Revista Brasileira de Ensino de Física, vol. 44, e20220115 (2022)
www.scielo.br/rbef
DOI: https://doi.org/10.1590/1806-9126-RBEF-2022-0115

Elliptic Helmholtz coil

David Romero-Abad*[1], José-Pedro Reyes-Portales[1], José-Luis La Rosa-Navarro[1], Roberto Suárez-Córdova[2]

[1]Universidad San Ignacio de Loyola, Facultad de Ingeniería, Grupo de Investigación en Física, Lima, Perú.
[2]Universidad Nacional de Ingeniería, Facultad de Ciencias, Lima, Perú.

Received on April 20, 2022. Revised on May 25, 2022. Accepted on May 31, 2022.

Using the Biot-Savart law, we compute the magnetic field produced by steady currents that circulate in the same direction through two parallel ellipse-shaped loops. The calculation was performed along the axis that passes across the center of the ellipses. Then, we show that this arrangement produces a nearly uniform magnetic field for particular values of the semi-axis of the ellipse and the separation distance between the loops. This outcome is obtained as a consequence of numerically solving the second derivative condition. In addition, we show that, for the values reached, the elliptical Helmholtz coil produces results that are compatible with those of the classical circular Helmholtz coil. Finally, we show how the uniformity of the magnetic field varies for regions off the axis of the ellipses. The target readers of the paper are students pursuing physics at the intermediate undergraduate level.

Keywords: Helmholtz coil, elliptic loop current, elliptic integrals, Biot-Savart law.

1. Introduction

A Helmholtz coil is a geometrical structure of two identical, parallel, circular, coaxial and same direction current-carrying coils whose midplane separation is equal to the radius of the coils. This device is designed to produce a nearly uniform region of a magnetic field, needed for many laboratory applications [1–4]. Usually, the exercise to calculate the magnetic field generated by a Helmholtz coil with a circular cross-section is a typical assignment for an introductory-level course in electricity and magnetism [5–8]. The computation of the magnetic field for coils with geometries other than the circular could be cumbersome because of the vectorial nature of the Biot-Savart law [17]. Different designs inspired by Helmholtz coils have been implemented since their creation in an attempt to optimize their main characteristics, such as their region of homogeneity and the simplicity of their manufacture [9–16]. In this work, we consider an extension of the classical Helmholtz coils when the wires are shaped like an ellipse. To analyze this situation, it is necessary to calculate the magnetic field generated by the proposed arrangement, which requires the use of the Biot-Savart law. The calculation of the magnetic field for a current-carrying wire with an elliptic shape was treated by Miranda [17] but only in the center of the ellipse. The computation for the magnetic filed in the axis of the ellipse is obtained by reference [18].

The setting for homogeneity is obtained by numerically solving the condition of the second derivative for the magnetic field on the axis. During the work, we have been able to verify that, at least in the axis of the ellipse, the magnetic field generates results that are compatible with those of the classical Helmholtz coil.

One of the aspects of design that are apparently not very covered when new variants of Helmholtz coils are considered is the geometry of the area where the magnetic field is to be generated. This is an important aspect depending on the use that you want to give the device. For example, in magnetic resonance imaging, it is need particular points of view that can only be achieved with elliptical loops [19].

The extension presented in this article appears to be an interesting design that could be useful in an undergraduate physics course because it employs one of the fundamental laws of magnetism, such as the Biot-Savart law, introduces a problem with new geometry, uses special functions such as elliptical integrals, has practical applications such as magnetic resonance imaging and incorporates numerical calculus as the core point in the solution of the problem.

2. Magnetic Field of a Helmholtz Coil With Elliptic Loops

A Helmholtz coil is a device designed to produce a nearly uniform magnetic field region. In its original version, this scheme consists of two circular loops separated by a distance of $2D$. In this article, we generalize this definition to also include the case of elliptic loops.

