COMPUTATIONAL & APPLIED MATHEMATICS

Volume 28, N. 2, pp. 213–230, 2009

ISSN 0101-8205
www.scielo.br/cam

The bounded solutions to nonlinear fifth-order differential equations with delay

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Abstract. In this paper, we improve some boundedness results, which have been obtained with respect to nonlinear differential equations of fifth order without delay, to a certain functional differential equation with constant delay. We give an illustrative example and also verify our main result by means of Liaponov tecnique.

Mathematical subject classification: 34K20.

Key words: boundedness, Liapunov functional, nonlinear differential equations of fifth.

1 Introduction

Among the scores of articles on the qualitative theory of differential equations, the number of articles on boundedness of solutions to nonlinear fifth order differential equations with delay is significantly less than those on differential equations without delay. For those contributions on the boundedness of solutions of nonlinear fifth order differential equations without delay, one can refer to the papers of Abou-El Ela and Sadek [1], Chukwu [3], Sinha [14], Tunç [15, 16, 17], Yuan Hong [21] and some other references thereof. For those regarding the boundedness of solutions of nonlinear fifth order differential equations with delay, we cite Tunç ([18, 19]). All of the aforementioned contributions have focused on the Liapunov's second (direct) method [12] utilizing Liapunov functions and functionals.

#CAM-23/08. Received: 18/IX/08. Accepted: 09/II/09.

It is also worth mentioning that the construction of Liapunov functions and functionals for higher order nonlinear differential equations still remains as a general problem. In fact, the construction of Liapunov functional for delay differential equations of higher orders is more difficult than the derivation of Liapunov function for differential equations without delay. Since 1960 many excellent books, most of them in Russian, have been published on the qualitative behaviors of delay differential equations. See, for example, Burton [2], Èl'sgol'ts [4], Èl'sgol'ts and Norkin [5], Gopalsamy [6], Hale [7], Hale and Verduyn Lunel [8], Kolmanovskii and Myshkis [9], Kolmanovskii and Nosov [10], Krasovskii [11], Makay [13] and Yoshizawa [20] and the references listed in these books.

In this paper, we consider nonlinear fifth order delay differential equation

$$\begin{gathered} x^{(5)}(t) + f_1(t, x(t-r), x'(t-r), x''(t-r), x'''(t-r), \\ x^{(4)}(t-r))x^{(4)}(t) + \alpha_2 x'''(t) + \alpha_3 x''(t) + \alpha_4 x'(t) + f_5(x(t-r)) \\ = p(t, x(t-r), x'(t-r), x''(t-r), x'''^{(4)}(t-r)), \end{gathered} \tag{1}$$

which is equivalent to the system

$$\begin{aligned} & x'(t) = y(t), \quad y'(t) = z(t), \quad z'(t) = w(t), \quad w'(t) = u(t), \\ u'(t) = & - f_1(t, x(t-r), y(t-r), z(t-r), w(t-r), u(t-r))u(t) \\ & - \alpha_2 w(t) - \alpha_3 z(t) - \alpha_4 y(t) - f_5(x(t)) + \int_{t-r}^{t} f_5'(x(s))y(s)ds \\ & + p(t, x(t-r), y(t-r), z(t-r), w(t-r), u(t-r)), \end{aligned} \tag{2}$$

where f_1, f_5 and p are continuous functions for the arguments displayed explicitly in equation (1); r is a positive constant , that is, r is a constant delay; α_2, α_3 and α_4 are some positive constants. This fact guarantees the existence of the solution of delay differential equation (1) (see Èl'sgol'ts [4, p. 14]). It is assumed that the derivative $f_5'(x) \equiv \frac{df_5}{dx}$ exists and is continuous for all x and all solutions considered are assumed to be real valued. In addition, we assume that the right-hand side of system (2) satisfies a Lipschitz condition in $x(t)$, $y(t)$, $z(t)$, $w(t)$, $u(t)$, $x(t-r)$, $y(t-r)$, $z(t-r)$, $w(t-r)$ and $u(t-r)$. Then the solution is unique (see Èl'sgol'ts [4, p. 15]). Throughout the paper $x(t)$, $y(t)$,

$z(t)$, $w(t)$ and $u(t)$ are also abbreviated as x, y, z, w and u, respectively. It should be noted that the equation considered here, (1), is completely different than that investigated by Tunç ([18, 19]).

2 Preliminaries

In order to reach our main result, we will give some basic information for the general non-autonomous delay differential system, see also Burton [2], Èl'sgol'ts [4], Èl'sgol'ts and Norkin [5], Gopalsamy [6], Hale [7], Hale and Verduyn Lunel [8], Kolmanovskii and Myshkis [9], Kolmanovskii and Nosov [10], Krasovskii [11], Makay [13] and Yoshizawa [20]. Now, we consider the general non-autonomous delay differential system

$$x' = f(t, x_t), \quad x_t(\theta) = x(t+\theta), \quad -r \le \theta \le 0, \quad t \ge 0\,, \tag{3}$$

where $f : [0, \infty) \times C_H \to \Re^n$ is a continuous mapping, $f(t, 0) = 0$, and we suppose that f takes closed bounded sets into bounded sets of $\Re^n$. Here $(C, \|.\|)$ is the Banach space of continuous function $\phi\colon [-r, 0] \to \Re^n$ with supremum norm, $r > 0$; C_H is the open H-ball in C;

$$C_H := \left\{\phi \in \left(C\,[-r, 0]\,,\ \Re^n\right) : \|\phi\| < H\right\}\,.$$

