}\left(\vec{R}_{CM}^{(1)}(t)-\vec{r}_L(t)\right)+2\,m_L\frac{df(t-\bar{t})}{dt}\left(\vec{V}_{CM}^{(1)}(t)-\vec{v}_L(t)\right) \\
&+ \left[\sum_{k=1}^{L-1}\vec{F}_k^{(ext)}(t)+f(\bar{t}-t)\vec{F}_L^{(ext)}(t)\right]+\left[f(t-\bar{t})\sum_{k=1}^{L-1}\vec{F}_{L\to k}^{(int)}(t)+f(\bar{t}-t)\sum_{k=L+1}^{N}\vec{F}_{k\to L}^{(int)}(t)\right. \\
&+ \left.\sum_{k=1}^{L-1}\sum_{j=L+1}^{N}\vec{F}_{j\to k}^{(int)}(t)\right],
\end{aligned}
\tag{65}
$$

valid for $t \in (-\infty,+\infty)$. Alternatively, from (26) we have

$$
\frac{df(t-\bar{t})}{dt} = -\frac{1}{m_L}\frac{dM_1(t)}{dt}, \tag{66}
$$

$$
\frac{d^2 f(t-\bar{t})}{dt^2} = -\frac{1}{m_L}\frac{d^2 M_1(t)}{dt^2}, \tag{67}
$$

and (65) amounts to

$$
\begin{aligned}
M_1(t)\, \vec{a}_{CM}^{(1)}(t) &= \left[-\frac{d^2 M_1(t)}{dt^2}\left(\vec{R}_{CM}^{(1)}(t) - \vec{r}_L(t)\right) - 2\frac{dM_1(t)}{dt}\left(\vec{V}_{CM}^{(1)}(t) - \vec{v}_L(t)\right)\right] \\
&\quad + \left[\sum_{k=1}^{L-1}\vec{F}_k^{(ext)}(t) + f(\bar{t} - t)\vec{F}_L^{(ext)}(t)\right] + \left[f(t - \bar{t})\sum_{k=1}^{L-1}\vec{F}_{L\to k}^{(int)}(t) + f(\bar{t} - t)\sum_{k=L+1}^{N}\vec{F}_{k\to L}^{(int)}(t)\right] \\
&\quad + \sum_{k=1}^{L-1}\sum_{j=L+1}^{N}\vec{F}_{j\to k}^{(int)}(t) \Bigg].
\end{aligned}
\tag{68}
$$

The "particle" that is placed at the CM of S_1 is a fictitious one. During the transition time interval in which the L-th particle migrates from S_1 to S_2, the internal forces that act on the L-th particle contribute to the acceleration $\vec{a}_{CM}^{(1)}(t)$.

2.2.4. Interpreting the force terms in the dynamics of the CM of S_1

All the physical forces on the r.h.s. of (68) that contribute to the dynamics of the CM of S_1 are forces that act, at time t, on the physical particles that belong to this subsystem. When the number of particles in each subsystem does not vary in time, $\frac{dM_1(t)}{dt} = 0$, we recover Newton's Second Law, that rules the time variation of the vector position of the fictitious particle located at the CM of the set of the physical particles.

The three bracketed groups of terms in the r.h.s. of equation (68) account for distinct contributions to the dynamics of the CM of the subsystem S_1: the first from the time variation of its mass $M_1(t)$; the second from the external forces acting on its permanent particles and on the migrating particle; and the last one from the internal forces (of the system as a whole), but they are internal or external forces that act on the subsystem S_1 depending on the different intervals of time, that is, $t < \bar{t} - \frac{\Delta t}{2}$, $t \in [\bar{t} - \frac{\Delta t}{2}, \bar{t} + \frac{\Delta t}{2}]$ and $t > \bar{t} + \frac{\Delta t}{2}$. For $t < \bar{t} + \frac{\Delta t}{2}$ the L-th particle belongs to S_1 and the forces in the second term in the third bracket are external to S_1. For $t > \bar{t} + \frac{\Delta t}{2}$ the L-th particle does not belong to S_1 and the first set of forces in the third bracket are external forces to S_1 and the second set of forces in the same bracket does not act on any particle of subsystem S_1.

The term

$$
-\frac{d^2 M_1(t)}{dt^2}\left(\vec{R}_{CM}^{(1)}(t) - \vec{r}_L(t)\right)
\tag{69}
$$

on the r.h.s. of (68) can be interpreted as fictitious restoration or repelling force, depending on the sign of the coefficient $\frac{d^2 M_1(t)}{dt^2}$, acting on the CM, whereas the term

$$
-2\frac{dM_1(t)}{dt}\left(\vec{V}_{CM}^{(1)}(t) - \vec{v}_L(t)\right)
\tag{70}
$$

is a fictitious viscous damping force. Both forces are due to the mass variation of S_1, being non-vanishing only during the transition interval. (They are proportional to derivatives of the transition function (5), which vanish outside this interval.)

The term

$$
\sum_{k=1}^{L-1}\vec{F}_k^{(ext)}(t)
\tag{71}
$$

represents the external forces to the system, acting on the permanent particles of S_1, whereas

$$
\sum_{k=1}^{L-1}\sum_{j=L+1}^{N}\vec{F}_{j\to k}^{(int)}(t),
\tag{72}
$$

represents those external forces exchanged between the permanent particles of S_1 and the permanent particles of S_2.

We can say that the migrating L-th particle is no longer in S_1 after the transition interval is over $t > \bar{t} + \frac{\Delta t}{2}$. With that in mind, we can interpret the remaining terms. Before and during the transition interval, that is $t \in (-\infty, \bar{t} + \frac{\Delta t}{2}]$, the term

$$
f(\bar{t} - t)\,\vec{F}_L^{(ext)}(t)
\tag{73}
$$

accounts for the external force on the migrating particle while it belongs to S_1 as we see from the conditions (5) that defines the function $f(t)$. The term

$$
f(\bar{t} - t)\sum_{k=L+1}^{N}\vec{F}_{k\to L}^{(int)}(t)
\tag{74}
$$

on the r.h.s. of (68) describes the interaction of the permanent particles of the other subsystem S_2 with the migration particle while it belongs to S_1, which corresponds to external forces to S_1. During and after the transition, the term

$$
f(t - \bar{t})\sum_{k=1}^{L-1}\vec{F}_{L\to k}^{(int)}(t)
\tag{75}
$$

depicts the interaction of the migrating particle and the permanent particles of S_1 during the transition and after the L-th particle has left the subsystem S_1. These forces are internal to the set of particles composing S_1 for $t < \bar{t} + \frac{\Delta t}{2}$ and external otherwise.