A single circular loop with radius a, carrying a current I, extended over the xy-plane and whose center

* Correspondence email address: davidromeroabad@gmail.com

is at the origin, produces a magnetic field over any point on the z-axis according to [6, p. 931],[7, p. 775], as

$$B_z = \frac{\mu_0 I}{4\pi}\int_0^{2\pi}\frac{a^2 d\phi'}{(z^2+a^2)^{3/2}} = \frac{\mu_0 I}{2}\frac{a^2}{(z^2+a^2)^{3/2}} \tag{1}$$

We may construct a Helmoltz coil by considering the superposition of N coils located on a plane parallel and above the xy-plane, together with N coils located on a plane parallel and below the xy-plane. The azimuthal axis for both arrangements will be the z-axis, and the centers will be at $z = \pm D$. For this case, the magnetic field over the z-axis would be [20]

$$B_z = \frac{\mu_0 N I a^2}{2} \times \left[\frac{1}{(a^2+(z+D)^2)^{3/2}} + \frac{1}{(a^2+(z-D)^2)^{3/2}}\right] \tag{2}$$

Proceeding by similarity, we make an analogous calculation for an elliptic single coil having semi-axial dimensions a and b relative to the x-axis and y-axis respectively, extended over the xy-plane with geometric center at the origin. This would produce a field over any point on the z-axis according to

$$B_z = \frac{\mu_0 I ab}{4\pi}\int_0^{2\pi}\frac{d\phi'}{[a^2\cos^2\phi' + b^2\sin^2\phi' + z^2]^{3/2}} \tag{3}$$

which may be expressed in terms of the complete elliptic integral of the second kind, as

$$B_z = \frac{\mu_0\, I a\, b}{\pi}\frac{E\left(\frac{a^2-b^2}{a^2+z^2}\right)}{(b^2+z^2)\sqrt{a^2+z^2}} \tag{4}$$

where E is the complete elliptic integral of the second kind, defined as [21, 22]

$$E(k) = \int_0^{\pi/2}\sqrt{1-k^2\,\sin^2\,\theta}\,d\theta \tag{5}$$

The calculation of the magnetic field due to an elliptic loop in the axis of the ellipse was performed in [18]. We want to find the conditions that two identical elliptic loops a distance $2D$ apart and with semiaxis distance a and b must fulfill in order to produce an almost constant magnetic field in the region midway between them when steady currents I circulate in the same direction. The setup of the elliptic Helmholtz coil is shown in Figure 1. Using the results from the reference [18] and the superposition principle, we can obtain the magnetic field produced in the z-axis

$$B_z = \frac{\mu_0\, I a\, b}{\pi}\left\{\frac{E\left(\frac{a^2-b^2}{a^2+(D-z)^2}\right)}{(b^2+(D-z)^2)\sqrt{a^2+(D-z)^2}} + \frac{E\left(\frac{a^2-b^2}{a^2+(D+z)^2}\right)}{(b^2+(D+z)^2)\sqrt{a^2+(D+z)^2}}\right\} \tag{6}$$

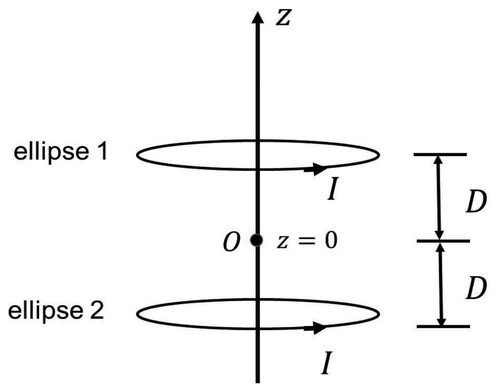

Figure 1: Helmholtz coil arrangement with elliptic loops.

Notice that the limit $a, b \to R$ imply $\lim_{k\to 0} E(k) \to 1$ and we recover the magnetic field of the Helmholtz coil with circular loops,

$$B_z = \frac{\mu_0\, I\, R^2}{2}\left\{\frac{1}{(R^2+(D-z)^2)^{3/2}} + \frac{1}{(R^2+(D+z)^2)^{3/2}}\right\}. \tag{7}$$

3. Homogeneity Conditions for the Hemlholtz Coil With Elliptic Loops

We can expand the expression of B_z obtained in equation (6) in a Taylor series around $z = 0$

$$B(z) = B(0) + z\left.\frac{\partial B}{\partial z}\right|_{z=0} + z^2\left.\frac{\partial^2 B}{\partial z^2}\right|_{z=0} + \ldots \tag{8}$$

From equation (8) it can be seen that the magnetic field on the axis is very close to a homogeneous magnetic field if the conditions $\partial B_z/\partial z|_{z=0} = 0$ and $\partial^2 B_z/\partial z^2|_{z=0} = 0$ are satisfied [23]. In that sense, we calculate the terms of the series starting with the constant term given by the magnetic field in $z = 0$

$$B_0 = B(0) = \frac{2\,\mu_0\, I a\, b}{\pi}\left\{\frac{E\left(\frac{a^2-b^2}{a^2+D^2}\right)}{(b^2+D^2)\sqrt{a^2+D^2}}\right\} \tag{9}$$