Standard existence theory, see Burton [2], shows that if $\phi \in C_H$ and $t \ge 0$, then there is at least one continuous solution $x(t, t_0, \phi)$ such that on $[t_0,\ t_0 + \alpha)$ satisfying equation (3) for $t > t_0$, $x_t(t, \phi) = \phi$ and α is a positive constant. If there is a closed subset $B \subset C_H$ such that the solution remains in B, then $\alpha = \infty$. Further, the symbol $|.|$ will denote the norm in $\Re^n$ with $|x| = \max_{1\le i\le n} |x_i|$.

Definition 1 (See [2]). A continuous function $W\colon [0, \infty) \to [0, \infty)$ with $W(0) = 0$, $W(s) > 0$ if $s > 0$, and W strictly increasing is a wedge. (We denote wedges by W or W_i, where i an integer.)

Definition 2 (See [2]). Let D be an open set in $\Re^n$ with $0 \in D$. A function $V\colon [0, \infty) \times D \to [0, \infty)$ is called positive definite if $V(t, 0) = 0$ and if there is a wedge W_1 with $V(t, x) \ge W_1(|x|)$, and is called decrescent if there is a wedge W_2 with $V(t, x) \le W_2(|x|)$.

Definition 3 (See [2]). Let $V(t,\phi)$ be a continuous functional defined for $t \geq 0$, $\phi \in C_H$. The derivative of V along solutions of (3) will be denoted by $\dot{V}$ and is defined by the following relation

$$\dot{V}(t,\phi) = \limsup_{h\to 0} \frac{V\big(t+h, x_{t+h}(t_0,\phi)\big) - V\big(t, x_t(t_0,\phi)\big)}{h},$$

where $x(t_0,\phi)$ is the solution of (3) with $x_{t_0}(t_0,\phi) = \phi$.

Example. Let us consider the nonlinear second order delay differential equation:

$$\begin{aligned} x''(t) + \varphi(t, x(t-r), x'(t-r))x'(t) + f(x(t-r)) \\ = \ p(t, x(t), x(t-r), x'(t), x'(t-r)) \end{aligned}$$

which is equivalent to the system

$$\begin{aligned} x' &= y, \\ y' &= -\,\varphi(t, x(t-r), y(t-r))y - f(x) + \int_{t-r}^{t} f'(x(s))y(s)ds \\ &\quad + p(t, x, x(t-r), y, y(t-r)). \end{aligned}$$

We assume that the functions φ and p are continuous and satisfy the following:

$$\varphi(t,x(t-r), y(t-r)) \geq \alpha_1$$

and

$$\big|p(t, x, x(t-r), y, y(t-r))\big| \leq q(t)$$

for all t, $t \in [0,\infty)$, $x, x(t-r), y$ and $y(t-r)$, where r is a positive constant, constant delay, which will be determined later; $\max q(t) < \infty$ and $q \in L^1(0,\infty)$; f is continuously differentiable satisfying the conditions: $x^{-1}f(x) \geq \alpha_2$, $(x \neq 0)$, and $\big|f'(x)\big| \leq L$ for all x; α_1, α_2 and L are some positive constants. In particular, let us take

$$p(t, x, x(t-r), y, y(t-r)) = \frac{1}{1 + t^2 + x^2 + x^2(t-r) + y^2 + y^2(t-r)}.$$

Then, it follows that

$$\frac{1}{1+t^2+x^2+x^2(t-r)+y^2+y^2(t-r)} \leq \frac{1}{1+t^2}\,.$$

If we choose

$$q(t) = \frac{1}{1+t^2}\,,$$

then we have

$$\int_0^\infty q(s)ds = \int_0^\infty \frac{1}{1+s^2}ds = \frac{\pi}{2} < \infty,$$

that is, $q \in L^1(0,\infty)$. Now, we introduce the Liapunov functional

$$2V(t,x_t,y_t) = \left[2\int_0^x f(s)ds + y^2 + 2\lambda\int_{-r}^0 \int_{t+s}^t y^2(\theta)d\theta ds\right]^{\frac{1}{2}},$$

where λ is a positive constant which will be determined later. It is clear that the functional $V(t,x_t,y_t)$ is positive definite:

$$V(t,0,0) = 0,$$

$$V(t,x_t,y_t) = \frac{1}{2}\left[2\int_0^x f(s)ds + y^2 + 2\lambda\int_{-r}^0 \int_{t+s}^t y^2(\theta)d\theta ds\right]^{\frac{1}{2}}$$

$$\geq \frac{1}{2}\left[2\int_0^x \frac{f(s)}{s}sds + y^2\right]^{\frac{1}{2}} \geq \frac{1}{2}(\alpha_2 x^2 + y^2)^{\frac{1}{2}} > 0$$

for all $x(\neq 0)$ and y. Along a trajectory of the equation, we have

$$\begin{aligned}\frac{d}{dt}V(t,x_t,y_t) \;=\; &\frac{1}{4}\left[2\int_0^x f(s)ds + y^2 + 2\lambda\int_{-r}^{0}\int_{t+s}^{t} y^2(\theta)d\theta ds\right]^{\frac{-1}{2}}\\ &\times\left[-2\varphi(t,x(t-r),y(t-r))y^2 + 2y\int_{t-r}^{t} f'(x(s))y(s)ds\right.\\ &+\frac{2y}{1+t^2+x^2+x^2(t-r)+y^2+y^2(t-r)}\\ &\left.+\,2\lambda y^2 r - 2\lambda\int_{t-r}^{t} y^2(s)ds\right].\end{aligned}$$