The results (71) – (74) show which forces in the subsystem S_1 account for the movement of the CM of S_1,

besides the forces (69) and (70) due to the variation of the total number of physical particles belonging to S_1.

Notice that, according to Eq. (68), during time interval in which the L-th particle goes from S_1 to S_2, we may have $\vec{a}_{CM}^{(1)} \neq 0$ even if all N particles of the system are free (with $\vec{F}_k^{(ext)} = \vec{0}$ and $\vec{F}_{k \to L}^{(int)} = \vec{0}$, for all $l, k \in \{1, 2, \ldots, N\}$). Even though observed from an inertial reference frame, the fictitious particle placed at the CM of S_1 may have non-null acceleration, even if all physical forces that act on the particles of S_1 vanish.

2.3. A simple example

As a simple realization of a particle set where the number of particles varies in time, we consider two free particles that slide on a horizontal air rail. The two particles have the same value m of mass. They leave from the same position $x = 0$ at $t = 0s$, but they slide in opposite directions in their one-dimensional motion along the air rail.

The positions and the velocities of the particles 1, $x_1(t)$ and $v_1(t)$, and 2, $x_2(t)$ and $v_2(t)$, are

$$x_1(t) = t \times \left(\frac{m}{s}\right) \Rightarrow v_1(t) = 1 \times \left(\frac{m}{s}\right), \quad (76)$$

$$x_2(t) = -t \times \left(\frac{m}{s}\right) \Rightarrow v_1(t) = -1 \times \left(\frac{m}{s}\right). \quad (77)$$

Initially both particles belong to the set S. For $t \gtrsim 3.25s$ particle 2 leaves this particle set. To describe this transition of particle 2, we choose the function $f(3s - t)$, see eq.(B9),

$$f(3s - t) = \frac{1}{2}\left[1 + tanh(10(3s - t))\right]. \quad (78)$$

Following (10), we define the position of the CM of the set S to be

$$X_{CM}(t) \equiv \frac{m x_1(t) + f(3s - t) \cdot m x_2(t)}{m + f(3s - t)m}. \quad (79)$$

The mass of the fictitious particle located at the CM of S is given by (12) with $L = 2$, that is,

$$M_{CM}(t) = m + f(3s - t)m \quad (80)$$

$$= \frac{m}{2}\left[3 + tanh(10(3s - t))\right]. \quad (81)$$

Substituting the expressions of $x_i(t)$, with $i = 1, 2$, and the transition function $f(3s - t)$ on the r.h.s. of (79), we obtain the position of the CM of S during the whole movement,

$$X_{CM}(t) = t \cdot \frac{[1 - tanh(10(3s - t))]}{\frac{1}{2}[3 + tanh(10(3s - t))]}. \quad (82)$$

In Fig.1a we plot the positions of the particles 1 and 2 (while the latter remains in the set S), and the CM of S. For $t \lesssim 3.25s$ the CM of S remains at the origin $x = 0$, since both particles have the same mass, they travel in opposite directions and with the same modulus of velocity. During the transition time the location of the CM of S goes from $x = 0$ up to the position of particle 1. After particle 2 leaves S, the position of the CM coincides with the position of particle 1 as we can see in Fig.1a.

The velocity of the CM of S, $V_{CM}(t)$, is the time variation of $X_{CM}(t)$, that is

$$V_{CM}(t) = \frac{dX_{CM}(t)}{dt} = \frac{1 + 40t - 2sinh(10(3s - t)) \cdot cosh(10(3s - t)) + 2cosh^2(10(3s - t))}{[3cosh(10(3s - t)) + sinh(10(3s - t))]^2}. \quad (83)$$

Fig.1b shows the velocities of the two particles while they belong to the set S and the CM of the latter. Before $t = 2.75s$, when particle 2 begins to leave S, the CM of the particle set remains at rest. During the interval of time when particle 2 migrates from S, the velocity of the CM suffers a sudden variation in time up it reaches the velocity of particle 1, that is kept constant during the whole movement.

Finally the dynamics of the fictitious particle located at the CM of S is driven by Newton's second law for a particle with varying mass, $M_{CM}(t)$,

$$\frac{d}{dt}\left[M_{CM}(t) V_{CM}(t)\right] = F_R(t), \quad (84)$$

where $F_R(t)$ is the resultant force applied to the fictitious particle at the CM.

Substituting the expressions (81) and (83) on the l.h.s. of (84) we can derive the expression of the force $F_R(t)$ after a simple but tedious calculation.

Fig.1c represents the curve $F_R(t) \times t$ and shows that the movement of the fictitious particle at the CM has a free evolution before and after the transition interval of

time when particle 2 leaves S. During the time interval $t \in (2.75s, 3.25s)$ a non null force $F_R(t)$ acts on the fictitious particle. In this example we consider the case when in eqs.(60) and (61) we do not have the contributions of physical forces to the motion of the two particles.

3. Center of mass of a system with of particles with variable mass

In this section we will consider a system of N physical particles of masses $\{m_1, m_2, \ldots m_N\}$, in which $N \geq 2$, so that the N-th particle has a variable mass $m_N(t)$. The movement of this set of particles will be observed from

an inertial frame of reference (in which Newton's Second Law is valid),

$$\frac{d\vec{p}_i(t)}{dt} = \frac{d}{dt}\left(m_i(t)\vec{v}_i(t)\right) = \sum_{\substack{j=1 \\ j\neq i}}^{N} \vec{F}_{j\to i}^{(int)}(t) + \vec{F}_i^{(ext)}(t)$$

(85)

in which $i \in \{1, 2, \ldots, N\}$, $\vec{r}_i(t)$ is the position of the i-th particle, m_i its mass — fixed for all particles, $i \in \{1, 2, \ldots N-1\}$ but for the N-th one, $m_N = m_N(t)$ — and $\vec{v}_i(t)$ is its velocity. $\vec{p}_i(t)$ is the linear momentum of the i-th particle, $\vec{F}_{j\to i}^{(int)}$ is the (internal) force that the j-th particle applies on it, whereas $\vec{F}_i^{(ext)}$ is the resultant of all external forces applied to that i-th particle. We point out that for the N-th particle, the one whose mass varies in time, the physical forces that act on it are responsible for the time variation of the product $m_N(t) \cdot \vec{v}_N(t)$, whereas the physical forces that act on the other $N-1$ particles in the set only drive the time dependence of their respective vector velocities.