Then, we determine the first derivative from equation (6)

$$\begin{aligned}\frac{\partial B_z}{\partial z} = &\frac{8C\left(a^2+b^2+2(D-z)^2\right)(D-z)\,E\left(\frac{a^2-b^2}{a^2+(D-z)^2}\right)}{(a^2+(D-z)^2)^{3/2}\,(b^2+(D-z)^2)^2} \\ &- \frac{8C\left(a^2+b^2+2(D+z)^2\right)(D+z)\,E\left(\frac{a^2-b^2}{a^2+(D+z)^2}\right)}{(a^2+(D+z)^2)^{3/2}\,(b^2+(D+z)^2)^2} \\ &+ \frac{4CK\left(\frac{a^2-b^2}{a^2+(D-z)^2}\right)(z-D)}{(a^2+(D-z)^2)^{3/2}\,(b^2+(D-z)^2)} \\ &+ \frac{4CK\left(\frac{a^2-b^2}{a^2+(D+z)^2}\right)(z+D)}{(a^2+(D+z)^2)^{3/2}\,(b^2+(D+z)^2)}\end{aligned} \tag{10}$$

where $C = \mu_0 \, I \, ab/4\,\pi$ and K is the complete elliptic integral of the first kind, defined as [21, 22]

$$K(k) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2 \, \sin^2 \, \theta}} \tag{11}$$

In (10) we have used the following relation

$$\frac{dE}{dk} = \frac{E(k) - K(k)}{k} \tag{12}$$

From this result, it is straightforward to verify that the first derivative is canceled when $z = 0$. In addition, the second derivative evaluated in $z = 0$, give the following expression:

$$\left.\frac{\partial^2 B_z}{\partial z^2}\right|_{z=0} = C_2 \times F(a, b, D) \tag{13}$$

where

$$C_2 \equiv - \frac{2C}{(a^2 + D^2)^{5/2} \, (b^2 + D^2)^3} \tag{14}$$

$$\begin{aligned} F(a,b,D) \equiv & \left[2a^4 \left(b^2 - 3D^2\right) + a^2 \left(b^2 D^2 + 2b^4 - 17D^4\right)\right. \\ & \left. - \; d^2 \left(17b^2 D^2 + 6b^4 + 19D^4\right)\right] \\ & \times E\left(\frac{a^2 - b^2}{a^2 + D^2}\right) + \left(b^2 + D^2\right) \\ & \times \left[-a^2 \left(b^2 - 3D^2\right) + 3b^2 D^2 + 7D^4\right] \\ & \times K\left(\frac{a^2 - b^2}{a^2 + D^2}\right) \end{aligned} \tag{15}$$

4. Finding the Loci Where the Second Derivative is Zero

Equating the equation (15) to zero defines an equivalent condition by which the same requirement for the Helmoltz coil would be fulfilled. That is, the first three derivatives would have to be null at the origin. To investigate whether the three-variable function in (15) is useful to achieve this condition, we fix the distance D and make a plot of F mapping the variables a and b. If we set the distance D to a unit value, we do not lose any generality, since in D the scale factor of the elliptical coil would be bound. Thus, with $D = 1$, a reasonable domain for a and b is set to be the interval $[1, 2.5]$. According to this scheme, a and b will be examined in a neighborhood of the circular coil for which the condition of the Helmholtz coil stands, namely $a = b = 2D$.

This plot can be seen in Figure 2. As it is evident, the two-variable function $F(a, b, D = 1)$ maps the variables a and b to both a positive and negative range. This allows us to find the locus for which F equals zero. The method we follow consists of iteratively fixing the value of D and finding numerically the curve corresponding to this value. The numerical method is necessary since F is expressed in terms of the elliptic integrals of the first and second kind, and there is no way to obtain an analytic solving for a or b. If for each D a curve is obtained as a sequence of points, we can proceed to vary D and obtain a family of curves. This would verify that for each D, the corresponding curve is referred to a specific scale, and all the curves represent equivalent situations.

Once the domain for a and b has been established, we can construct a mesh of points in rectangular coordinates and for each point a_i a bracketing interval can be established over which we find the solution b_i that satisfies the condition $F(a_i, b_i, D_n) = 0$ for the value of D_n considered in the i-th iteration. The results after continuing with this process can be seen in Figure 3, and the numeric values can be found in the Appendix.