Hence, we get

$$\begin{aligned}V(t,x_t,y_t)\frac{d}{dt}V(t,x_t,y_t) &= -\frac{1}{4}\big[\varphi(t,x(t-r),y(t-r))\big]y^2\\ +\frac{1}{4}y\int_{t-r}^{t} f'(x(s))y(s)ds &+ \frac{y}{4(1+t^2+x^2+x^2(t-r)+y^2+y^2(t-r))}\\ &+\frac{1}{4}\lambda y^2 r - \frac{1}{4}\lambda\int_{t-r}^{t} y^2(s)ds.\end{aligned}$$

In view of afore mentioned assumptions, the inequality $2|uv| \le u^2 + v^2$ and the fact

$$\frac{y}{1+t^2+x^2+x^2(t-r)+y^2+y^2(t-r)} \le \frac{|y|}{1+t^2}$$

we find

$$\begin{aligned}V(t,x_t,y_t)\frac{d}{dt}V(t,x_t,y_t) \;\le\; &-\frac{1}{8}\big[2\alpha_1-(L+2\lambda)r\big]\,y^2 + \frac{|y|}{4(1+t^2)}\\ &+\frac{1}{8}(L-2\lambda)\int_{t-r}^{t} y^2(s)ds.\end{aligned}$$

If we choose $\lambda = \dfrac{L}{2}$, then we have

$$V(t,x_t,y_t)\frac{d}{dt}V(t,x_t,y_t) \le -\frac{1}{4}\big[\alpha_1 - Lr\big]\,y^2 + \frac{|y|}{4(1+t^2)}\,.$$

Therefore, it follows that

$$\begin{aligned} V(t,x_t,y_t)\frac{d}{dt}V(t,x_t,y_t) &\leq -\frac{1}{4}\alpha\, y^2 + \frac{|y|}{4(1+t^2)} \\ &\leq \frac{|y|}{4(1+t^2)} \\ &\leq \frac{1}{2(1+t^2)}V(t,x_t,y_t)\,. \end{aligned}$$

for some constant $\alpha > 0$ provided that $r < \dfrac{\alpha_1}{L}$. Thus, we get

$$\frac{d}{dt}V(t,x_t,y_t) \leq \frac{1}{2(1+t^2)}\,.$$

Now, integrating this inequality from 0 to t and using the fact $\dfrac{1}{1+t^2} \in L^1(0,\infty)$, we have

$$V(t,x_t,y_t) \leq V(0,x_0,y_0) + \frac{\pi}{4}\,.$$

Therefore, one can conclude that

$$|x| \leq K, \quad |y| \leq K$$

for all $t \geq 0$. That is,

$$|x| \leq K, \quad \left|x'\right| \leq K$$

for all $t \geq 0$. This fact shows that all solutions of the equation considered are bounded.

3 Main result

We establish the following result.

Theorem. *In addition to the basic assumptions imposed on the functions f_1, f_5 and p appearing in equation* (1)*, we assume that there are positive constants α, α_1, α_2, α_3, α_4, ε, ε_0, ε_1 , δ and λ such that the following conditions hold:*

(i) $\alpha_1\alpha_2 - \alpha_3 > 0$, $(\alpha_1\alpha_2 - \alpha_3)\alpha_3 - (\alpha_1\alpha_4 - \alpha_5)\alpha_1 > 0$,

$$\delta_0 := (\alpha_3\alpha_4 - \alpha_2\alpha_5)(\alpha_1\alpha_2 - \alpha_3) - (\alpha_1\alpha_4 - \alpha_5)^2 > 0,$$

$$\Delta_1 := \frac{(\alpha_3\alpha_4 - \alpha_2\alpha_5)(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} - (\alpha_1\alpha_4 - \alpha_5) \geq 2\varepsilon\alpha_2 \quad \textit{and}$$

$$\Delta_2 := \frac{\alpha_3\alpha_4 - \alpha_2\alpha_5}{\alpha_1\alpha_4 - \alpha_5} - \frac{\alpha_1\alpha_4 - \alpha_5}{\alpha_1\alpha_2 - \alpha_3} - \frac{\varepsilon}{\alpha_1} > 0.$$

(ii) $\varepsilon_0 \leq f_1(t, x(t-r), y(t-r), z(t-r), w(t-r), u(t-r)) - \alpha_1 \leq \varepsilon_1$ *for all* t, $x(t-r)$, $y(t-r)$, $z(t-r)$, $w(t-r)$ *and* $u(t-r)$, *where the constant* ε_1 *satisfies the inequality*

$$\varepsilon_1 \leq \min\left\{\frac{\varepsilon\alpha_2(\alpha_1\alpha_4 - \alpha_5)^2}{16\alpha_4^2(\alpha_1\alpha_2 - \alpha_3)^2}, \frac{\varepsilon}{4\alpha_1^2}, \frac{\varepsilon\alpha_4}{4\delta^2}\right\}.$$

