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The wave beating produced by circular plane pistons

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The physics textbooks presents the fundamental concepts of wave beating from an interference process of two plane waves at the time domain. In this work, we propose a teaching practice, based on computational simulation, at undergraduate level, combining the beating caused by waves transmitted from two plane circular pistons vibrating on thickness mode with the resulting acoustic field. The proposal leads to the perception of the beating at the spatial domain. The results show that the pressure amplitude generated from the beating has periodic variations along the central axis from a boundary location. This boundary is determined by the number of extreme values and the ratio between source radius and wavelength.

Keywords: Rayleigh integral, acoustic wave, interference.

1. Introduction

The study of the acoustic field generated by the surface of a vibrating body is carried out from the model based on the distribution of point sources. From a practical point of view, a plane circular piston is a vibrating body with geometry of large interest. For example, many piezoelectric transducers used for the ultrasonic generation in applications of industry and medical equipments has this usual shape.

The vibration of plane circular piston is considerated uniform and normal to the surface and the acoustic field generated is determined by the phasorial sum of the wave transmitted by each one of the point sources distributed on its plane surface [1, 2].

The acoustic field is divided in two parts: the nearfield and the farfield. In the nearfield, localized in the neighborhood of the vibrating surface, the pressure presents many minima and maxima values resulting of a complex interference process. On the other hand, in the farfield, localized beyond the nearfield, the pressure decays smoothly. Conventionally, the position of the last maximum is considered the location of boundary between these regions.

The determination of the acoustic field is performed by computers running a numerical algorithm that calculates Rayleigh integral [3]. There are many methods to calculate the fields [4], further comprising methods with didactic purpose [5]. However, the pressure on the central axis perpendicular to the surface of the plane circular source is a particular case whose field can be determined by an analytical expression.

The interference of two waves with sligthly different frequencies is named beating. The beating results in a wave whose amplitude varies along the time at a frequency defined by the difference between the frequencies of the original waves.

The beating with sound waves is useful technique for the tuning of the musical instruments. When two devices emits tones at slightly different frequencies the beating can be perceived, accusing untuning. Also the beating is used to determine poisoned gases in mines. By draining out two identical pipes, one filled with pure air and other with gas from a mine, the beating generation indicates a divergence of composition, therefore a possible contamination of the gas. Systems that work based on Doppler effect such as the radar used to measure the speed of vehicles in a highway (in this case the beating is caused by electromagnetic waves) and ultrasonic probes that detects the blood move are important examples of the beating detection. Electronic circuits of both systems compare waves transmitted and received with differents frequencies and generated an electric signal proportional to the speed of the movement (vehicles or blood).

The textbooks of basic physics introduce the beating straightforwardly from the interference of two plane waves in the time domain [6, 7]. In addition, in the practice of physics teaching laboratories, we find several experimental proposals for the beating detection in the time domain. Commonly, ilustrative experiments with diapasons give us a good qualitative evaluation of the beating [8], but the introduction of technology in the experiments increases the understanding and also adds a positive motivation for the studies [9–12]. However, we have not found explanations about the beating produced by extended vibrating bodies.

In this work, we investigate the beating and its acoustic field caused by waves generated by discs vibrating on thickness mode. We consider two superimposed discs vibrating at somewhat different frequencies and

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determine the field along the central axis normal to the disc surface. We propose an alternative approach for enhancing and attaching concepts of acoustic field to beating caused by a more complex interference process and not only by a simple plane wave as dealt in the introductory physics textbooks. Further, we also show the beating in the spatial domain. The study was carried out by computational simulations, since the experimental set to detect the acoustic field is very expensive and not accessible to the most teaching laboratories.

2. Theory

Figure 1 shows a sketch of the geometry we have used to determine the acoustic field. We consider an acoustic source as a plane circular piston of radius a disposed in the infinite, flat and rigid baffle on the plane $z = 0$. This extended source is composed of a distribution of point sources on the area S in sinusoidal vibration, in phase, at angular frequency ω and amplitude U_0 perpendicular to the surface. The propagation media is homogeneous, isotropic, lossless, density ρ, speed of waves propagation c, and each point P has coordinates r, θ and z.