From this family of equivalent curves, we select the one with the highest number of points obtained for the proposed domain. In this case, such curve corresponds to $D = 0.6$. This curve represents a continuous bundle of elliptical coils whose semi-axis a and b have values in the interval $[1, 2.5]$ and for which the first three derivatives vanish at the central point of symmetry, for each one of them (i.e. the center of each coil). This curve is shown in the Figure 4. Similarly, the ellipses corresponding to this curve are drawn in the Figure 5.

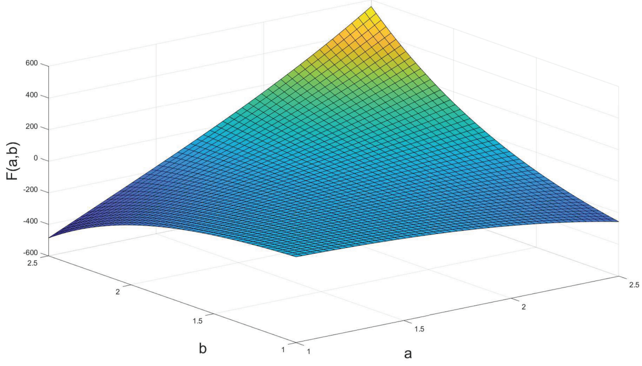

Figure 2: Plot of $F(a, b)$ for $D = 1$.

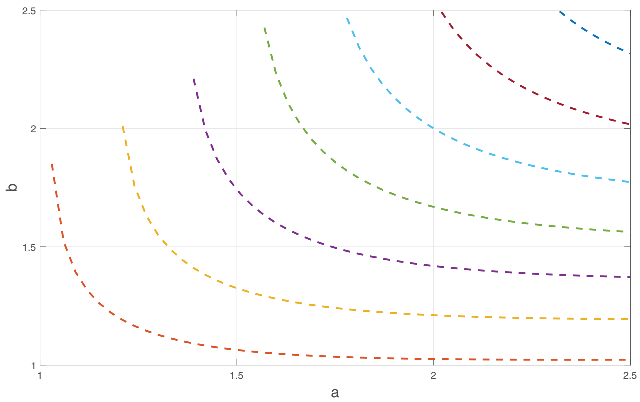

Figure 3: Loci for which the second derivative of the magnetic field is zero in the center of each coil. D ranges from 0.6 (lowest curve) to 1.2 (highest curve).

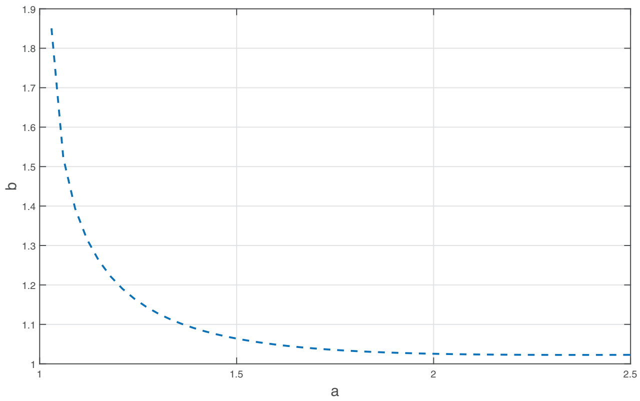

Figure 4: Curve a vs b when $D = 0.6$ for which the second derivative of the Magnetic field is zero at the center of the coil.

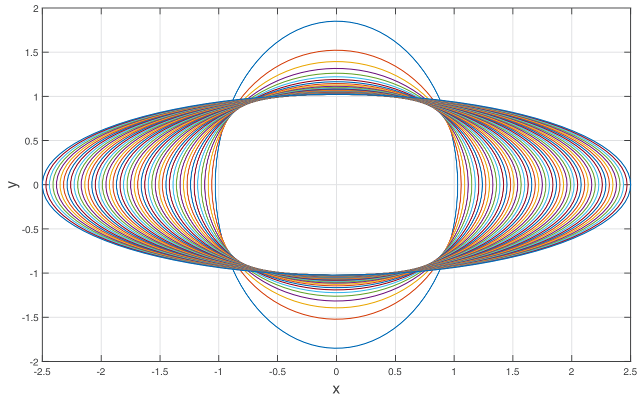

Figure 5: Set of ellipses corresponding to $D = 0.6$.