Nowadays we do not know physical (non-relativistic) particles whose masses vary in time. However, this should not stop us from asking questions about the Physics that governs particles with varying mass; it is part of the role of Theoretical Physics presenting challenges for Experimental Physics. Moreover, one is reminded that a "particle" can be understood as any physical body whose dimensions are ignored for the sake of a given context.

The position of the CM of this set of N particles is defined as

$$\vec{R}_{CM}(t) \equiv \frac{m_1\,\vec{r}_1(t) + m_2\,\vec{r}_2(t) + \cdots + m_N(t)\,\vec{r}_N(t)}{M_{CM}(t)},$$

(86)

in which

$$M_{CM}(t) \equiv m_1 + m_2 + \cdots + m_N(t) \qquad (87)$$

is the mass of the fictitious particle whose position coincides with the location CM at any time t, and this mass is equal to the total mass of the system, similar to (1). The time dependence of $\vec{R}_{CM}(t)$ comes not only from the movement of particles itself (i.e., from the trajectories $\vec{r}_i(t)$), but also from the time dependence of the mass of the N-th particle. Using (86) as a definition for the CM position of the set formed by the N particles, the location for the application of their total weight remains in $\vec{R}_{CM}(t)$.

To the ficticious particle located at the CM we assign a linear momentum given by

$$\vec{P}_{CM}(t) = M_{CM}(t)\,\vec{V}_{CM}(t), \tag{88}$$

in which $\vec{V}_{CM}(t)$ is the CM velocity,

$$\begin{aligned}
\vec{V}_{CM}(t) &= \frac{d\vec{R}_{CM}(t)}{dt} = \frac{d}{dt}\left[\frac{m_1\,\vec{r}_1(t) + m_2\,\vec{r}_2(t) + \cdots + m_N(t)\,\vec{r}_N(t)}{M_{CM}(t)}\right] \\
&= \frac{m_1\,\vec{v}_1(t) + m_2\,\vec{v}_2(t) + \cdots + m_N(t)\,\vec{v}_N(t)}{M_{CM}(t)} \\
&\quad - \frac{1}{M_{CM}(t)^2}\frac{dM_{CM}(t)}{dt}\left[m_1\,\vec{r}_1(t) + m_2\,\vec{r}_2(t) + \cdots + m_N(t)\,\vec{r}_N(t)\right] + \frac{1}{M_{CM}(t)}\frac{dm_N(t)}{dt}\,\vec{r}_N(t). \quad (89)
\end{aligned}$$

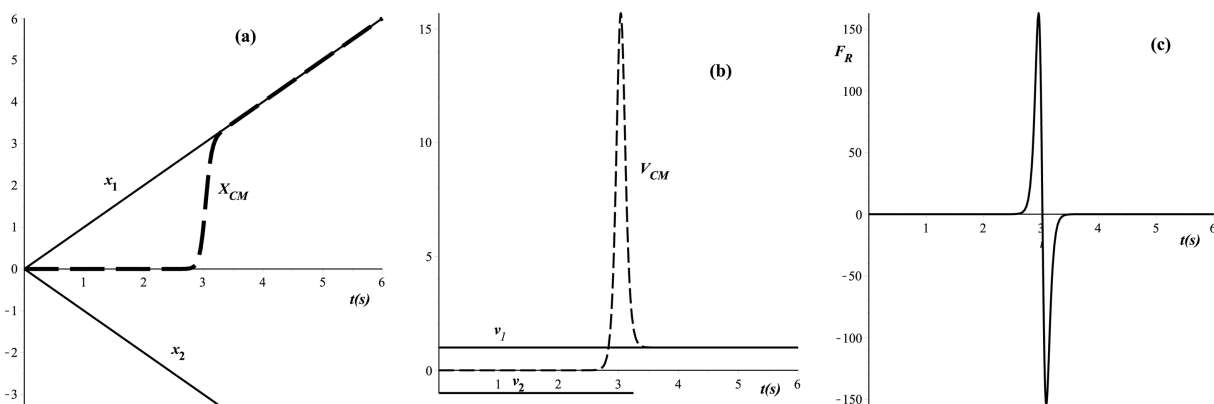**Figure 1:** a) The position of the particles 1 and 2 (solid lines), and the CM (dashed line) of S as functions of t. b) The time dependence of the velocities of the particles (solid lines) that belong to S and its CM (dashed line). c) The curve of $F_R \times t$, $F_R(t)$ being the resultant force applied on the fictitious particle at the CM during the whole movement.

However, from (86),

$$-\frac{1}{M_{CM}(t)^2}\frac{dM_{CM}(t)}{dt}\left[m_1\,\vec{r}_1(t)+m_2\,\vec{r}_2(t)+\cdots+m_N(t)\,\vec{r}_N(t)\right]=-\frac{1}{M_{CM}(t)}\left(\frac{dM_{CM}(t)}{dt}\right)\,\vec{R}_{CM}(t),\quad(90)$$

so that

$$\vec{V}_{CM}(t)=\frac{m_1\,\vec{v}_1(t)+m_2\,\vec{v}_2(t)+\cdots+m_N(t)\,\vec{v}_N(t)}{M_{CM}(t)}+\frac{1}{M_{CM}(t)}\left[\frac{dm_N(t)}{dt}\,\vec{r}_N(t)-\frac{dM_{CM}(t)}{dt}\,\vec{R}_{CM}(t)\right].\quad(91)$$

From (87) we have

$$\frac{dM_{CM}(t)}{dt}=\frac{dm_N(t)}{dt},\quad(92)$$

which results in

$$\vec{V}_{CM}(t)=\frac{m_1\,\vec{v}_1(t)+m_2\,\vec{v}_2(t)+\cdots+m_N(t)\,\vec{v}_N(t)}{M_{CM}(t)}+\frac{1}{M_{CM}(t)}\frac{dm_N(t)}{dt}\left[\vec{r}_N(t)-\vec{R}_{CM}(t)\right].\quad(93)$$

The first term in the r.h.s. of (93) is the ordinary result for a system of particles with constant mass. The second term accounts for the dependence of the velocity of the CM on the variation of the mass of the N-th particle. This previous result coincides with (42) when $N=L$, and $m_L(t)=m_L\cdot f(\bar{t}-t)$, in which m_L is a constant value and $f(\bar{t}-t)$ is the transition function described in subsection 2.1. To verify that both results are equal, we should notice that the coefficient of the last term on the r.h.s. of (42) can be rewritten as

$$-\frac{m_L}{M_1(t)}\frac{df(t-\bar{t})}{dt}=\frac{1}{M_1(t)}\frac{dm_L(t)}{dt}.\quad(94)$$