The pressure in the point P of the propagation media is determined by Eq. (1) (Rayleigh integral), where dS is an element of area, j is the complex number, λ is the wavelength, $k = 2\pi/\lambda$ is the number of wave, t is the time, and r' is the distance from an area element dS to P.

$$p(r,\theta,t) = \frac{j\rho c}{\lambda} \iint_S \frac{U_0(\sigma,\varphi)}{r'} e^{j(\omega t - kr')} dS \tag{1}$$

The surface integral is defined in polar coordinates, where the area element is $\mathrm{d}S = \sigma \mathrm{d}\sigma \mathrm{d}\varphi$. All point sources vibrate at same amplitude and phase, therefore $U_0(\sigma,\varphi)$ is a constant U_0 and Eq. (1) is rewritten as

$$p(r,\theta,t) = \frac{j\rho c}{\lambda} U_0 \iint_S \frac{1}{r'} e^{j(\omega t - kr')} \sigma d\sigma d\varphi \tag{2}$$

Rayleigh integral is calculated by numerical integration [3]. However, an analytical solution can be obtained

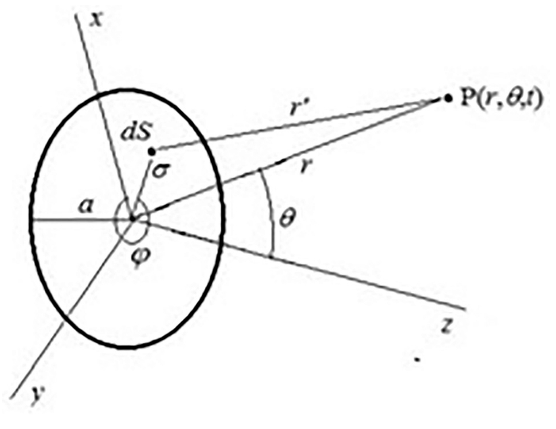

Figure 1: Coordinate system used in calculating the acoustic field.

when the the pressure is evaluated along the axis z, $\theta = 0$, and thus $r = z$. In this case, the integrand is not dependent on φ and the integral relative to this variable is 2π. Therefore, Equation (2) can be simplified [1] resulting in the integral of Eq. (3)

$$p(r,0,t) = j\rho c \frac{U_0}{\lambda} \int_0^a \frac{exp(-jk\sqrt{r^2+a^2})}{\sqrt{r^2+a^2}} 2\pi\sigma d\sigma \tag{3}$$

whose solution is shown in (4)

$$p(r,0,t) = \rho c U_0 e^{j(\omega t - kr)}[1 - \exp(-jk(\sqrt{r^2+a^2} - r))] \tag{4}$$

We can point out that when $r \gg a$, (4) is approximated as (5)

$$p(r,0,t) = \rho c U_0 e^{j(\omega t - kr)} \tag{5}$$

The beating is obtained from the wave interference produced by two sources vibrating at sligthly different angular frequencies ω_1 and $\omega_2 = \omega_1 + \Delta\omega$, where $\Delta\omega$ is an increment of the angular frequency [6]. The resulting wave oscillates at angular frequency ω (Eq. (6)). The evolution of this wave along the time presents periodic amplitude variation defined by a beating frequency (Eq. (7)).

$$\omega = \frac{1}{2}(\omega_2 + \omega_1) \tag{6}$$

$$\omega_{beat} = \omega_2 - \omega_1 \tag{7}$$

The field produced as result of the beating is calculated by phasorial adding of the pressures generated by each single circular plane piston.

3. Methodology

We have used a simple computer code written in Matlab to calculate the complex pressures of the transmitted waves (Eq. (4)). Firstly, we determined the fields for each source at ω_1 and ω_2. The amplitude of both vibrations are the same. The radius of the plane circular piston we choose is $a = 0.01$ m. We choose to take a medium with similar characteristics to the water (speed wave propagation and density of $c = 1500$ m/s and $\rho = 1000$ kg/m^3, respectively). The simulations were performed for a ratio $a/\lambda = 4.0$, considering the lower angular frequency ω_1 which results in 3.7699×10^6 rad/s (or frequency $f_1 = 600.0$ kHz). The second angular frequency ω_2 had increments of 2.5% ($f_2 = 15.0$ kHz), 5.0% ($f_2 = 30.0$ kHz) and 10.0% ($f_2 = 60.0$ kHz) in relation to f_1.