(iii) $f_5(0) = 0$, $f_5(x) \neq 0$ *if* $x \neq 0$, $x^{-1}f_5(x) \geq \alpha(x \neq 0)$, *and* $f_5'(x) \leq \alpha_5$ *for all* x.

$$\left(f_5'(x) - \alpha_5\right)^2 < \min\left\{\frac{\varepsilon^2\alpha_4}{4}, \frac{\varepsilon^2\alpha_2\alpha_4}{16}\right\}.$$

(iv) $|p(t, x(t-r), y(t-r), z(t-r), w(t-r), u(t-r))| \leq q(t)$ *for all* t, $x(t-r)$, $y(t-r)$, $z(t-r)$, $w(t-r)$ *and* $u(t-r)$, *where* $\max q(t) < \infty$ *and* $q \in L^1(0, \infty)$.

Then, there exists a finite positive constant K *such that the solution* $x(t)$ *of equation* (1) *defined by the initial functions*

$$x(t) = \phi(t),\ \ x'(t) = \phi'(t),\ \ x''(t) = \phi''(t),\ \ x'''(t) = \phi'''(t),\ \ t_0 - r \leq t \leq t_0$$

satisfies the inequalities

$$|x(t)| \leq K, \quad \left|x'(t)\right| \leq K, \quad \left|x''(t)\right| \leq K, \quad \left|x'''(t)\right| \leq K, \quad \left|x^{(4)}(t)\right| \leq K$$

for all $t \geq t_0$, *where* $\phi \in C^4\left(\left[t_0 - r,\ t_0\right],\ \Re\right)$, *provided that*

$$r < \min\left\{\frac{\varepsilon_0}{2\alpha_5}, \frac{\varepsilon}{2\alpha_1\alpha_5}, \frac{7\varepsilon\alpha_2(\alpha_1\alpha_4 - \alpha_5)}{8\alpha_4\alpha_5(\alpha_1\alpha_2 - \alpha_3)}, \frac{\varepsilon\alpha_4}{2(\delta\alpha_5 + 2\lambda)}\right\}.$$

Proof. For the proof, we introduce the Liapunov functional $V_0 = V_0(t, x_t, y_t, z_t, w_t, u_t)$:

$$
\begin{aligned}
2V_0 = {} & u^2 + 2\alpha_1 uw + \frac{2\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} uz \\
& + 2\delta uy + \left[\alpha_1^2 + \alpha_2 - \frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5}\right] w^2 \\
& + 2\left[\alpha_3 + \frac{\alpha_1\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} - \delta\right] wz + 2\alpha_1\delta wy + 2\alpha_4 yw + 2wf_5(x) \\
& + \alpha_1\alpha_3 z^2 + \left[\frac{\alpha_2\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} - \alpha_4 - \alpha_1\delta\right] z^2 + 2\delta\alpha_2 yz + 2\alpha_1\alpha_4 yz \\
& - 2\alpha_5 zy + 2\alpha_1 zf_5(x) + \frac{\alpha_4^2(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} y^2 + (\delta\alpha_3 - \alpha_1\alpha_5)y^2 \\
& + \frac{2\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} yf_5(x) + 2\delta\int_0^x f_5(\xi)d\xi + 2\lambda\int_{-r}^{0}\int_{t+s}^{t} y^2(\theta)d\theta ds \\
& + 2\mu\int_{-r}^{0}\int_{t+s}^{t} z^2(\theta)d\theta ds + 2\rho\int_{-r}^{0}\int_{t+s}^{t} w^2(\theta)d\theta ds,
\end{aligned}
\tag{4}
$$

where λ, μ and ρ are some positive constants which will be determined later in the proof and δ is also a positive constant satisfying

$$
\delta := \frac{\alpha_5(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} + \varepsilon\,. \tag{5}
$$

Now, in view of (4), it follows that

$$
\begin{aligned}
2V_0 = {} & \left[u + \alpha_1 w + \frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} z + \delta y\right]^2 + \frac{\alpha_4\delta_0}{(\alpha_1\alpha_4 - \alpha_5)^2}\left(z + \frac{\alpha_5}{\alpha_4} y\right)^2 \\
& + \frac{(\alpha_1\alpha_4 - \alpha_5)}{(\alpha_1\alpha_2 - \alpha_3)}\left[\frac{\alpha_1\alpha_2 - \alpha_3}{\alpha_1\alpha_4 - \alpha_5} f_5(x) + \frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} y + \alpha_1 z + w\right]^2 \\
& + \Delta_2\,(w + \alpha_1 z)^2 + \frac{\varepsilon}{\alpha_1} w^2 + 2\varepsilon\left(\frac{\alpha_3\alpha_4 - \alpha_2\alpha_5}{\alpha_1\alpha_4 - \alpha_5}\right) yz + 2\lambda\int_{-r}^{0}\int_{t+s}^{t} y^2(\theta)d\theta ds \\
& + 2\mu\int_{-r}^{0}\int_{t+s}^{t} z^2(\theta)d\theta ds + 2\rho\int_{-r}^{0}\int_{t+s}^{t} w^2(\theta)d\theta ds + \sum_{i=1}^{2} V_i,
\end{aligned}
$$

where

$$V_1 := 2\delta \int_0^x f_5(\xi)d\xi - \frac{\alpha_1\alpha_2 - \alpha_3}{\alpha_1\alpha_4 - \alpha_5} f_5^2(x),$$

$$V_2 := \left[\delta\alpha_3 - \alpha_1\alpha_5 - \frac{\alpha_5^2\delta_0}{\alpha_4(\alpha_1\alpha_4 - \alpha_5)^2} - \delta^2\right] y^2.$$