The linear momentum (88) then amounts to

$$\vec{P}_{CM}(t)=m_1\,\vec{v}_1(t)+m_2\,\vec{v}_2(t)+\cdots+m_N(t)\,\vec{v}_N(t)+\frac{dm_N(t)}{dt}\left[\vec{r}_N(t)-\vec{R}_{CM}(t)\right]\quad(95)$$

or

$$\vec{P}_{CM}(t)=\vec{P}(t)+\frac{dm_N(t)}{dt}\left[\vec{r}_N(t)-\vec{R}_{CM}(t)\right]\quad(96)$$

in which $\vec{P}(t)$ is the total linear momentum of the system,

$$\vec{P}(t)=\vec{p}_1(t)+\vec{p}_2(t)+\cdots+\vec{p}_N(t)=\sum_{i=1}^{N}m_i(t)\vec{v}_i(t).\quad(97)$$

Eq. (96) shows that the linear momentum of the CM and the total momentum of the system only coincide under at least one of two situations: when all particle masses are constant; or when the positions of the CM and of the particle with variable mass are the same for all t. Newton's Second Law for the fictitious particle located at the CM and with varying mass, $M_{CM}(t)$, is

$$\frac{d(\vec{P}_{CM}(t))}{dt}=\vec{F}_R^{(CM)}(t)\quad(98)$$

$$=\frac{d(M_{CM}(t)\,\vec{V}_{CM}(t))}{dt},\quad(99)$$

in which $\vec{F}_R^{(CM)}(t)$ is the resulting force acting on the fictitious particle located at the CM of the set of N particles.

The equation of motion for the fictitious particle at the CM is obtained by differentiating (96), and taking (88) into account,

$$M_{CM}(t)\vec{A}_{CM}(t)=\frac{d\vec{P}(t)}{dt}+\frac{d^2m_N(t)}{dt^2}\left[\vec{r}_N(t)-\vec{R}_{CM}(t)\right]$$

$$+\frac{dm_N(t)}{dt}\left[\vec{v}_N(t)-\vec{V}_{CM}(t)\right]-\frac{dM_{CM}(t)}{dt}\vec{V}_{CM}(t),\quad(100)$$

in which

$$\vec{A}_{CM}(t)=\frac{d\vec{V}_{CM}(t)}{dt}\quad(101)$$

is the acceleration of the CM.

As

$$\frac{d\vec{P}(t)}{dt} = \sum_1^N \frac{d\vec{p}_i(t)}{dt} \tag{102}$$

we may insert the equation of motion (85) of the i-th particle into (100) we obtain

$$M_{CM}(t)\vec{A}_{CM}(t) = \sum_{i=1}^N \left[\sum_{\substack{j=1 \\ j \neq i}}^N \vec{F}_{j \to i}^{(int)}(t) + \vec{F}_i^{(ext)}(t) \right] + \frac{d^2 m_N(t)}{dt^2} \left[\vec{r}_N(t) - \vec{R}_{CM}(t) \right]$$

$$+ \quad \frac{d m_N(t)}{dt} \left[\vec{v}_N(t) - \vec{V}_{CM}(t) \right] - \frac{d M_{CM}(t)}{dt} \vec{V}_{CM}(t). \tag{103}$$

As in (54), Newton's Third Law and a little algebra shows that the contribution of internal forces vanishes,

$$\sum_{i=1}^N \sum_{\substack{j=1 \\ j \neq i}}^N \vec{F}_{j \to i}^{(int)}(t) = \vec{0}, \tag{104}$$

so that, for the case of particles with varying mass, internal forces do not contribute to the dynamics of the CM. Together with (92), that leads to the equation of motion of the CM

$$M_{CM}(t)\,\vec{A}_{CM}(t) = \sum_1^N \vec{F}_i^{(ext)}(t)$$

$$- \frac{d^2 m_N(t)}{dt^2} \left(\vec{R}_{CM}(t) - \vec{r}_N(t) \right)$$

$$- \frac{d m_N(t)}{dt} \left(2\,\vec{V}_{CM}(t) - \vec{v}_N(t) \right). \tag{105}$$

One of the differences between the previous result (105) and (68) is the absence of the term $+\frac{d m_N(t)}{dt} \cdot \vec{v}_N(t)$. This difference happens because in this section the mass of the N-th particles is time dependent and the physical forces that act on it are responsible for the time variation of the product of the mass by the vector velocity. In subsection 2 the mass of the migration particles does not vary and the physical forces only drive the time dependence of the vector velocity of the particle.

If the mass $m_N(t)$ of the N-th particle is of the form

$$m_N(t) = m_N^{(0)}\, f(t - \bar{t}) + m_N^{(f)}, \tag{106}$$

in which $m_N^{(0)}$ and $m_N^{(f)}$ are constants and $f(t)$ is as in (5) (see Appendix B for a particular example of the transition function $f(t)$), then the extra terms of the equation of motion (105) are non-vanishing only during the transition interval, for which

$$\frac{d m_N(t)}{dt} \neq 0 \tag{107}$$

and

$$\frac{d^2 m_N(t)}{dt^2} \neq 0. \tag{108}$$

The term linear in $\left(\vec{R}_{CM}(t) - \vec{r}_N(t) \right)$ on the r.h.s. (105) of could be interpreted as a repelling or restoring force, either pulling the CM away from or pushing it towards its "normal" position, according to the instantaneous sign of $\frac{d^2 m_N(t)}{dt^2}$. The term in (105) proportional to $\left(2\,\vec{V}_{CM}(t) - \vec{v}_N(t) \right)$ could be interpreted as a damping force acting on the CM. Notice that this term differs from its correspondent one in Eq. (68).

The solution of (105) would require the knowledge of $\vec{r}_N(t)$ and $\vec{m}_N(t)$ for all t. In section 4 we will exploit some simple numerical solutions for the unidimensional case of free particles.