The results were presented as relative magnitude of pressure (p/p_{max}) as function of relative distance, where p_{max} is the largest pressure amplitude. In all cases the relative distance, normalized by a^2/λ, was considered in the range comprising 0 and 10.

After that, the field produced by the beating was obtained by adding of complex pressure obtained for

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each frequency. Further, the positions of the relative minima and maxima of the relative pressures were identified.

To determine the acoustic field in any point of the propagation mediu, we perform a numerical two-dimensional integration of Eq. (1). However, there are simpler methods that transform Eq. (1) in a one-dimensional integral [13] saving computational work. We have implemented an algorithm based on geometrical considerations to determine the acoustic field generated by the sources with circular shapes. A fine description of the algorithm is found in [5].

According to Fig. 1, the circular source is placed in xy-plane where all point sources are vibrating with same amplitude and phase. The pressure in any point $P(x_P, y_P, z_P)$ belonging to the propagation medium is determined by adding the individual vibration generated by the point sources. Point sources equidistant from P generate parcels of the pressure proportional to the velocity phasor, $u_i = U_0 e^{j\omega r'/c}$. These sources are disposed in arcs, lenght L, which are determined considering the intersection between the inner area of surface of the circular source and an imaginary sphere whose radius grows centered in P. Therefore, each phasor is multiplied by the respective arc lenght. Further, each parcel of the adding is divided by the distance (r') from the arc to P. Equation (8) gives the summation to determine the pressure in P

$$p(x_P, y_P, z_P) = A \sum_i u_i \frac{L_i}{r'_i} \tag{8}$$

where A is a constant dependent on physical properties of the propagation medium;
i is the index of the each arc.

We may wonder a sphere centered in P with radial growing. The intersection of this sphere with xy-plane is a circunference with center in the P' and radius ρ. There are three cases to be considered:

(a) the circunference does not intersect the circle where the source is defined, therefore the arc is not drawn and none point source adds pressure in P (Fig. 2a);
(b) the circunference is fully contained in the source, so the arc lenght is determined by Equation (9) (Fig. 2b);
(c) Equation (10) is used to determine the arc when the circunference is partially inner to the source (Fig. 2c).

$$L = 2\pi\rho \tag{9}$$

$$L = 2\rho \; cos^{-1}\left(\frac{\rho^2 + x_P^2 + y_P^2 - a^2}{2\rho(x_P^2 + y_P^2)}\right) \tag{10}$$

A code written in Matlab has been used to determine the field. The basic instructions for running the algorithm are:

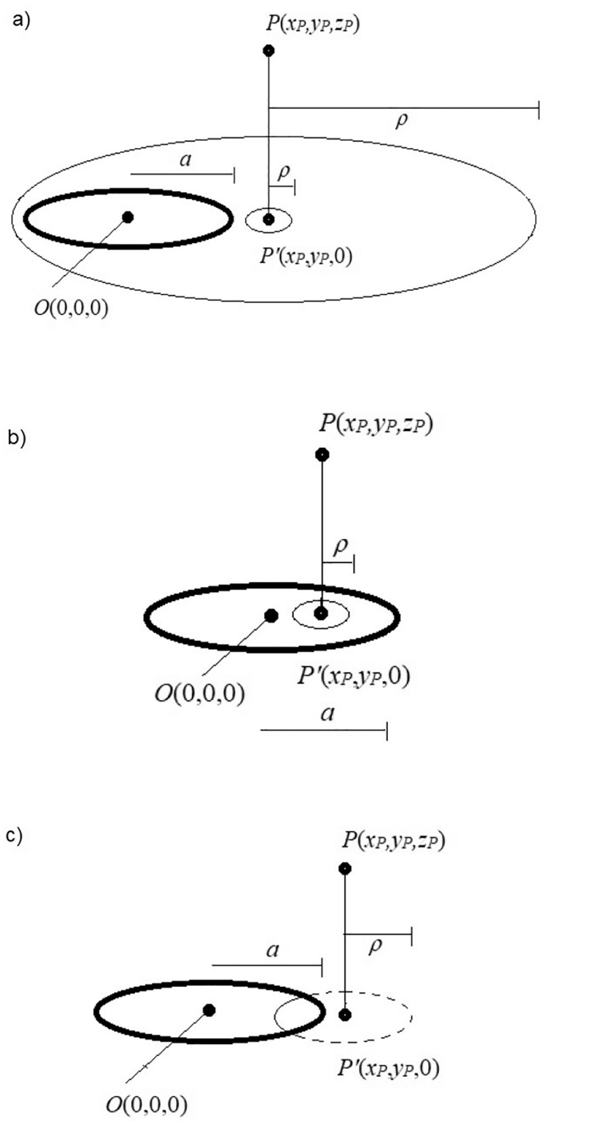

Figure 2: a) Both circunference with radius signed as ρ do not intersect the circular source, thus the velocity phasor is zero. b) The circunference with radius ρ is fully inside the circular source, thus $L = 2\pi\rho$. c) The circunference with radius ρ intersect 2 points of the circular source, thus the arc inside the source (full line) is L, defined by Eq. 10.

(a) take a point P,
(b) project P on the xy-plane to obtain P',
(c) P' is a center of a circunference whose radius ρ is incremented;
(d) for each radius ρ, L is determined and the parcels of the pressure are calculed using Eqs. (8), (9) and (10) according to the case pictured;
(e) the final pressure in P is obtained when a full sweeping on the circular source is concluded;
(f) return to a).

After running this algorithm at ω_1 and ω_2, the acoustic field generated by the beating is determined by the phasorial adding of the pressures in each point of the propagation medium as shown in Eq. (11).

$$p(x_P, y_P, z_P) = p_1(x_P, y_P, z_P) + p_2(x_P, y_P, z_P) \quad (11)$$

where

p is the pressure phasor in (x_P, y_P, z_P) into the beating field;
p_1 is the pressure phasor in (x_P, y_P, z_P) calculated at ω_1;
p_2 is the pressure phasor in (x_P, y_P, z_P) calculated at ω_2.

The resulting field was determined on a plane perpendicular to the x-axis ($x = 0$) (Figure 1) in the ranges $z/a^2/\lambda$ (between 0 and 10) and $y/a^2/\lambda$ (between 0 and 2). Both coordinates were discretized in 200 points. The results of these calculations were represented in color maps which also allow us "see" the field in lateral regions to the central axis and, therefore, get a qualitative evaluation of the directional factor [1, 3]. From a simplyfied physical point of view, the directional factor expresses how much the pressure beam is narrow and therefore how the acoustic energy is distributed in the field.

4. Results and Discussion

The results are presented as function of normalized coordinates. Therefore, although the calculations have been performed with ultrasonic frequencies, the results can be interpreted for lower frequencies, at audible range for example, taking in account the ratio a/λ. Further, the spacial coordinates (axis z and y) are also normalized agregating generalization to the interpretation of the results. Thus, the choice of dimensions and vibration frequencies of the sources does not limit the interpretation of the results to a specific case, but allows us evaluate the results comprehensively.

Figure 3 shows an example of relative magnitude of pressure at ω_1 and ω_2 when $\Delta\omega$ means a increment of 5.0% in relation to ω_1. The nearfield is identified as the region between the source ($z = 0$) and the last maximum (around $z/a^2/\lambda = 1.0$). The number of minima and maxima in this region depends on a/λ, so that the simulations considering larger values of a/λ will set a nearfield with more complexity, as seen in [3].

Figure 4 shows the relative pressure resulting of the beating at the time $t = 0$ s considering $\Delta\omega$ of 5.0% too. We can see that from $z/a^2/\lambda \approx 1.2$ the pressure waves assume a spatial periodic shape. This position stands just above the last maximum produced by the single sources shown in Fig. 3. In fact, hereafter, this position will be dealt as a reference that establishes the boundary where the beating begins to ocurr according to the usual definition found in the textbooks, but with gradual amplitude decrease.