The assumptions $f_5(0) = 0$, $f_5(x)\operatorname{sgn} x > 0$, $x^{-1}f_5(x) \geq \alpha$ $(x \neq 0)$, $f_5'(x) \leq \alpha_5$ and (5) imply that

$$\begin{aligned} V_1 &= 2\varepsilon \int_0^x f_5(\xi)d\xi + \frac{2(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} \int_0^x f_5(\xi)\left[\alpha_5 - f_5'(\xi)\right] \\ &\geq 2\varepsilon \int_0^x f_5(\xi)d\xi \geq 2\varepsilon \int_0^x \alpha\xi d\xi = \varepsilon\alpha x^2 \end{aligned}$$

and

$$\begin{aligned} V_2 &= \left[\frac{\alpha_5\delta_0}{\alpha_4(\alpha_1\alpha_4 - \alpha_5)} - 2\varepsilon\left(\varepsilon + \frac{2\alpha_5(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} - \alpha_3\right)\right] y^2 \\ &\geq \frac{\alpha_5\delta_0}{2\alpha_4(\alpha_1\alpha_4 - \alpha_5)} y^2 \end{aligned}$$

provided that

$$\frac{\alpha_5\delta_0}{4\alpha_4(\alpha_1\alpha_4 - \alpha_5)} \geq \varepsilon\left[\varepsilon + \frac{2\alpha_5(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} - \alpha_3\right],$$

which we now assume. Now, the estimates related to V_1 and V_2 yield

$$\begin{aligned} 2V_0 \geq& \left[u + \alpha_1 w + \frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} z + \delta y\right]^2 \\ &+ \frac{\alpha_4\delta_0}{(\alpha_1\alpha_4 - \alpha_5)^2}\left(z + \frac{\alpha_5}{\alpha_4} y\right)^2 + \varepsilon\alpha x^2 + \Delta_2(w + \alpha_1 z)^2 \\ &+ \frac{\alpha_5\delta_0}{2\alpha_4(\alpha_1\alpha_4 - \alpha_5)} y^2 \frac{\varepsilon}{\alpha_1} w^2 + 2\varepsilon\left(\frac{\alpha_3\alpha_4 - \alpha_2\alpha_5}{\alpha_1\alpha_4 - \alpha_5}\right) yz \\ &+ 2\lambda \int_{-r}^{0}\int_{t+s}^{t} y^2(\theta)d\theta ds + 2\mu \int_{-r}^{0}\int_{t+s}^{t} z^2(\theta)d\theta ds \\ &+ 2\rho \int_{-r}^{0}\int_{t+s}^{t} w^2(\theta)d\theta ds. \end{aligned} \tag{6}$$

Following a similar way as that of Tunç [18], one can easily conclude from (6) that

$$V_0 \geq D_8\big(x^2 + y^2 + z^2 + w^2 + u^2\big) \tag{7}$$

for a sufficiently small positive constant D_8. Now, let

$$\frac{d}{dt} V_0\big(t, x_t, y_t, z_t, w_t, u_t\big) = \dot{V}_0 \quad \text{and} \quad \big(x(t), y(t), z(t), w(t), u(t)\big)$$

be a solution of system (2). Then, a direct computation along this solution shows that

$$
\begin{aligned}
&\frac{d}{dt} V_0(t, x_t, y_t, z_t, w_t, u_t) = \\
&- \big[f_1(t, x(t-r), y(t-r), z(t-r), w(t-r), u(t-r)) - \alpha_1\big]\, u^2 \\
&- \left[\alpha_1\alpha_2 - \left\{\alpha_3 + \frac{\alpha_1\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} - \delta\right\}\right] w^2 \\
&- \left[\frac{\alpha_3\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} - \{\delta\alpha_2 + \alpha_1\alpha_4 - \alpha_5\}\right] z^2 - \left[\delta\alpha_4 - \frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} f_5'(x)\right] y^2 \\
&- \alpha_1 \big[f_1(t, x(t-r), y(t-r), z(t-r), w(t-r), u(t-r)) - \alpha_1\big]\, wu \\
&- \frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} \big[f_1(t, x(t-r), y(t-r), z(t-r), w(t-r), u(t-r)) - \alpha_1\big]\, zu \\
&- \delta \big[f_1(t, x(t-r), y(t-r), z(t-r), w(t-r), u(t-r)) - \alpha_1\big]\, yu \\
&- \big[\alpha_5 - f_5'(x)\big]\, yw - \alpha_1 \big[\alpha_5 - f_5'(x)\big]\, yz + u \int_{t-r}^{t} f_5'(x(s)) y(s)\, ds \\
&+ \alpha_1 w \int_{t-r}^{t} f_5'(x(s)) y(s)\, ds + \frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} z \int_{t-r}^{t} f_5'(x(s)) y(s)\, ds \\
&+ \delta y \int_{t-r}^{t} f_5'(x(s)) y(s)\, ds + \left[u + \alpha_1 w + \frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} z + \delta y\right] \\
&\times p(t, x(t-r), y(t-r), z(t-r), w(t-r), u(t-r)) \\
&+ \lambda y^2 r - \lambda \int_{t-r}^{t} y^2(s)\, ds + \mu z^2 r - \mu \int_{t-r}^{t} z^2(s)\, ds + \rho w^2 r - \rho \int_{t-r}^{t} w^2(s)\, ds.
\end{aligned}
$$

