Revista Brasileira de Ensino de Física, vol. 43, e20210126 (2021)
www.scielo.br/rbef
DOI: https://doi.org/10.1590/1806-9126-RBEF-2021-0126

Exact analysis of spontaneous phase synchronization of two identical coupled electrical linear LC oscillators

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Received on March 31, 2021. Revised on April 27, 2021. Accepted on April 27, 2021.

This paper provides contributions to the study of the phase synchronization transient of two identical linear LC oscillators, coupled through a resistor. The main contribution is obtaining an exact equation that represents, as a function of time, the evolution of the phase angle between the two oscillators, from the initial condition until the synchronization. It is demonstrated that the system has two equilibrium points, one unstable when the phase angle is equal to π, and the other stable when it is equal to zero and to which the phase converges for any initial value other than π. It is also demonstrated that the dynamics of the two coupled oscillators can be described by the Kuramoto model. An analysis of the energy dissipated during synchronization is presented and the concept of synchronization energy is introduced. The results of the theoretical analysis are verified with the use of computer simulation.
Keywords: Coupled oscillators, Kuramoto, linear LC oscillators, phase synchronization.

1. Introduction

The synchronization between oscillators mutually coupled by some type of interconnection is a concept that is widespread in science and engineering. Currently, it is perhaps the most studied dynamic concept in biological systems, neuroscience, chemistry, physics, astronomy and engineering. Popular examples include synchronization of metronomes, clocks, heartbeats, fireflies, neurons, and earthquakes. These phenomena are universal and can be understood within a common structure based on nonlinear dynamics [1–5].

The electrical or electronic oscillators are suitable for the experimental verification of theoretical studies, as they are easy to build and the electrical quantities are easily accessible and observable.

A review of the technical literature published in recent decades reveals a large number of publications dealing with the synchronization of these oscillators, but always focusing on non-linear oscillators, particularly the Van de Pol oscillator.

In [6], where a behavioral analysis of two linear LC oscillators associated in series with a non-linear resistor is reported, it is shown that with an appropriate parametric combination, the two oscillators can operate synchronously. In [7], the possibility of simultaneous multimode oscillations in an arbitrary number of van by Pol oscillators with the same natural oscillating frequency, mutually coupled by inductances or capacitances, is investigated. The conditions required for the phase synchronization of systems of identical oscillators are discussed in [8]. The results of an investigation on the stability of various oscillatory modes of an inductively coupled ring of Van der Pol oscillators are reported in [9]. A piecewise-linear dynamical system two-coupled relaxation oscillator is studied in [10]. The analysis of N Wien-bridge oscillators with the same natural frequency mutually coupled by one resistor in presented in [11]. The study of N oscillators which have the same natural frequency mutually coupled by one negative resistor is described in [12]. The investigation of resistively coupled nonlinear oscillators with different frequencies is reported in [13]. The stability of operation of two coupled and synchronized Van deer Pol oscillators, using the quasilinear approximation method, is addressed in [14]. The synchronization of two mutually coupled second-order Duffing-type nonlinear oscillators is investigated in [15]. Experimental results for the synchronization of several Van der Pol oscillators coupled in various forms are reported in [16]. An analysis of the synchronization and amplitude death phenomena in a pair of coupled Van der Pol oscillators under two types of coupling is given in [17].

All of the results reported in these publications were obtained from the study of non-linear electrical or electronic oscillators with a focus on the conditions required for the synchronization. None of the aforementioned authors present an exact analysis of the phase behavior as a function of time, mainly because it is difficult (or impossible) to obtain analytical solution, due to the non-linearity of the circuits.

A comprehensive analysis of the collective behavior of identical linear harmonic oscillators coupled by both restorative and dissipative components is provided

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in [18], and synchronization in an array of identical linear oscillators of arbitrary order, coupled through a dynamic network comprising dissipative connectors and restorative connectors is studied in [19]. In both publications, the main focus is to present necessary and sufficient condition for synchronization.