Figures 5a to 5c show the relative magnitude of pressure for the three frequency deviations. The distance between consecutive maximum tends to a constant value

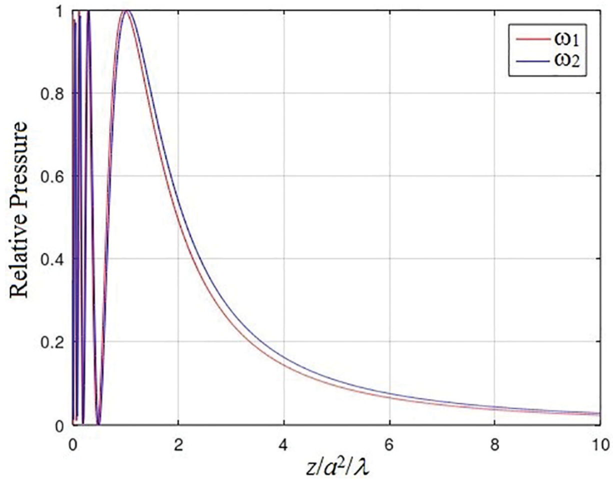

Figure 3: Axial relative pressure amplitude as function of $z/a^2/\lambda$ for ω_1 (red line) and ω_2 (blue line).

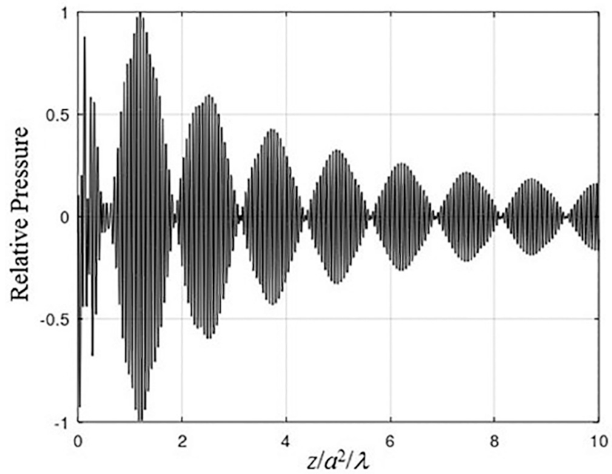

Figure 4: Axial relative pressure as function of $z/a^2/\lambda$ at $t = 0$s of the wave beating obtained from the percentual variation of frequency of 5.0%.

as far as the distance from the source grows. The same feature is also observed for the minimum pressure values. We calculated the difference among consecutive minima (Δz_{min}) and maxima (Δz_{max}). This distance between consecutive extreme values is the wavelength referent to the beating frequencies calculed by $c/\Delta z_{min}$ and $c/\Delta z_{max}$. Figure 6 shows that the relative beating frequency normalized by f_1 are 2.5% – blue line, 5.0% – red line, 10.0% – green line, as expected.

The distance between the extreme values is constant after the fifth maximum/minimum. Therefore, the beating effectivelly begins after a reference defined by the number of extreme values, being unperturbed by the percentual frequency variation. The number of extreme values in the anterior region to the reference remains unchanged. This number is defined for the a/λ ratio, similar to that verified in fields produced

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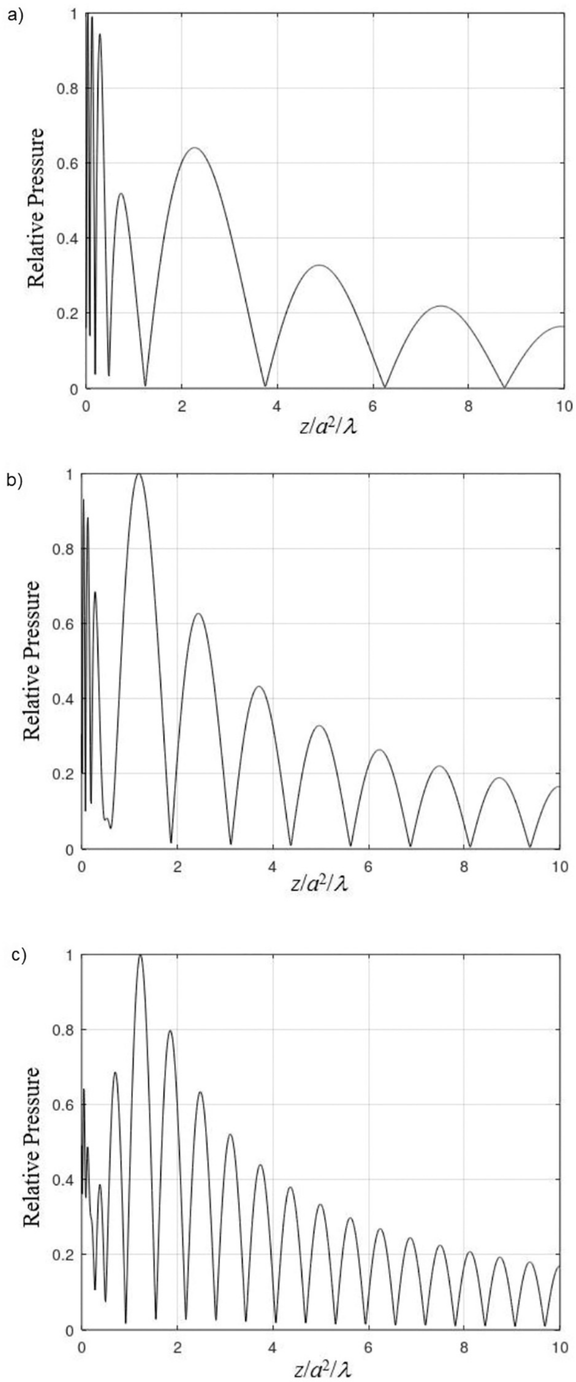