4. Examples: the CM of a set of three particles

The simplest example of the behavior of the CM of subsystem left by a migrating particle is provided by considering a unidimensional system with three free particles. These three particles could be sliding over a horizontal air rail with negligible friction. The three particles leave from the same space point on the air rail, but with different velocities. Let us imagine that those particles have positions given by

$$x_1(t) = t, \tag{109}$$

$$x_2(t) = \frac{t}{2}, \tag{110}$$

$$x_3(t) = -t. \tag{111}$$

in convenient units of length and time. Let the subsystem S_1 be composed of permanent particles 1 and 3, with masses $m_1 = 1$ and $m_3 = 2$, respectively, and the migrating particle be the particle 2, with mass equal to $m_2 = 1$. All the three particles have constant mass. Let the transition function that describes the movement of the particle 2 out of the subsystem S_1 be $f(5 - t)$, in which

$$f(\tau) = \frac{1}{2} + \frac{1}{2} \tanh\left(2\,\tau \right). \tag{112}$$

The transition occurs about the instant $\bar{t} = 5$, with interval amplitude $\Delta t \approx 1$, as Fig. 2 shows. The variable t is in some suitable unit of time.

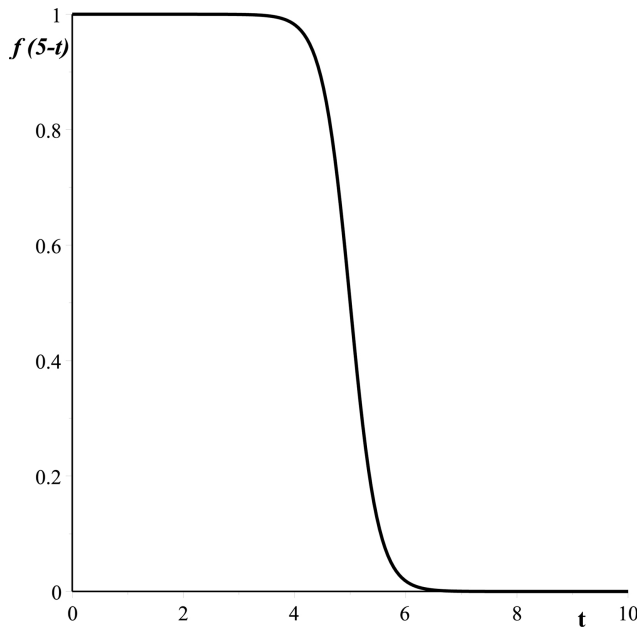The mass of the CM of the subsystem S_1, at each time t, is

Figure 2: The transition function $f(5-t)$ that describes the location of the particle 2 at each moment t. During the time interval $t \in [4, 6]$, approximately, the particle 2 leaves the subsystem S_1.

$$M_1(t) = m_1 + f(5-t) \cdot m_2 + m_3$$
$$= \frac{1}{2} \left(7 + \tanh(2(5-t)) \right). \quad (113)$$

The position of the CM of S_1, $X_{CM}^{(1)}(t)$, is

$$X_{CM}^{(1)}(t) = \frac{m_1\, x_1(t) + f(5-t) \cdot m_2\, x_2(t) + m_3\, x_3(t)}{m_1 + f(5-t) \cdot m_2 + m_3}$$
$$= \frac{t}{2} \times \left(\frac{\tanh\left(2\left(5-t\right) \right) - 3}{\tanh\left(2\left(5-t\right) \right) + 7} \right). \quad (114)$$

Fig. 3 shows the value of the position of the three particles along the x-axis and that of the CM of the system of the particles that belong to the set S_1 at each time t, in the plane (t, x). The curve of the CM of the particle set shows the position of the application of their total weight at instant t.

In the case discussed up to now, the masses of the three particles are constant. Applying Newton's Second Law, that drives the time evolution of theses particles, we obtain

$$m_i \frac{d^2 x_i(t)}{dt^2} = F_R^{(i)}(t), \quad i = 1, 2, 3, \quad (115)$$

where $F_R^{(i)}(t)$ is the resultant force that acts on the i-th particle at time t, and $\dfrac{d^2 x_i(t)}{dt^2} = a_i(t)$, where $a_i(t)$ is the

instant acceleration of that particle. From (109) - (111) we obtain

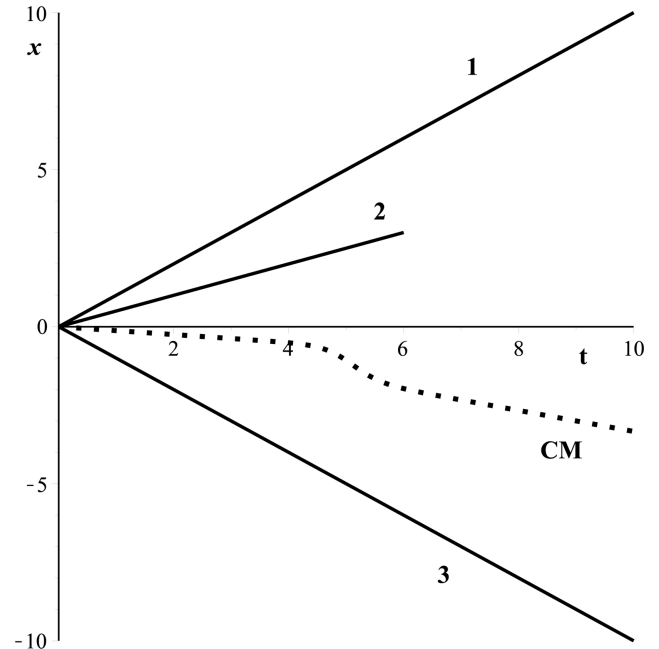$$\frac{d^2 x_i(t)}{dt^2} = 0, \quad i = 1, 2, 3, \quad (116)$$

Figure 3: Positions of the 3 particles (solid lines) as a function of the time t while they belong to the subsystem S_1, and of the CM of subsystem S_1 (dotted line), shown in the (t, x) plane. The permanent particles are 1 and 3; as particle 2 leaves the system, the location of the CM as a function of time is also shown. The transition interval when the mass of particle 2 vanishes corresponds approximately to $t \in [4, 6]$. For $t \gtrsim 6$, particle 2 does not belong to S_1 anymore.

showing that

$$F_R^{(i)}(t) = 0, \quad i = 1, 2, 3, \quad (117)$$

at any instant t. Consequently the three mentioned particles are free particles with null resultant forces acting on each one of them. During the whole time interval $t \in [0, +\infty)$, the plot of the positions $x_i(t)$, for $i \in \{1, 2, 3\}$ in the (t, x) plane is made of straight lines, as shown in Fig. 3. After $t = 6$, particle 2 no longer belongs to the particle system that is under observation.