In this article, the results on the theoretical studies of the phenomenon of spontaneous synchronization between two particular types of linear electric oscillators are reported. with emphasis on the phase behavior between the two oscillators as a function of time. Based on a literature search, the exact analysis of the spontaneous phase synchronization of the two identical coupled linear LC oscillators studied, does not appear to have been previously reported.

2. Analysis

Figure 1 shows a system formed by two ideal linear oscillators, coupled by the resistor R from the moment the switch S is closed. Each oscillator is formed by a resonant LC circuit in parallel, with $L_1 = L_2 = L$ and $C_1 = C_2 = C$. The peak values of the voltages on the capacitors, before the synchronization transient, are identical and equal to V_p. The energies initially stored in each LC pair are identical to each other. Before closing the switch S, the two oscillators oscillate freely, with identical frequencies equal to $f = \frac{1}{2\pi\sqrt{LC}}$. At the instant $t = 0$ the switch S is closed, with an initial phase angle ϕ_o between the voltages V_{c1} and V_{c2} assuming any value in the interval $0 \leq \phi_o \leq \pi$.

2.1. Analysis of voltage and power in the resistor R

The equations that describe the circuit are

$$\frac{di_{L1}}{dt} = \frac{v_{C1}}{L} \tag{1}$$

$$\frac{di_{L2}}{dt} = \frac{v_{C2}}{L} \tag{2}$$

$$\frac{dv_{C1}}{dt} = -\frac{i_{L1}}{C} - \frac{1}{CR}(v_{C1} - v_{C2}) \tag{3}$$

$$\frac{dv_{C2}}{dt} = -\frac{i_{L2}}{C} + \frac{1}{CR}(v_{C1} - v_{C2}) \tag{4}$$

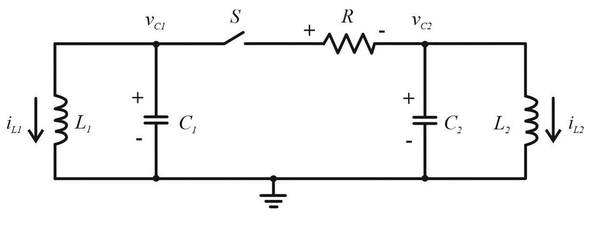

Figure 1: Two coupled identical linear LC oscillators.

Let us define

$$v = v_{C1} - v_{C2} \tag{5}$$

$$i = i_{L1} - i_{L2} \tag{6}$$

Subtracting (2) from (1) we find

$$\frac{di_{L1}}{dt} - \frac{di_{L2}}{dt} = \frac{1}{L}(v_{C1} - v_{C2}) \tag{7}$$

The substitution of (5) and (6) in (7) yields

$$\frac{di}{dt} = \frac{v}{L} \tag{8}$$

Subtraction of (4) from (3) gives

$$\frac{dv_{C1}}{dt} - \frac{dv_{C2}}{dt} = -\frac{1}{C}(i_{L1} - i_{L2}) - \frac{2}{CR}(v_{C1} - v_{C2}) \tag{9}$$

The substitution of (5) and (6) from (9) yields

$$\frac{dv}{dt} = -\frac{i}{C} - \frac{2v}{CR} \tag{10}$$

Equations (8) and (10) describe the equivalent circuit shown in Figure 2.