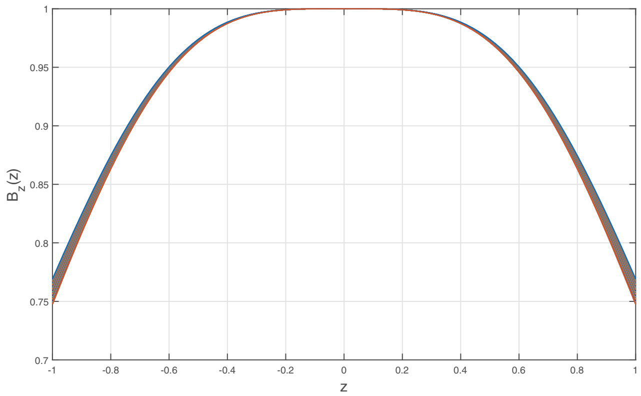

Figure 6: Set of curves representing the magnetic field on the $z-$axis for all the ellipses with $D = 0.6$.

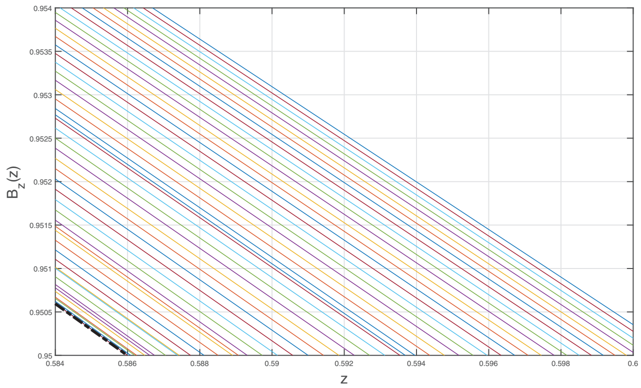

Figure 7: Detailed close region showing the curves with the magnetic field on the $z-$axis for all the ellipses with $D = 0.6$. The black dashed curve corresponds to the Helmholtz coil for $D = 0.6$.

An immediate result of obtaining this family of elliptical coils is the evaluation of the magnetic field produced by them on the z-axis. The z component of this field can be calculated using the expression (6) and this result is compared with that produced by a Helmoltz coil with equal distance between symmetrical coils. Prior to the calculation we can infer that it is necessary that there be a continuity in the magnetic field produced by the family of elliptical coils corresponding to $D = 0.6$ as a function of a and b over the curve in which the second derivative vanishes, and the field produced by the coil of Helmholtz corresponding to the same distance D. This is a consequence of the simple fact that Helmholtz's circular coil is just a special case of elliptical coils. However, the equation (6) is unable to evaluate the circular case because the elliptic integrals of the first and second kind exclude the zero value in their domains by default, this being precisely the case in which $a = b$. By making the graph of B_z explicit for all elliptical coils corresponding to the curve where $D = 0.6$, together with the same component of the magnetic field for the Helmholtz coil corresponding to the same distance D, this continuity is clear. This plot is shown in the Figure 6. The field produced by the Helmholtz coil is calculated separately by the standard formula. In the Figure 7 a very fine zoom of the graph is shown and it can be seen how the Helmholtz coil corresponds to a situation of a minimal condition for the value of the B_z component within this spectrum of elliptical coils. Of course, to show this continuity, the magnetic field has been normalized by its magnitude at the origin. It is clear that in an actual situation, to maintain this condition, a greater energy would be necessary, via a greater current, to conserve such magnitude in the center of symmetry while at the same time using coils with greater area.

5. Off-axis Magnetic Field for an Elliptic Helmholtz Coil

We use the Biot-Savart law and the configuration shown in Figure 8 to calculate the off-axis magnetic field due to an elliptical loop. The infinitesimal element of length is:

$$d\vec{\ell} = -a \sin\theta' d\theta' \; \hat{e}_x + b \cos\theta' d\theta' \; \hat{e}_y \tag{16}$$

where a and b are the semi-axis of the ellipse [18]. To obtain the magnetic field at a position $\vec{r} = x \; \hat{e}_x +$

 DOI: https://doi.org/10.1590/1806-9126-RBEF-2022-0115

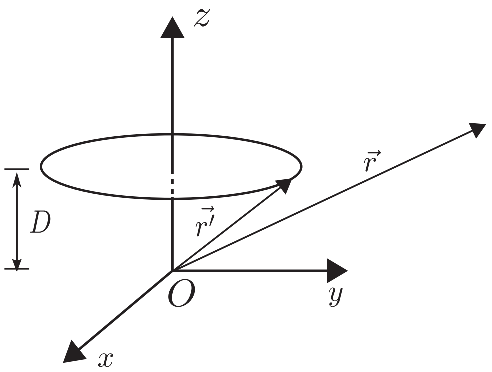

Figure 8: Eliptic coil arrangement to compute the off-axis magnetic field.