Figure 5: a) Axial relative pressure amplitude as function of $z/a^2/\lambda$ for the wave beating obtained from the percentual variation of frequency of 2.5%. b) Axial relative pressure amplitude as function of $z/a^2/\lambda$ for the wave beating obtained from the percentual variation of frequency of 5.0%. c) Axial relative pressure amplitude as function of $z/a^2/\lambda$ for the wave beating obtained from the percentual variation of frequency of 10.0%.

by circular plane pistons [2]. Therefore, in the case of a comparison with a commercial device, the number of pressure maximums and minimums can be changed without any loss of generality.

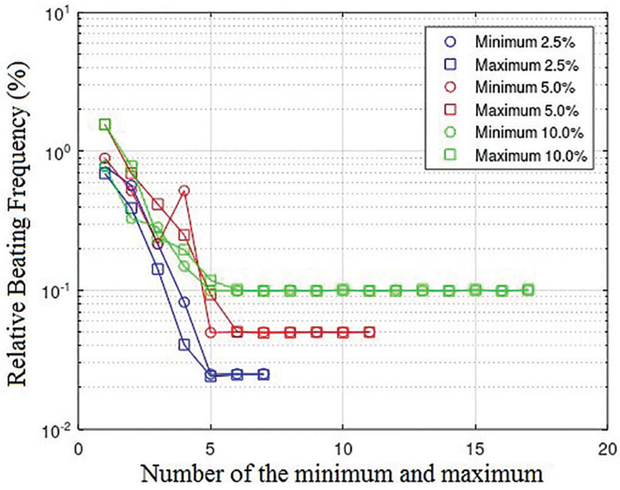

Figure 6: Relative beating frequencies as function of the number of minima and maxima for each percentual difference of frequency (2.5% – blue line, 5.0% – red line, 10.0% – green line) for $z/a^2/\lambda$ in the range from 0 to 10.

The spatial periodicity of the field in positions above the reference is a typical and peculiar characteristic of the wave beating. In the same range of relative distance ($z/a^2/\lambda$), when the difference of the beating frequencies grows the variations along the farfield become less spaced, indicating that the wave groups are smaller.

In circular plane piston vibrating at a single frequency, when z becomes much higher than a, the term ($\sqrt{r^2+a^2}-r$)), in Eq. (3), tends to 0, therefore reaching the plane wave condition which leads to the behavior of the farfield. According to the theoretical approach denoted in the physics textbooks, the beating is achived by the interference between plane waves. Thus, this is the reason of the beating arises when this condition is observed.

Figures 7a to 7c show the color maps of the acoustic field on the plane $x = 0$. The normalized radius of the circular source is 0.25. The darker colors identify regions with low pressures. Considering the pressures as a function of the coordinate θ (see Fig. 1), a visual analysis of the color maps did not reveal substantial changes in the directional factor. However, the lateral lobes present the same spatial frequency found in the region located in front of the vibration source. As expected, in the region near to the source, we also find the complex interference process peculiar to the nearfield.

We have developed this work based on acoustic waves emitted by circular sources. This shape is very popular in ultrasonic probes, loudspeakers and musical instruments. Therefore, this shape is present in practical situations which may stimulate the students in the study of the beating. It is worthy point out that other shapes (line, rectangular, for example) [1] can be used to study the behavior of the acoustic fields generated by beating and also link to the sound sources met in practical applications.