Differentiating (10) with respect to time and rearranging the terms we find

$$\frac{d^2v}{dt^2} + \frac{2}{RC}\frac{dv}{dt} + \frac{1}{C}\frac{di}{dt} = 0 \tag{11}$$

Substituting (8) in (11), we find

$$\frac{d^2v}{dt^2} + \frac{2}{RC}\frac{dv}{dt} + \frac{1}{LC}v = 0 \tag{12}$$

The natural frequency of the lossless circuit is

$$\omega_o = \frac{1}{\sqrt{LC}} \tag{13}$$

Substitution of (13) in (12) gives

$$\frac{d^2v}{dt^2} + \frac{2}{RC}\frac{dv}{dt} + \omega_o^2 v = 0 \tag{14}$$

The solution of (14) for $i(0) = 0$ is

$$v = V_{po}e^{-\alpha t}\cos(\beta t) \tag{15}$$

where β, which represents the oscillation frequency of the voltage v, is given by

$$\beta^2 = \omega_o^2 - \alpha^2 \tag{16}$$

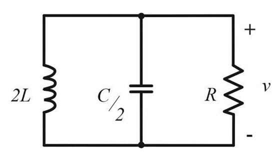

Figure 2: Equivalent circuit described by (8) and (10).

 DOI: https://doi.org/10.1590/1806-9126-RBEF-2021-0126

with

$$\alpha = \frac{1}{RC} \tag{17}$$

The coupling factor is defined by $K = \frac{1}{R}$, which substituted in (16) and (17) results in (18) and (19), respectively.

$$\beta^2 = \omega_o^2 - \left(\frac{K}{C}\right)^2 \tag{18}$$

$$\alpha = \frac{K}{C} \tag{19}$$

Let us consider the case of a weak coupling between the two oscillators in which $0 \leq K \ll 1$. Thus, the decay of the peak value of the voltage v is very slow in relation to the oscillation period of the circuit. The instantaneous power dissipated in the resistor R is given by

$$p = Kv^2 \tag{20}$$

Substituting (15) in (20) we find

$$p = KV_{po}^2 e^{-2\alpha t}\cos(\beta t)^2 \tag{21}$$

The quasi instantaneous average power is defined by

$$P = \frac{K}{2\pi}\int_0^{2\pi} V_{po}^2 e^{-2\alpha t}\cos(\beta t)^2 d(\beta t) \tag{22}$$

Integration of (22) gives

$$P = \frac{K}{2}{V_{po}}^2 e^{-2\alpha t} \tag{23}$$

The voltages across the capacitors before closing the switch S can be described by

$$v_{C1} = V_p \sin\left(\omega t + \frac{\phi_o}{2}\right) \tag{24}$$

and

$$v_{C2} = V_p \sin\left(\omega t - \frac{\phi_o}{2}\right) \tag{25}$$

The voltage v is defined by

$$v = v_{C1} - v_{C2} \tag{26}$$

Substituting (24) and (25) in (26) and rearranging the terms we find

$$V_{po} = 2V_p \sin\left(\frac{\phi_o}{2}\right) \tag{27}$$

Substitution of (27) in (23) yields

$$P = 2K{V_p}^2 \sin\left(\frac{\phi_o}{2}\right)^2 e^{-2\alpha t} \tag{28}$$

Typical waveforms of the voltages across the capacitors, the resistor and the instantaneous power dissipated in the resistor are shown in Figure 3.

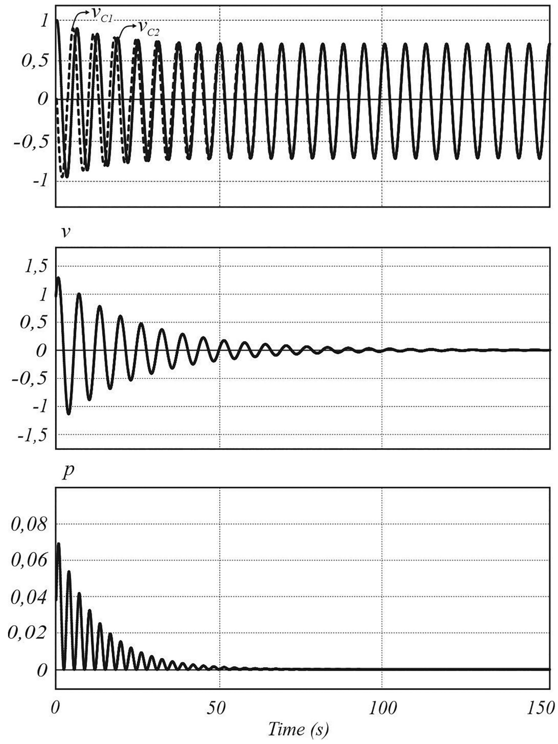

Figure 3: Typical waveforms for the voltages v_{C1} and v_{C2} across the capacitors, the voltage v across the coupling resistor R and the instantaneous power p dissipated in R.