In view of the assumptions of theorem and expression (5), we have that

$$\left[\alpha_1\alpha_2 - \left\{\alpha_3 + \frac{\alpha_1\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5} - \delta\right\}\right] \geq \varepsilon,$$

$$\frac{\alpha_3\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5} - (\delta\alpha_2 + \alpha_1\alpha_4 - \alpha_5) > \varepsilon\alpha_2$$

and

$$\delta\alpha_4 - \frac{\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5} f_5'(x) \geq \varepsilon\alpha_4.$$

By the assumption $f_5'(x) \leq \alpha_5$ and inequality $2\,|uv| \leq u^2 + v^2$, we obtain the following:

$$u\int_{t-r}^{t} f_5'(x(s))y(s)ds \leq \frac{\alpha_5}{2}ru^2 + \frac{\alpha_5}{2}\int_{t-r}^{t} y^2(s)ds,$$

$$\alpha_1 w\int_{t-r}^{t} f_5'(x(s))y(s)ds \leq \frac{\alpha_1\alpha_5}{2}rw^2 + \frac{\alpha_1\alpha_5}{2}\int_{t-r}^{t} y^2(s)ds,$$

$$\frac{\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}z\int_{t-r}^{t} f_5'(x(s))y(s)ds \leq \frac{\alpha_4\alpha_5(\alpha_1\alpha_2-\alpha_3)}{2(\alpha_1\alpha_4-\alpha_5)}rz^2 + \frac{\alpha_4\alpha_5(\alpha_1\alpha_2-\alpha_3)}{2(\alpha_1\alpha_4-\alpha_5)}\int_{t-r}^{t} y^2(s)ds$$

and

$$\delta y\int_{t-r}^{t} f_5'(x(s))y(s)ds \leq \frac{\delta\alpha_5}{2}ry^2 + \frac{\delta\alpha_5}{2}\int_{t-r}^{t} y^2(s)ds.$$

Making use of these inequalities, we get

$$\begin{aligned}
\dot{V}_0 \leq & -[f_1(t,x(t-r),y(t-r),z(t-r),w(t-r),u(t-r)) - \alpha_1]\; u^2 \\
& - \varepsilon w^2 - (\varepsilon\alpha_2)z^2 - (\varepsilon\alpha_4)y^2 \\
& - \alpha_1\,[f_1(t,x(t-r),y(t-r),z(t-r),w(t-r),u(t-r)) - \alpha_1]\,wu \\
& - \frac{\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}\,[f_1(t,x(t-r),y(t-r),z(t-r),w(t-r),u(t-r)) - \alpha_1]\,zu \\
& - \delta\,[f_1(t,x(t-r),y(t-r),z(t-r),w(t-r),u(t-r)) - \alpha_1]\,yu
\end{aligned}$$

$$
\begin{aligned}
&+ \left[f_5'(x) - \alpha_5\right] yw + \alpha_1 \left[f_5'(x) - \alpha_5\right] yz + \left| u + \alpha_1 w + \frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} z + \delta y \right| \\
&\times |p(t, x(t-r), y(t-r), z(t-r), w(t-r), u(t-r))| \\
&+ \frac{\alpha_5}{2} r u^2 + \frac{\alpha_1\alpha_5}{2} r w^2 + \frac{\alpha_4\alpha_5(\alpha_1\alpha_2 - \alpha_3)}{2(\alpha_1\alpha_4 - \alpha_5)} r z^2 + \frac{\delta\alpha_5}{2} r y^2 \\
&+ \lambda\, r y^2 + \mu z^2 r + \rho w^2 r + \mu \int\limits_{t-r}^{t} z^2(s) ds - \rho \int\limits_{t-r}^{t} w^2(s) ds \\
&+ \left[\frac{\alpha_4\alpha_5(\alpha_1\alpha_2 - \alpha_3)}{2(\alpha_1\alpha_4 - \alpha_5)} + \frac{\alpha_5}{2} + \frac{\alpha_1\alpha_5}{2} + \frac{\delta\alpha_5}{2} - \lambda \right] \int\limits_{t-r}^{t} y^2(s) ds \\
&\le -\frac{1}{4} \left[(f_1(t, x(t-r), y(t-r), z(t-r), w(t-r), u(t-r)) - \alpha_1) - 2\alpha_5 r \right] u^2 \\
&- \frac{1}{4} \left[2\varepsilon - \left(\rho + \frac{\alpha_1\alpha_5}{2} \right) r \right] w^2 - \left[\frac{7\varepsilon\alpha_2}{8} - \left(\mu + \frac{\alpha_4\alpha_5(\alpha_1\alpha_2 - \alpha_3)}{2(\alpha_1\alpha_4 - \alpha_5)} \right) r \right] z^2 \\
&- \left[\frac{\varepsilon\alpha_4}{4} - \left(\frac{\delta\alpha_5}{2} + \lambda \right) r \right] y^2 + \left| u + \alpha_1 w + \frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} z + \delta y \right| \\
&\times |p(t, x(t-r), y(t-r), z(t-r), w(t-r), u(t-r))| \\
&+ \left[\frac{\alpha_4\alpha_5(\alpha_1\alpha_2 - \alpha_3)}{2(\alpha_1\alpha_4 - \alpha_5)} + \frac{\alpha_5}{2} + \frac{\alpha_1\alpha_5}{2} + \frac{\delta\alpha_5}{2} - \lambda \right] \int\limits_{t-r}^{t} y^2(s) ds \\
&- \mu \int\limits_{t-r}^{t} z^2(s) ds - \rho \int\limits_{t-r}^{t} w^2(s) ds - \sum_{k=4}^{8} V_k, \qquad (8)
\end{aligned}
$$