We apply (68), that drives the dynamics of the position of the CM of S_1, to derive the position of $X_{CM}^{(1)}(t)$ during the time interval $t \in (4, 6)$, when $X_{CM}^{(1)}(t)$ does not follow a straight line. The acceleration of the CM of the subsystem S_1, $a_{CM}^{(1)}(t)$, is obtained from (68), and in the absence of forces applied to the physical particles, see (117), it becomes,

$$M_1(t)a_{CM}^{(1)}(t) = -\left(\frac{d^2 M_1(t)}{dt^2}\right) \cdot (X_{CM}(t) - x_2(t)) - 2\left(\frac{dM_1(t)}{dt}\right) \cdot (V_{CM}(t) - v_2(t)), \tag{118}$$

where $M_1(t)$ is given by (113), $V_{CM}^{(1)}(t)$ is the velocity of the CM of the subset S_1,

$$V_{CM}^{(1)}(t) = \frac{dX_{CM}^{(1)}(t)}{dt} = \frac{-1 - 20t + 4\sinh(2(5-t))\cosh(2(5-t)) - 20\cosh^2(2(5-t))}{2\left(7\cosh(2(5-t)) + \sinh(2(5-t))\right)^2}. \tag{119}$$

We plot in Fig.4 the time dependence of the velocity of the CM of S_1 during the time interval $t \in [0, 10]$. For $t \lesssim 4$ the velocity of the CM of S_1 is approximately constant, $V_{CM} \approx -0.125$, and for $t \gtrsim 6$ $V_{CM}^{(1)}(t)$, becomes approximately constant again, $V_{CM} \approx -0.333$, but with a different value from that of the beginning of the movement. During that time interval, the velocity $V_{CM}(t)$ varies in time, increasing its modulus for $t \in [4, 5]$ and decreasing for $t \in (5, 6)$ up to returning to a new constant value. Notice that the kinetic energy of the fictitious particle located at the CM is not conserved, although the physical particles that belong to the set are all free particles. Remember that we are creating the movement of a fictitious particle from the information of the physical particles but it does not mean that they have the same behaviour.

The expression of $x_2(t)$ is given by (110) and $v_2(t)$ is the instantaneous velocity of particle 2

$$v_2(t) = \frac{dx_2(t)}{dt} = \frac{1}{2}. \tag{120}$$

From (113) and (112), we obtain

$$-\frac{d^2 M_1(t)}{dt^2} = -\frac{d^2 f(5-t)}{dt^2}$$
$$= 4\tanh(2(5-t)), \tag{121}$$

and

$$-2\frac{dM_1(t)}{dt} = -2\frac{df(5-t)}{dt}$$
$$= 2\,\text{sech}(2(5-t)). \tag{122}$$

In Figs. 5a and 5b we plot the coefficients $-\frac{d^2 M_1(t)}{dt^2}$ and $-2\frac{dM_1(t)}{dt}$, respectively. From these plots we verify that the two terms on the r.h.s. on (118) are non-null in the interval $t \in (4, 6)$. It is during this time interval that the movement of the CM of the subset S_1 does not describe a free particle with constant mass.

From Fig.5a we obtain that $-\frac{d^2 M_1(t)}{dt^2} > 0$ for $t \in (4, 5)$, such that the linear term $(X_{CM}(t) - x_2(t))$ in

(118) is a repelling force during this interval of time, when the CM of this system leaves the first straight line for $t \lesssim 4$. This interpretation is compatible with Fig.4, since the modulus of the velocity of the CM of the system, $|V_{CM}(t)|$, increases during this interval of time. For the time interval $t \in (5, 6)$, Fig.5a shows that $-\frac{d^2 M_1(t)}{dt^2} < 0$ and the linear term in (118) becomes a restoration force, whose coupling coefficient depends on time.
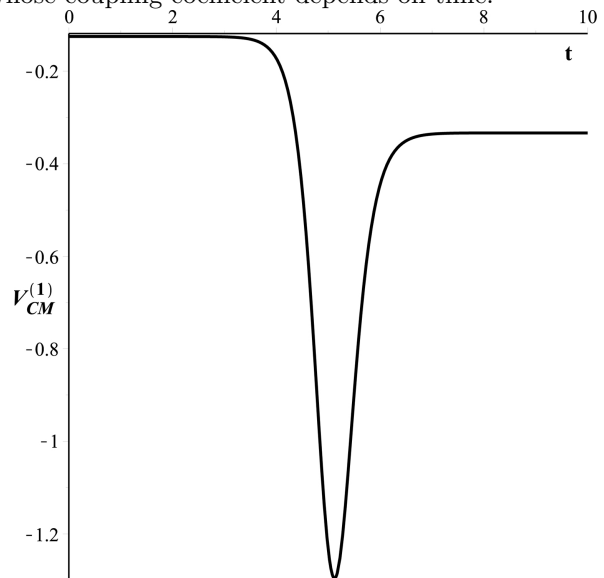

Figure 4: The time dependence of the velocity of the CM of subsystem S_1, $V_{CM}^{(1)}(t)$, during the time interval $t \in [0, 10]$.

Fig.5b shows that during the interval $t \in (4, 6)$ we have $-2\frac{dM_1(t)}{dt} > 0$ and that the movement of the CM of S_1 is under the action of an effective dissipative force that decreases the modulus of its velocity, $|V_{CM}^{(1)}(t)|$. After $t \gtrsim 5$, when the effective linear force on the CM of S_1 becomes a restoring force, the modulus of $V_{CM}^{(1)}(t)$ diminishes.