2.2. Voltages across the capacitors after the synchronization transient

Let us define two new quantities, i_x and v_x, by

$$i_x = \frac{i_{L1} + i_{L2}}{2} \tag{29}$$

and

$$v_x = \frac{v_{C1} + v_{C2}}{2} \tag{30}$$

Differentiating (29) and (30) with respect to time, we find respectively

$$\frac{di_x}{dt} = \frac{1}{2}\left(\frac{di_{L1}}{dt} + \frac{di_{L2}}{dt}\right) \tag{31}$$

and

$$\frac{dv_x}{dt} = \frac{1}{2}\left(\frac{dv_{C1}}{dt} + \frac{dv_{C2}}{dt}\right) \tag{32}$$

Substitution of (1) and (2) in (31) and (3) and (4) in (32), gives respectively

$$\frac{di_x}{dt} = \frac{v_x}{L} \tag{33}$$

and

$$\frac{dv_x}{dt} = -\frac{i_x}{C} \tag{34}$$

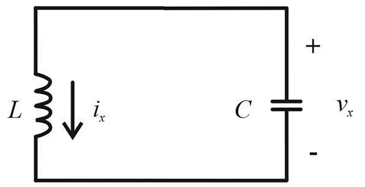

Figure 4: Equivalent circuit described by the second order differential equation (35).

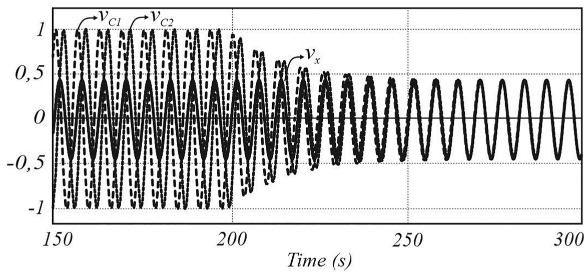

Figure 5: Voltages v_{C1}, v_{C2} and v_x as a function of time, during the phase synchronization transient.

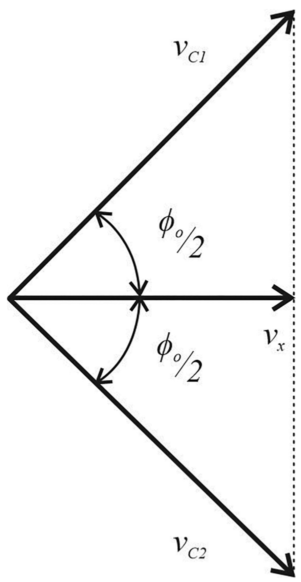

Figure 6: Representation of v_{C1}, v_{C2} and v_x at the instant $t = 0$.

Combining (33) and (34) we find

$$\frac{d^2 v_x}{dt^2} + \omega_o^2 v_x = 0 \tag{35}$$

which describes the equivalent circuit shown in Figure 4.

Figure 5 shows the voltages v_{C1}, v_{C2} and v_x for a typical parametric combination. We observe that the voltages across the two capacitors, when phase synchronization occurs, become equal to the voltage v_x. In addition, as equation (35) indicates, and which is confirmed by the simulation, the amplitude and the voltage phase remain unchanged during the synchronization transient. For this reason, the phase of v_x in this analysis will be considered null and will serve as a reference for the phases of the voltages v_{C1} and v_{C2}.