where

$$
\begin{aligned}
V_4 = {} & \frac{1}{4}\left[f_1(t, x(t-r), y(t-r), z(t-r), w(t-r), u(t-r)) - \alpha_1 \right] u^2 \\
& + \frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} \left[f_1(t, x(t-r), y(t-r), z(t-r), w(t-r), u(t-r)) - \alpha_1 \right] zu \\
& + \frac{\varepsilon\alpha_2}{16} z^2,
\end{aligned}
$$

$$
\begin{aligned}
V_5 &= \frac{1}{4}\big[f_1(t, x(t-r), y(t-r), z(t-r), w(t-r), u(t-r)) - \alpha_1\big]\, u^2 \\
&\quad + \alpha_1\big[f_1(t, x(t-r), y(t-r), z(t-r), w(t-r), u(t-r)) - \alpha_1\big] w\, u \\
&\quad + \frac{\varepsilon}{4} w^2, \\
V_6 &= \frac{1}{4}\big[f_1(t, x(t-r), y(t-r), z(t-r), w(t-r), u(t-r)) - \alpha_1\big]\, u^2 \\
&\quad + \delta\big[f_1(t, x(t-r), y(t-r), z(t-r), w(t-r), u(t-r)) - \alpha_1\big] y u \\
&\quad + \frac{\varepsilon\alpha_4}{4} y^2, \\
V_7 &= \frac{\varepsilon}{4}\, w^2 - \big[f_5'(x) - \alpha_5\big] y w + \frac{\varepsilon\alpha_4}{4} y^2, \\
V_8 &= \frac{\varepsilon\alpha_2}{16} z^2 - \alpha_1\big[f_5'(x) - \alpha_5\big] y z + \frac{\varepsilon\alpha_4}{4} y^2.
\end{aligned}
$$

Now, subject to the assumptions (ii) and (iii) of theorem, one easily finds that

$$
V_4 \geq 0, \quad V_5 \geq 0, \quad V_6 \geq 0, \quad V_7 \geq 0, \quad V_8 \geq 0,
$$

respectively. Gathering the above discussion into (8) and making use of the assumption (ii), it follows that

$$
\begin{aligned}
\dot{V}_0 \leq & -\frac{1}{4}\left[\varepsilon_0 - 2\alpha_5 r\right]\, u^2 - \frac{1}{4}\left[2\varepsilon - (4\rho + 2\alpha_1\alpha_5) r\right]\, w^2 \\
& - \left[\frac{7\varepsilon\alpha_2}{8} - \left(\mu + \frac{\alpha_4\alpha_5(\alpha_1\alpha_2 - \alpha_3)}{2(\alpha_1\alpha_4 - \alpha_5)}\right) r\right] z^2 - \left[\frac{\varepsilon\alpha_4}{4} - \left(\frac{\delta\alpha_5}{2} + \lambda\right) r\right] y^2 \\
& + \left[\frac{\alpha_4\alpha_5(\alpha_1\alpha_2 - \alpha_3)}{2(\alpha_1\alpha_4 - \alpha_5)} + \frac{\alpha_5}{2} + \frac{\alpha_1\alpha_5}{2} + \frac{\delta\alpha_5}{2} - \lambda\right] \int_{t-r}^{t} y^2(s)ds \\
& + \left| u + \alpha_1 w + \frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} z + \delta y \right| \\
& \times \left| p(t, x(t-r), y(t-r), z(t-r), w(t-r), u(t-r)) \right| \\
& - \mu \int_{t-r}^{t} z^2(s)ds - \rho \int_{t-r}^{t} w^2(s)ds.
\end{aligned} \tag{9}
$$