$y\ \hat{e}_y + z\ \hat{e}_z$, produced by the element $d\vec{\ell}$ located at $\vec{r}' = a\cos\theta'\ \hat{e}_x + b\sin\theta'\ \hat{e}_y + z'\ \hat{e}_z$, we calculate:

$$\vec{r} - \vec{r}' = (x - a\cos\theta')\ \hat{e}_x + (y - b\sin\theta')\ \hat{e}_y + (z - z')\ \hat{e}_z \tag{17}$$

$$|\vec{r} - \vec{r}'| = \left[(x - a\cos\theta')^2 + (y - b\sin\theta')^2 + (z - z')^2\right]^{1/2} \tag{18}$$

$$\begin{aligned} d\vec{\ell} \times (\vec{r} - \vec{r}') = {} & b\,(z' - z)\ \cos\theta'\ d\theta'\ \hat{e}_x \\ & + a\,(z' - z)\ \sin\theta'\ d\theta'\ \hat{e}_y \\ & + (-ay\sin\theta' - bx\cos\theta' + ab)\ d\theta'\ \hat{e}_z \end{aligned} \tag{19}$$

Replacing in the Biot-Sarvat law and setting $z' = D$, we obtain:

$$B_x(x, y, z, D) = \frac{\mu_0\, I}{4\,\pi} \int \frac{b\,(D - z)\ \cos\theta' d\theta'}{\left[(x - a\cos\theta')^2 + (y - b\sin\theta')^2 + (z - D)^2\right]^{3/2}} \tag{20}$$

$$B_y(x, y, z, D) = \frac{\mu_0\, I}{4\,\pi} \int \frac{a\,(D - z)\ \sin\theta' d\theta'}{\left[(x - a\cos\theta')^2 + (y - b\sin\theta')^2 + (z - D)^2\right]^{3/2}} \tag{21}$$

$$B_z(x, y, z, D) = \frac{\mu_0\, I}{4\,\pi} \int \frac{(-ay\sin\theta' - bx\cos\theta' + ab)\, d\theta'}{\left[(x - a\cos\theta')^2 + (y - b\sin\theta')^2 + (z - D)^2\right]^{3/2}} \tag{22}$$

To obtain the magnetic field in a point of space (x, y, z) due to an elliptic Helmholtz coil as shown in Figure 1, we have to use the superposition principle and the expressions (20), (21) and (22) for one loop located in $z' = D$ and for other loop located in $z' = -D$, then

$$B_{T_x} = B_x(x, y, z, D) + B_x(x, y, z, -D) \tag{23}$$

$$B_{T_y} = B_y(x, y, z, D) + B_y(x, y, z, -D) \tag{24}$$

$$B_{T_z} = B_z(x, y, z, D) + B_z(x, y, z, -D) \tag{25}$$

The Figure 9 shows the magnetic field B_{T_z} of the elliptical coil in the XY midplane ($z = 0$) perpendicular to the z axis with $D = 0.6$. It can be seen that the magnetic field produced by the coil presents a flat and elongated region following the contour of the elliptical coil. However, unlike the circular Helmholtz coil, in this case the magnetic field has a slight deviation towards the ends.

The Figure 10 shows the variation of the magnetic field according to the transverse direction xz. As expected, the homogeneity of the magnetic field is very similar to that of the circular Helmholtz coil. The figure compares the homogeneity of the Helmholtz coil with some cases of elliptical coils in the z-axis. As can be seen, the behavior in this axis is almost identical.

Figures 11 and 12 show the behavior of the magnetic field B_{T_z} in the directions of the y axis and the x axis. In these graphs the homogeneity of the elliptical coils

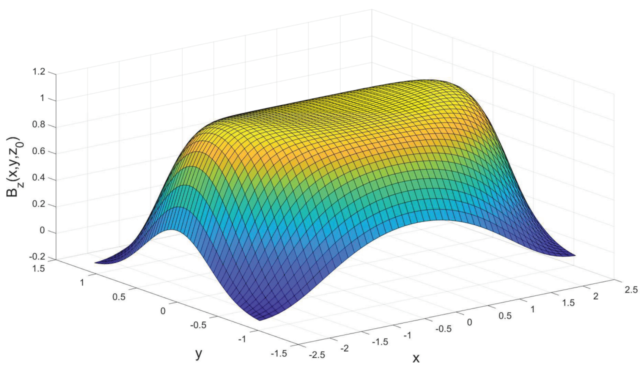

Figure 9: Magnetic field for an elliptic coil at the XY-midplane for $a = 2.5$, $b = 1.228$, and $D = 0.6$.