2.3. Determination of the phase angle between the capacitor voltages

Figure 6 shows the phasor representations of the voltages v_{C1}, v_{C2} and v_x at the instant $t = 0$, when the switch S is closed. The initial phase angles of v_{C1} and v_{C2} with respect to v_x are defined by $\frac{\phi_o}{2}$ and $-\frac{\phi_o}{2}$, respectively.

The voltages v_{C1} and v_{C2} across the two capacitors are given respectively by

$$v_{C1} = V_p F(t) \cos\left(\theta + \frac{\phi}{2}\right) \tag{36}$$

and

$$v_{C2} = V_p F(t) \cos\left(\theta - \frac{\phi}{2}\right) \tag{37}$$

where the angle ϕ is variable with time in the interval $\phi_o \leq \phi \leq 0$.

Substitution of (36) and (37) in (30) gives

$$v_x = V_p F(t) \cos\left(\frac{\phi}{2}\right) \cos\theta \tag{38}$$

As follows from Figure 6,

$$v_x = V_p \cos\left(\frac{\phi_o}{2}\right) \cos\theta \tag{39}$$

Equating (38) and (39) and rearranging the terms we find

$$F(t) = \frac{\cos\left(\frac{\phi_o}{2}\right)}{\cos\left(\frac{\phi}{2}\right)} \tag{40}$$

The substitution of (40) in (36) and (37) results in (41) and (42), respectively.

$$v_{C1} = V_p \frac{\cos\left(\frac{\phi_o}{2}\right)}{\cos\left(\frac{\phi}{2}\right)} \cos\left(\theta + \frac{\phi}{2}\right) \tag{41}$$

$$v_{C2} = V_p \frac{\cos\left(\frac{\phi_o}{2}\right)}{\cos\left(\frac{\phi}{2}\right)} \cos\left(\theta - \frac{\phi}{2}\right) \tag{42}$$

The instantaneous power dissipated in the resistor R is defined by

$$p = K(v_{C1} - v_{C2})^2 \tag{43}$$

Substituting (41) and (42) in (43) we obtain

$$p = 4K{V_p}^2 \sin(\theta)^2 \cos\left(\frac{\phi_o}{2}\right)^2 \left(\frac{1}{\cos\left(\frac{\phi}{2}\right)^2} - 1\right) \tag{44}$$

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The quasi instantaneous average power dissipated in the resistor is defined by

$$P = \frac{1}{2\pi}\int_0^{2\pi} p d\theta \tag{45}$$

Substituting (44) in (45) and performing the integration we obtain

$$P = 2K{V_p}^2 \cos\left(\frac{\phi_o}{2}\right)^2 \left(\frac{1}{\cos\left(\frac{\phi}{2}\right)^2} - 1\right) \tag{46}$$

Equating (28) and (46) we find

$$\cos\left(\frac{\phi_o}{2}\right)^2 \left(\frac{1}{\cos\left(\frac{\phi}{2}\right)^2} - 1\right) = \sin\left(\frac{\phi_o}{2}\right)^2 e^{-2\alpha t} \tag{47}$$

Solving (47) for $\cos\left(\frac{\phi}{2}\right)$ we obtain

$$\cos\left(\frac{\phi}{2}\right) = \frac{\cos\left(\frac{\phi_o}{2}\right)}{\sqrt{\cos\left(\frac{\phi_o}{2}\right)^2 + e^{-2\alpha t}\sin\left(\frac{\phi_o}{2}\right)^2}} \tag{48}$$

With the appropriate trigonometric substitutions and algebraic manipulation, from (48) we find

$$\tan\left(\frac{\phi}{2}\right) = \tan\left(\frac{\phi_o}{2}\right) e^{-\alpha t} \tag{49}$$

Solving (49) for the angle ϕ as function of time we obtain

$$\phi = 2a\tan\left[\tan\left(\frac{\phi_o}{2}\right) e^{-\alpha t}\right] \tag{50}$$

Figure 7 shows the behavior of the angle ϕ as a function of time, for different values of the initial phase ϕ_o. With the exception of the case in which $\phi_o = \pi$, which is an unstable equilibrium point, that is, for any other value of ϕ_o the phases of the two oscillators are synchronized.