If we let

$$\lambda = \left[\frac{\alpha_4\alpha_5(\alpha_1\alpha_2 - \alpha_3)}{2(\alpha_1\alpha_4 - \alpha_5)} + \frac{\alpha_5}{2} + \frac{\alpha_1\alpha_5}{2} + \frac{\delta\alpha_5}{2}\right],$$

$$\mu = \frac{\alpha_4\alpha_5(\alpha_1\alpha_2 - \alpha_3)}{2(\alpha_1\alpha_4 - \alpha_5)} \quad \text{and} \quad \rho = \frac{\alpha_1\alpha_5}{2},$$

then subject to the assumption (i) of theorem the inequality (9) implies that

$$\begin{aligned}\dot{V}_0 \;\le\; & -\frac{1}{4}\left[\varepsilon_0 - 2\alpha_5 r\right]\, u^2 - \frac{1}{4}\left[2\varepsilon - 4(\alpha_1\alpha_5)r\right]\, w^2 \\ & - \left[\frac{7\varepsilon\alpha_2}{8} - \frac{\alpha_4\alpha_5(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} r\right] z^2 - \left[\frac{\varepsilon\alpha_4}{4} - \left(\frac{\delta\alpha_5}{2} + \lambda\right)\, r\right] y^2 \\ & + \left|u + \alpha_1 w + \frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} z + \delta y\right| \\ & \times \left|p(t, x(t-r), y(t-r), z(t-r), w(t-r), u(t-r))\right|.\end{aligned}$$

Thus, one easily obtains that

$$\begin{aligned}&\frac{d}{dt} V_0(t, x_t, y_t, z_t, w_t, u_t) \le -\tau(y^2 + z^2 + w^2 + u^2) \\ &\quad + \left|u + \alpha_1 w + \frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} z + \delta y\right| \\ &\quad \times \left|p(t, x(t-r), y(t-r), z(t-r), w(t-r), u(t-r))\right|\end{aligned}$$

for some constant $\tau > 0$ provided that

$$r < \min\left\{\frac{\varepsilon_0}{2\alpha_5}, \frac{\varepsilon}{2\alpha_1\alpha_5}, \frac{7\varepsilon\alpha_2(\alpha_1\alpha_4 - \alpha_5)}{8\alpha_4\alpha_5(\alpha_1\alpha_2 - \alpha_3)}, \frac{\varepsilon\alpha_4}{2(\delta\alpha_5 + 2\lambda)}\right\}.$$

Clearly, we have

$$\begin{aligned}&\frac{d}{dt} V_0(t, x_t, y_t, z_t, w_t, u_t) \le \left|u + \alpha_1 w + \frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} z + \delta y\right| \\ &\quad \times \left|p(t, x(t-r), y(t-r), z(t-r), w(t-r), u(t-r))\right| \\ &\quad \le \left(|u| + \alpha_1 |w| + \frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} |z| + \delta |y|\right) q(t) \\ &\quad \le D_7 \left(|y| + |z| + |w| + |u|\right) q(t),\end{aligned}$$

where

$$D_7 = \max\left\{1,\ \alpha_2,\ \frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5},\ \delta\right\}.$$

Using the fact $|y| < 1 + y^2$, $|z| < 1 + z^2$, $|w| < 1 + w^2$ and $|u| < 1 + u^2$, we see that

$$\frac{d}{dt} V_0(t, x_t, y_t, z_t, w_t, u_t) \leq D_7 \left[4 + \left(y^2 + z^2 + w^2 + u^2\right)\right] q(t).$$

In view of (7), it follows that

$$\frac{d}{dt} V_0\big(t, x_t, y_t, z_t, w_t, u_t\big) \leq 4D_7 q(t) + D_7 D_8^{-1} V_0\big(t, x_t, y_t, z_t, w_t, u_t\big) q(t).$$

Finally, integrating this inequality from t_0, $(t_0 \geq 0)$, to t and using the assumption $q \in L^1(0, \infty)$ and Gronwall-Reid-Bellman inequality, we conclude that

$$\begin{aligned}
V_0(t, x_t, y_t, z_t, w_t, u_t) &\leq V_0(t_0, x_{t_0}, y_{t_0}, z_{t_0}, w_{t_0}, u_{t_0}) \\
&\quad + 4D_7 A + D_7 D_8^{-1} \int_{t_0}^{t} (V_0(s, x_s, y_s, z_s, w_s, u_s)) q(s) ds \\
&\leq \left[V_0(t_0, x_{t_0}, y_{t_0}, z_{t_0}, w_{t_0}, u_{t_0}) + 4D_7 A\right] \exp\left(D_7 D_8^{-1} \int_{t_0}^{t} q(s) ds\right) \\
&\leq \left[V_0(t_0, x_{t_0}, y_{t_0}, z_{t_0}, w_{t_0}, u_{t_0}) + 4D_7 A\right] \exp\left(D_7 D_8^{-1} A\right) = K_1 < \infty,
\end{aligned}$$

where $K_1 > 0$ is a constant,

$$K_1 = \left[V_0\big(t_0, x_{t_0}, y_{t_0}, z_{t_0}, w_{t_0}, u_{t_0}\big) + 4D_7 A\right] \times \exp\left(D_7 D_8^{-1} A\right),$$

and

$$A = \int_0^{\infty} q(s) ds.$$

Now, the inequality (7) and the last inequality together give that

$$x^2 + y^2 + z^2 + w^2 + u^2 \leq D_8^{-1} V(t, x_t, y_t, z_t, w_t, u_t) \leq K,$$

where $K = K_1 D_8^{-1}$. This fact completes the proof of theorem. □

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