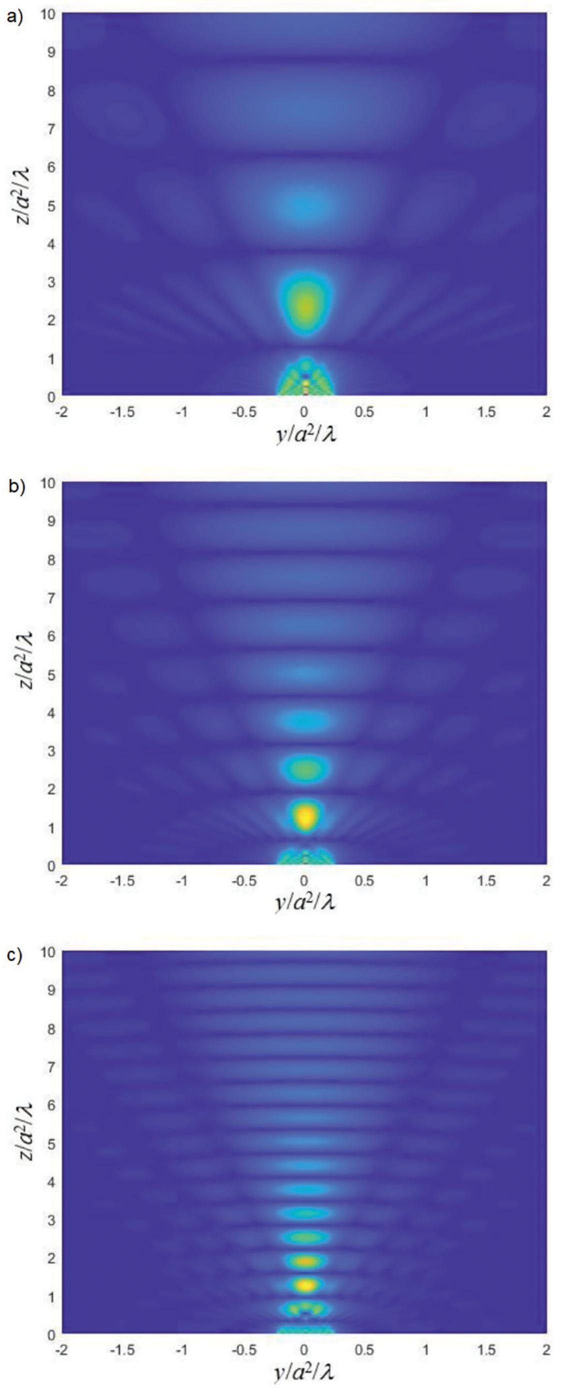

Figure 7: a) Color map of pressure on the yz-plane for the wave beating obtained from the percentual variation of frequency of 2.5%. b) Color map of pressure on the yz-plane for the wave beating obtained from the percentual variation of frequency of 5.0%. c) Color map of pressure on the yz-plane for the wave beating obtained from the percentual variation of frequency of 10.0%.

5. Conclusion

The pressure amplitudes of the acoustic fields generated by the plane circular pistons present abrupt variations at the nearfield, but decay smoothlly at the farfield. On the other hand, when the field produced by these sources results from the beating process, it is distinguished by a regular spatial sequence of minimum and maximum pressure values. Both distances between minima as maxima tend to the wavelength of the beating frequency. In fact, the effective beating arises in distances greater than a boundary found when the waves produced by the sources becomes approximately planes. This boundary depends on the number of the extreme values, which in turn depends on a/λ.

This work shows a study with simulation of acoustic fields generated by two superimposed plane circular pistons causing beating. We have approached the beating caused by extended sources and introduced its spatial dependence. Thus, our purpose goes beyond the conventional textbooks used em introductory physics courses where the beating is dealt in the regular activities of physics teaching as plane waves only in the time domain.

We also introduce the fundamentals for understanding the acoustic fields, topic not always adressed in the preliminar studies of physics, which may be used as complement in the study of the phasors and elementar concepts of computational programming language. Thus, our propose combines two important topics covered in the acoustics studies aiming support new activities for the physics teaching at undergraduate level.

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