Expression (49) is the solution of the first-order non-linear differential equation

$$\frac{d\phi}{dt} = -\alpha\sin(\phi) \tag{51}$$

which is a particular case of the Kuramoto model for two coupled oscillators [20].

Note that the circuit formed by the two linear oscillators coupled by a resistor, shown in Fig. 1, is linear and that the relationships between currents and voltages are described by the system of linear differential equations (1)–(4). However, the behavior of the phase angle between voltages (or currents) is described by the nonlinear differential equation (51), for which the solution is the transcendental equation (50).

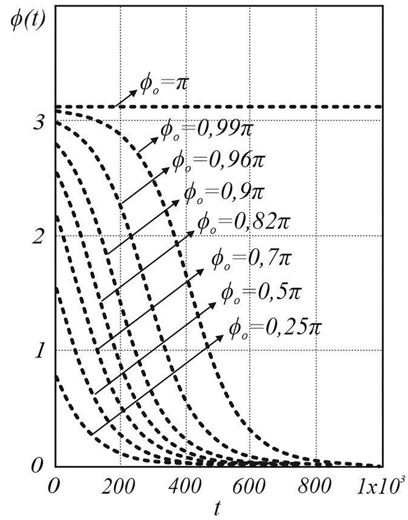

Figure 7: Representation of the angle ϕ as a function of time, with different values of ϕ_o and $\alpha = 0.01$.

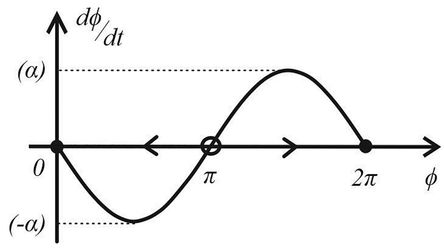

Figure 8: Graphical representation of (51), showing the equilibrium points.

The graphical representation of (51) is shown in Figure 8 for positive values of ϕ. For $0 \leq \phi \leq \pi$, the equation has two equilibrium points, one stable in $\phi = 0$ and the other unstable in $\phi = \pi$ [21]. This means that when $\phi_o = \pi$, phase synchronization between v_{C1} and v_{C2} does not occur, and it remains constant and equal to π. For any condition in which $\phi_o < \pi$, the phase described by the angle ϕ spontaneously synchronizes, according to the expression (50).

In Figure 9, the voltages v_{C1} and v_{C2}, and the phase angle between them, are shown, for $\phi_o = 2.23rd$, $R = 50\Omega$, $C = 1F$, $L = 1H$ and $V_p = 1V$, obtained by computational simulation.

Figure 10 shows the voltages v_{C1} and v_{C2}, for $\phi_o = \pi$, $R = 20\Omega$, $C = 1F$, $L = 1H$ and $V_p = 1V$. As mentioned above, $\phi = \pi$ is a point of equilibrium, although unstable. The initial phase angle remains throughout the transient period and phase synchronization does not occur.

2.4. Energy dissipated in resistor R during the phase synchronization

During the phase synchronization process, part of the energy initially stored in the capacitors is dissipated in