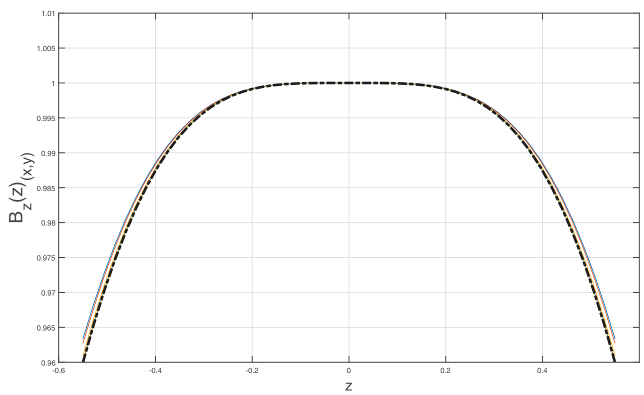

Figure 10: Magnetic field for some optimized elliptic coils and the Helmholtz coil (in dashed curve) for $D = 0.6$.

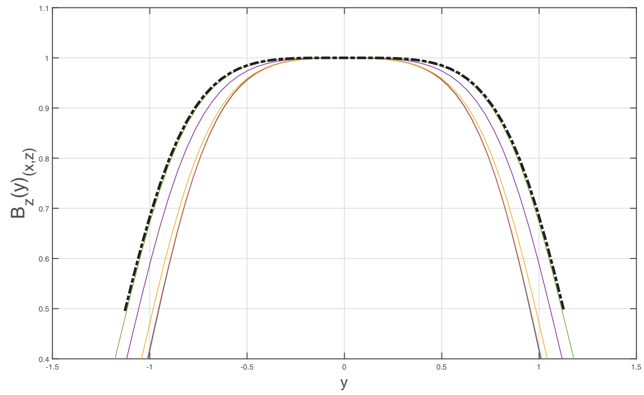

Figure 11: Magnetic Field comparison in the y direction.

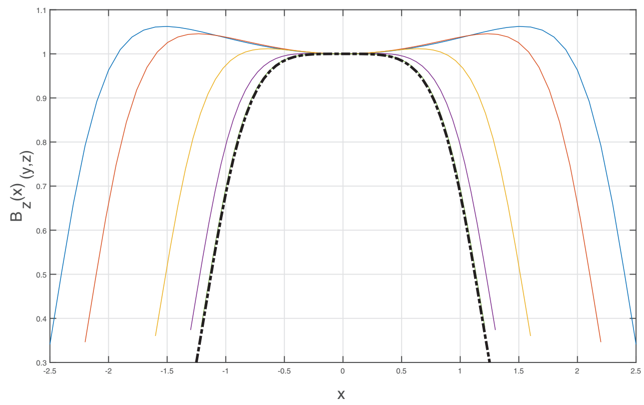

Figure 12: Magnetic Field comparison in the x direction.

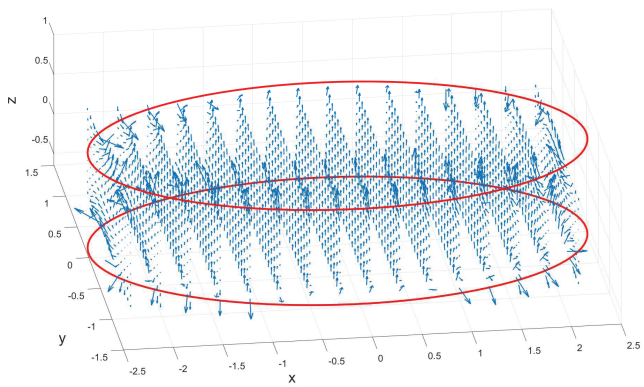

Figure 13: Vectorial plot of the magnetic field for the elliptic coil in three dimensions.

can be observed, for some values of the eccentricities. It is clear that, given the normalization adopted for the magnetic field of each array, the plateau region lengthens in the direction of the major axis of each ellipse and shortens in the direction of the minor axis of each ellipse.

In the Figure 13 we can see a vectorial description of the magnetic field produced by the elliptical coil. The purpose of this pictorial description corresponds more to a check of the correctness of the numerical calculation established for the elliptical coil. However, the fundamental characteristic of this calculation corresponds eminently to the determination of the locus for which the elliptical coil has a behavior homologous to that of the Helmholtz coil.