The expression (114) can also give the location of the CM of a set of three particles, whose positions $x_i(t)$, with $i = 1, 2, 3$, are given by eqs.(109) - (111), and with
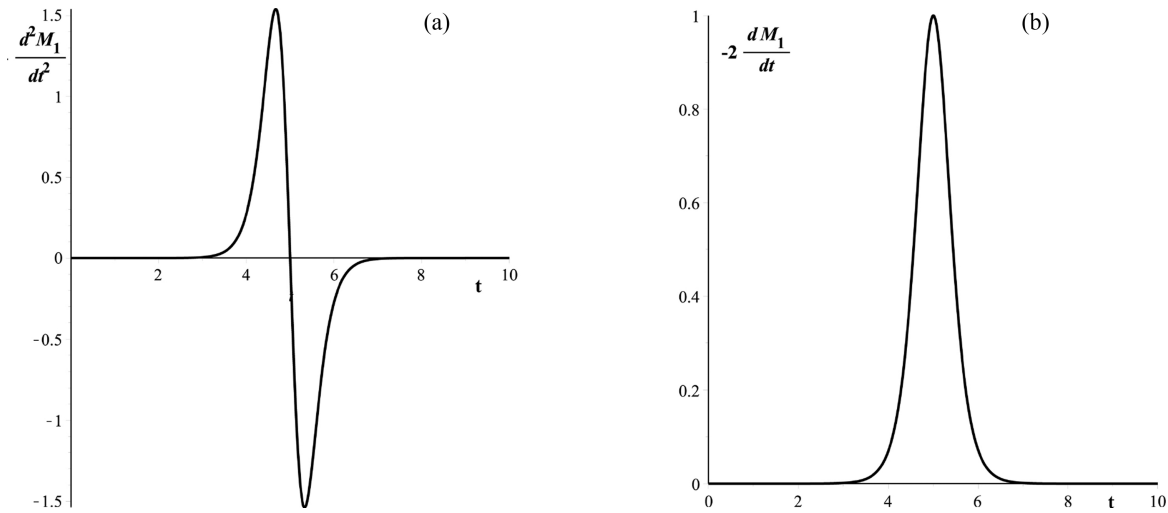
Figure 5: *a*) The plot of the coefficient $-\dfrac{d^2 M_1(t)}{dt^2}$ as a function of time t. The expression of this coefficient is given by (121). *b*) The plot of the coefficient $-2\,\dfrac{dM_1(t)}{dt}$ as a function of time t, see (122).

respective masses

$$m_1 \;=\; 1, \tag{123}$$

$$m_2(t) \;=\; \frac{1}{2} + \frac{1}{2}\,\tanh(2(5-t)), \tag{124}$$

$$m_3 \;=\; 2. \tag{125}$$

The plot of $m_2(t)$ coincides with the curve $f(2(5-t))$ shown in Fig.2. The positions of these three particles, and the CM of this set of particles, are shown in Fig.3.

In section 3 we pointed out that Eq. (105), that drives the dynamics of the CM of the system when the mass of one of its particles varies in time, differs from Eq. (68), that determines the instantaneous acceleration of the CM of a system with a migrating particle. How is it possible that the position of the CM of the subsystem S_1 with a migrating particle 2, discussed previously, be the same as the CM of a set of the three particles with masses (123) - (125) that follow the paths (109) - (111)?

Let us discuss in detail the movement of the three particles that follow the positions (109) - (111), whose masses are given by (123) - (125). We call system S this set of three particles. Newton's Second Law written for the particles 1 and 3 reads

$$m_i \frac{d^2 x_i(t)}{dt^2} = F_R^{(i)}(t), \quad i = 1,3. \tag{126}$$

Form (109) and (111), we obtain

$$F_R^{(i)}(t) = 0, \quad i = 1,3; \tag{127}$$

that means that these two particles are free particles during the whole movement. The dynamics of particle 2, whose mass varies in time, see (124), is

$$\frac{d(m_2(t)\,v_2(t))}{dt} = F_R^{(2)}(t), \tag{128}$$

where from (110) we have

$$v_2(t) = \frac{dx_2(t)}{dt} = \frac{1}{2}, \tag{129}$$

and v_2 has some suitable unit of velocity.
Substituting the value of $m_2(t)$, see (124), and the result (129) on the r.h.s. of (128), we have

$$
\begin{aligned}
F_R^{(2)}(t) &= \frac{d}{dt}\left[\frac{1}{2} + \frac{1}{2}\,\tanh(2(5-t))\right] \\
&= \frac{1}{2}\,\frac{1}{\cosh^2(2(5-t))} \quad \Rightarrow \\
\Rightarrow \quad F_R^{(2)}(t) &= -\frac{1}{2}\,\mathrm{sech}^2(2(5-\mathrm{t})), \tag{130}
\end{aligned}
$$

From Fig.6 we verify that during the time interval $t \in (4,6)$, the force that acts on particle 2 is non null and the movement of this particles in this time interval is not of a free particle. The force $F_R^{(2)}(t)$ is responsible for the particle 2 to maintain a constant velocity, see (120), including the time interval (4,6) when its mass is varying in time. Outside this time interval the movement of particle 2 coincides with that of a free particle.

The sum of the masses (123) - (125) gives the mass of the CM of system S, that is

$$
\begin{aligned}
M_{CM}(t) &= m_1 + m_2(t) + m_3 \\
&= \frac{1}{2}\left(7 + \tanh(2(5-t))\right) \\
&= M_1(t). \tag{131}
\end{aligned}
$$

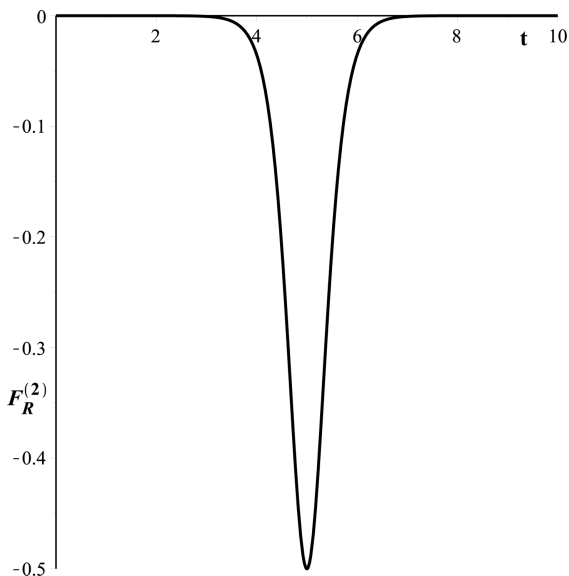The dynamics of the CM of the system S is given by (105), that in the present case becomes

Figure 6: The curve $F_R^{(2)}(t)$ as a function of the time t. $F_R^{(2)}(t)$ is the force that acts on particle 2 during its movement.

$$M_{CM}(t)\,A_{CM}(t) = F_R^{(2)}(t) - \frac{d^2 m_2(t)}{dt^2}$$
$$\times\,[X_{CM}(t) - x_2(t)] - \frac{dm_2(t)}{dt}\,[2V_{CM}(t) - v_2]. \quad (132)$$

From (124) we obtain

$$-\frac{d^2 m_2(t)}{dt^2} \;=\; 4\,\tanh(2(5-t)) \qquad (133)$$

$$-\frac{dm_2(t)}{dt} \;=\; \operatorname{sech}^2(2(5-7)). \qquad (134)$$

Note that $F_R^{(2)}(t)$, see (128), can also be written as

$$F_R^{(2)}(t) = \frac{dm_2(t)}{dt}\,v_2, \qquad (135)$$

that being substituted into the r.h.s. of (132) yields

$$M_{CM}(t)\,A_{CM}(t) = -\frac{d^2 m_2(t)}{dt^2}\,[X_{CM}(t) - x_2(t)]$$
$$-2\,\frac{dm_2(t)}{dt}\,[V_{CM}(t) - v_2]. $$
$$\qquad (136)$$

Since $M_{CM}(t) = M_1(t)$ and

$$\frac{d^2 m_2(t)}{dt^2} = \frac{d^2 M_1(t)}{dt^2} \qquad (137)$$

$$\frac{dm_2(t)}{dt} = \frac{dM_1(t)}{dt}, \qquad (138)$$

which shows that the differential equations (118) and (132) are identical, and the CM of the two systems of particles described in this paper follow the same trajectory.