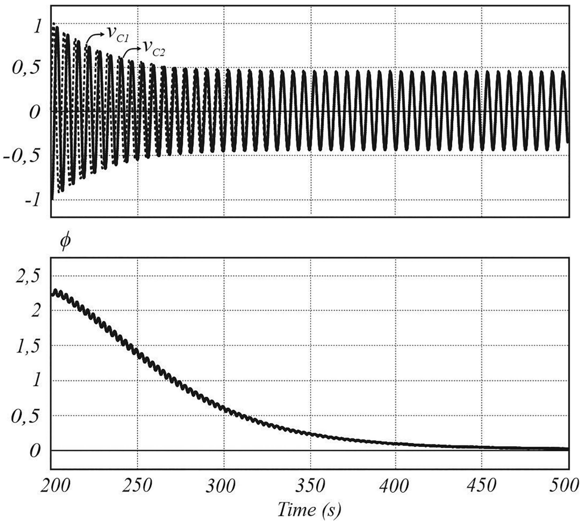

Figure 9: Representation of voltages v_{C1} and v_{C2} across the capacitors and ϕ for $\phi_o = 2.23rd$, with $R = 50\Omega$, $C = 1F$, $L = 1H$ and $V_p = 1V$.

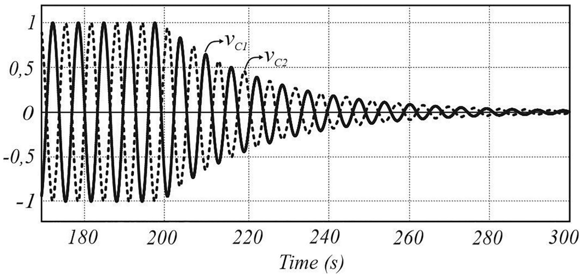

Figure 10: Representation of voltages v_{C1} and v_{C2} across the capacitors for $\phi_o = \pi$, with $R = 20\Omega$, $C = 1F$, $L = 1H$ and $V_p = 1V$.

the resistor R. From the moment the synchronization between the voltages v_{C1} and v_{C2} is carried out, these voltages become identical in phase and amplitude and the voltage across the resistor becomes equal to zero, and the dissipation of energy ceases. In this section, we determine an expression that describes the energy dissipated in the resistor R during the synchronization transient.

The energy initially stored in each oscillator before closing the switch S is given by

$$E = \frac{1}{2} C {V_p}^2 \tag{52}$$

Therefore, the total initial energy is

$$E_i = C {V_p}^2 \tag{53}$$

The energy stored in the two oscillators after synchronization is given by

$$E_f = C {V_p}^2 \cos\left(\frac{\phi_o}{2}\right)^2 \tag{54}$$

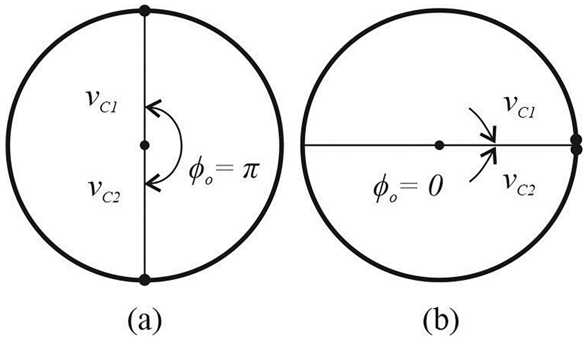

Figure 11: Representation of initial phase between voltages v_{C1} and v_{C2}. (a) $\phi_o = \pi$ and (b) $\phi_o = 0$.

The energy dissipated in the resistor is defined by

$$\Delta E = E_i - E_f \tag{55}$$

Substituting (53) and (54) in (55) we obtain

$$\Delta E = C {V_p}^2 \left(1 - \cos\left(\frac{\phi_o}{2}\right)^2\right) \tag{56}$$

Dividing (56) by (53), we obtain the energy dissipated in the resistor R, normalized in relation to the initial energy E_i, given by

$$\frac{\Delta E}{E_i} = 1 - \cos\left(\frac{\phi_o}{2}\right)^2 \tag{57}$$

For $\phi_o = \pi$, as shown in Figure 11(a), $\frac{\Delta E}{E_i} = 1$, which means that all the energy initially stored in the oscillators is dissipated in the resistor R. Despite this, as it is a point of equilibrium, the phase angle between the two voltages remains constant and equal to π, as mentioned above. For $\phi_o = 0$, as shown in Figure 11(b), the two oscillators are initially synchronized and $\frac{\Delta E}{E_i} = 0$, which means that no energy is dissipated in the process. We can interpret the energy lost in the resistor as a synchronization energy.