6. Conclusions

The most important conclusion of the present investigation corresponds to the fundamental fact that an elliptical coil formed by two symmetrical windings similar to the Helmhotz coil will have an optimized behavior similar to said coil for a locus in which the first three derivatives cancel in the center of symmetry.

Unlike the case of the Helmholtz coil for which this condition represents a fixed relationship between the radius and the distance between the windings, in the case of the elliptical coil there is a continuous locus that depends on the eccentricity of the coil.

It is notable that this locus represents a continuous family of symmetric coils for which the Helmhotz coil is a special case. It is even more remarkable that the homogeneity in the azimuthal axis of symmetry is practically not altered when the eccentricity of the coils varies. This fact is surprising and leads us to think about the possibility of expanding the magnetic field towards elongated shapes, maintaining the original advantages of homogeneity of the Helmholtz coil but optimizing the homogeneity of the magnetic field in the areas far from said axis.

We conclude that it is feasible to have a function that determines the relationship between the axes of an elliptical coil, that is, the eccentricity, in such a way that it is always possible to build an elliptical coil that sustains said optimization. A question that remains to be investigated based on the current conclusions corresponds to the fact of whether it is possible to increase the homogeneity in the cross-sectional region.

Supplementary material

The following online material is available for this article:
Appendix – Data for different a and b.

References

[1] A.E. Ruark and M.F. Peters, J. Opt. Soc. Am. **13**, 205 (1926).

[2] E.M. Purcell, Am. J. Phys **57**, 18 (1989).

[3] M.S. Crosser, S. Scott, A. Clark and P.M. Wilt, Rev. Sci. Instrum. **81**, 084701 (2010).

[4] E.R. Huggins and J.J. Lelek, Am. J. Phys. **47**, 992 (1979).

[5] J. Walker, D. Halliday and R. Resnick, *Fundamentals of physics* (John Wiley and Sons, Inc., New Jersey, 2018), p. 861.

[6] H.D. Young, R.A. Freedman, A.L. Ford and F.W. Sears, *Sears and Zemansky's university physics: With modern physics* (Pearson Addison Wesley, San Francisco, 2016), p. 952.

[7] R.A. Serway and J.W. Jewett, *Physics for Scientists and Engineers* (Cengage, Bostom, 2019), p. 794.

[8] D.J. Griffiths, *Introduction to Electrodynamics* (Pearson Education, New Jersey, 2013), p. 259.

[9] R.D. Strattan and F.J. Young, Appl. Sci. Res. **9**, 117 (1961).

[10] A.H. Firester, Rev. Sci. Instrum. **37**, 1264 (1966).

[11] R. Merritt, C. Purcell and G. Stroink, Rev. Sci. Instrum. **54**, 879 (1983).

[12] I. Sasada and Y. Nakashima, J. Appl. Phys. **99**, 08D904 (2006).

[13] R. Beiranvand, Rev. Sci. Instrum. **85**, 055115 (2014).

[14] J.J. Abbott, Rev. Sci. Instrum. **86**, 054701 (2015).

[15] F.J. Salazar, F.J. Nieves, A. Bayón and F. Gascón, Rev. Sci. Instrum. **88**, 095107 (2017).

[16] Y. Jin, D. Pan, Z. Sun, Z. Zou and L. Li, Rev. Sci. Instrum. **91**, 074708 (2020).

[17] J.A. Miranda, Am. J. Phys. **68**, 254 (2000).

[18] D. Romero-Abad, Phys. Educ. **56**, 025014 (2021).

[19] G. Giovannetti, A. Flori, D. de Marchi, G. Matarazzo, F. Frijia, S. Burchielli, D. Montanaro, G.D. Aquaro, L. Menichetti, Concepts Magn Reson Part B **47B**, e21361 (2017).

[20] J.R. Reitz, F.J. Milford and R.W. Christy, *Foundations of electromagnetic theory* (Addison-Wesley, Wilmington, 1996), p. 201.

[21] G.B. Arfken, H.J. Weber and F.E. Harris, *Mathematical Methods for Physicists: A Comprehensive Guide* (Academic Press, Cambridge, 2013), p. 927.

[22] M. Abramowitz and I.A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables* (Dover, New York, 1972), p. 587.

[23] D.J. Griffiths, *Introduction to Electrodynamics* (Pearson Education, New Jersey, 2013).