Finally in Fig.7 we compare the linear momentum of the the CM of the system S, $P_{CM}(t)$,

$$P_{CM}(t) \equiv M_{CM}(t)\,V_{CM}(t), \qquad (139)$$

where $M_{CM}(t)$ and $V_{CM}(t)$ are given by (131) and (119), respectively, and the total linear momentum of the three particles, $P_T(t)$,

$$P_T(t) \equiv m_1 v_1 + m_2(t) v_2 + m_3 v_3, \qquad (140)$$

with $v_i(t) = \dfrac{dx_i(t)}{dt}$, $i = 1, 2, 3$, and the positions $x_i(t)$ given by (109) - (111).

From Fig.7 we verify that for $t \lesssim 4$ and $t \gtrsim 6$ the momentum of the CM of S, $P_{CM}(t)$, and the total linear momentum of three particles that belong to S, $P_T(t)$, coincide. But during the interval $t \in [4, 6]$ these two functions of linear momentum are quite distinct, as mentioned previously.
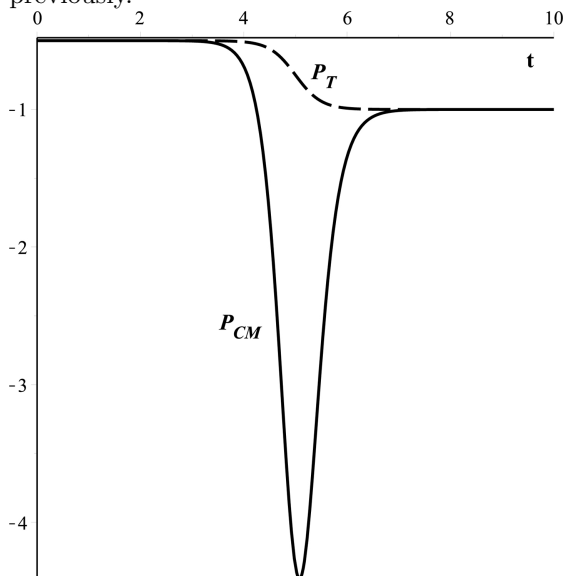

Figure 7: The curves of $P_{CM}(t)$ (solid line) of the CM of the system S and $P_T(t)$ (dashed line), the total linear momentum of the three particles in the set S, in the interval $t \in [0, 10]$.

The function $P_{CM}(t)$ and $P_T(t)$ coincide when the masses of the particles do not vary in time, or when the number of particles in the system is constant in time. We point out that during the whole movement the application of the total weight force of the particle set coincide with the position of $X_{CM}^{(1)}$.

5. Conclusions

A well-known result in the study of classical particle systems is that of the identity of the total momentum of a system and the momentum of the center of mass (CM), which is a fictitious particle with some defined properties to represent, in some sense, the system behavior.

When the number of particles of a system or their masses vary in time, however, this identity may not hold true, at least during the time interval in which variations occur.

We have examined two situations. The first is that of a system composed of two subsystems with non-overlapping permanent particles, but for a single particle that migrates from one subsystem to the other. We have obtained the equation of motion for the CM of one of the subsystems, showing that apart from the expected forces, other fictitious forces appear in the equation. Then we have studied the case of a single system in which one of its particles has a varying mass, obtaining a similar, but not identical, equation of motion for its CM. In both cases we have used smooth transition functions to describe continuous variations. The fictitious transient forces that appear are related not only to the time-dependence of the number/mass of particles, but also on the position and velocities of the CM, as well as on the migrating particle or variable-mass particle, analogous to instantaneous restoration and damping forces. We include in the presentation an example with three particles moving on an air rail, where we can describe the physical situation in terms of a migrating particle, or when the mass of one of the particles in the set varies in time.

Although we have extended the CM definition of a set of particles presented in Physics textbooks to include in this set a migration particle or a mass particle varying in time, the position where the total weight force of this system is applied coincides with that of our extended CM concept for these types of set.

It is interesting that students realize that they can raise new questions using the knowledge they have just learned in Newtonian Mechanics. Mainly they can find the answers to their own questions using the three Laws of Mechanics studied in the first classes of Physics.

Supplementary material

The following online material is available for this article:
Appendix A - The position of the CM of a system of particles with variable masses
Appendix B - An explicit example for the transition function $f(t)$

References

[1] H.M. Nussenzweig, *Curso de Física Básica* (Edgar Blücher, São Paulo, 2002), 4ª ed.

[2] J. Barcelos-Neto, *Mecânica Newtoniana, Lagrangiana e Hamiltoniana* (Ed. Livraria da Física, São Paulo, 2013), 2ª ed .

[3] N.A. Lemos, *Mecânica Analítica* (Ed. Livraria da Física, São Paulo, 2004), 1ª ed.

[4] L.D. Landau and E.M. Lifshitz, *Mechanics* (Butterworth-Heinenann, Oxford, 1976), 3ª ed.

[5] W. Greiner, *Classical Mechanics, Systems of Particles and Hamiltonian Dynamics* (Springer, New York, 2003).

[6] R.P. Feynman, R.B. Leighton and M. Sands, *The Feynman Lectures on Physics* (Addison-Wesley, Reading, 1965).

[7] W.T. Thomson, AIAA Journal **4**, 766 (1966).

[8] B. Samardžija and S. Šegan, Publ. Astron. Obs. Belgrade **91**, 97 (2012).

[9] F.O. Eke and T.C. Mao, Int. J. of Mechanical Engineering Education **30**, 123 (2002).

[10] A. Nanjangud and F.O. Eke, Int. Rev. of Aerospace Engineering **5**, 256 (2012).