3. Conclusions

The analysis of the phase synchronization transient between two linear LC oscillators coupled through a resistor allowed us to obtain an explicit expression for the phase angle of the two oscillators as a function of time and conclude that the phenomenon can also be described by a first order Kuramoto model.

It is shown that for any initial phase angle other than π, which is an unstable equilibrium point, phase synchronization occurs spontaneously. When the initial angle is π, all the energy initially stored in each oscillator is dissipated in the coupling resistor, however without changing the phase angle. For any other value of the initial phase angle, part of the energy is dissipated during synchronization.

 DOI: https://doi.org/10.1590/1806-9126-RBEF-2021-0126

Acknowledgments

The work of Evelyn Sagiorato, graduate student in electrical engineering with Federal University of Santa Catarina (UFSC), who collaborated in the manuscript editing, is gratefully acknowledged.

References

[1] J. Wu and X. Li, IEEE Circuits and Systems Magazine **20**, 46 (2020).

[2] A. Pikovsky, M. Rosenblum and J. Kurths, *Synchronization: A Universal Concept in Nonlinear Sciences*, (Cambridge University Press, Cambridge, 2001), v. 12.

[3] S.H. Strogazt, *Sync: the Emerging Science of Spontaneous Order* (Hyperion, New York, 2003).

[4] L.H.A. Monteiro, *Sistemas Dinâmicos* (Livraria da Física, São Paulo, 2011).

[5] R.L. Viana, Rev. Bras. Ensino Fís. **42**, e20200272 (2020).

[6] J. Schaffner, IRE Professional Group on Circuit Theory **CT-1**, 2 (1954).

[7] T. Endo and S. Mori, IEEE Trans. Circuits and Systems **23**, 100 (1976).

[8] M. Grattarola and V. Torre, IEEE Trans. on Circuits and Systems **24**, 209 (1977).

[9] T. Endo and S. Mori, IEEE Trans. on Circuits and Systems **25**, 7 (1978).

[10] T. Saito, IEEE Trans. on Circuits and Systems **35**, 1147 (1988).

[11] S. Moro, Y. Nishio and S. Mori, in: *Proceedings of IEEE International Symposium on Circuits and Systems – ISCAS '94* (IEE, London, 1994), v. 6, p. 213.

[12] H. Kanasugi, S. Moro, Y. Nishio and S. Mori, in: *Proceedings of IEEE International Symposium on Circuits and Systems – ISCAS '94* (IEE, London, 1994), v. 6, p. 205.

[13] Y. Setou, Y. Nishio and A. Uschida, IEICE Trans. Fundamentals **E79-A**, 1575 (1996).

[14] I.M. Filanovsky and C.J.M. Verhoeven, in: *2006 13th IEEE International Conference on Electronics, Circuits and Systems*, (IEE, Nice, 2006), p. 1253.

[15] C.K. Volos, I.M. Kyprianidis and I.N. Stouboulos, International Journal of Circuits, Systems and Signal Processing **1**, 274 (2007).

[16] T. Nagai, Y. Uwate and Y. Nischio, in: *IEEE Workshop on Nonlinear Circuit Network* (IEEE, Tokushima, 2012), p. 74.

[17] A.K. Singh and R.S. Yadava, Phys. Scr. **94**, 075201 (2019).

[18] S. Emre Tuna, Automatica **75**, 236 (2017).

[19] S. Emre Tuna, Automatica **117**, 109008 (2020).

[20] Y. Kuramoto, in: *International Symposium on Mathematical Problems in Theoretical Physics*, edited by H. Araki (Springer-Verlag, Berlin, 1975), v. 39, p. 420.

[21] S.H. Strogatz, *Nonlinear Dynamics and Chaos: With Applications to Physics Biology Chemistry Engineering* (Westview, Boulder, 2001).