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Sharpness of Muqattash-Yahdi problem

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Abstract. Let ψ denote the psi (or digamma) function. We determine the values of the parameters p, q and r such that

$$\psi(n) \approx \ln(n+p) - \frac{q}{n+r}$$

is the best approximations. Also, we present closer bounds for psi function, which sharpens some known results due to Muqattash and Yahdi, Qi and Guo, and Mortici.

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Key words: psi function, polygamma functions, rate of convergence, approximations.

The gamma function is usually defined for $x > 0$ by

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt\ .$$

The logarithmic derivative of the gamma function:

$$\psi(x) = \frac{d}{dx}\{\ln\Gamma(x)\} = \frac{\Gamma'(x)}{\Gamma(x)} \quad \text{or} \quad \ln\Gamma(x) = \int_1^x \psi(t)\,dt$$

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is known as the psi (or digamma) function. The successive derivatives of the psi function $\psi(x)$:

$$\psi^{(n)}(x) := \frac{d^n}{dx^n}\{\psi(x)\} \quad (n \in \mathbb{N})$$

are called the polygamma functions.

The following asymptotic formula is well known for the psi function:

$$\begin{aligned}\psi(x) &\sim \ln x - \frac{1}{2x} - \sum_{n=1}^{\infty} \frac{B_{2n}}{2nx^{2n}} \\ &= \ln x - \frac{1}{2x} - \frac{1}{12x^2} + \frac{1}{120x^4} - \frac{1}{252x^6} + \cdots \quad (x \to \infty)\end{aligned} \tag{1}$$

(see [1, p. 259]), where

$$B_0 = 1, \quad B_1 = -\frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_4 = -\frac{1}{30}, \quad B_6 = \frac{1}{42},$$
$$B_8 = -\frac{1}{30}, \quad B_{10} = \frac{5}{66}, \ldots, \quad \text{and} \quad B_{2n+1} = 0 \quad (n \in \mathbb{N})$$

are the Bernoulli numbers.

Recently, the approximations of the following form:

$$\psi(x) \approx \ln(x+a) - \frac{1}{x}, \quad a \in [0, 1], \quad x \in [2, \infty) \tag{2}$$

were studied by Muqattash and Yahdi [6]. They computed the error

$$\left|\psi(x) - \left(\ln(x+a) - \frac{1}{x}\right)\right| \le \ln\left(1 + \frac{1}{x}\right) \le \ln\left(\frac{3}{2}\right) = 0.4054651081\ldots,$$

and then the approximation (2) was compared with the approximation obtained by considering the first two terms of the series (1), that is

$$\left|\psi(x) - \left(\ln(x+a) - \frac{1}{x}\right)\right| \le \left|\psi(x) - \left(\ln(x) - \frac{1}{2x}\right)\right|.$$

Very recently, the family (2) was also discussed by Qi and Guo [7]. One of their main results is the following inequality on $x \in (0, \infty)$:

$$\ln\left(x + \frac{1}{2}\right) - \frac{1}{x} < \psi(x) < \ln(x + e^{-\gamma}) - \frac{1}{x}, \tag{3}$$

where $\gamma = 0.577215\ldots$ is the Euler–Mascheroni constant.

In the final part of the paper [6], the authors wonder whether there are profitable constants $a \in [0, 1]$ and $b \in [1, 2]$ for which better approximations of the form

$$\psi(x) \approx \ln(x+a) - \frac{1}{bx} \tag{4}$$

can be obtained. Mortici [3] solved this open problem and proved that the best approximations (4) appear for

$$a = \frac{1}{\sqrt{6}}, \quad b = 6 - 2\sqrt{6}$$

and

$$a = -\frac{1}{\sqrt{6}}, \quad b = 6 + 2\sqrt{6}.$$

Moreover, the author derived from [3, Theorem 2.1] the following symmetric double inequality: For $x > \frac{1}{\sqrt{6}} = 0.40824829\ldots$,

$$\ln\left(x - \frac{1}{\sqrt{6}}\right) - \frac{1}{(6+2\sqrt{6})x} \le \psi(x) \le \ln\left(x + \frac{1}{\sqrt{6}}\right) - \frac{1}{(6-2\sqrt{6})x}. \tag{5}$$

This double inequality is more accurate than the estimations (3) of Qi and Guo.

We define the sequence $(v_n)_{n\in\mathbb{N}}$ by

$$v_n = \psi(n) - \left(\ln(n+p) - \frac{q}{n+r}\right). \tag{6}$$

We are interested in finding the values of the parameters p, q and r such that $(v_n)_{n\in\mathbb{N}}$ is the *fastest* sequence which would converge to zero. This provides the best approximations of the form:

$$\psi(n) \approx \ln(n+p) - \frac{q}{n+r}. \tag{7}$$

Our study is based on the following Lemma 1, which provides a method for measuring the speed of convergence.

Lemma 1 (see [4] and [5]). *If the sequence $(\lambda_n)_{n\in\mathbb{N}}$ converges to zero and if there exists the following limit:*

$$\lim_{n\to\infty} n^k(\lambda_n - \lambda_{n+1}) = l \in \mathbb{R} \qquad (k > 1),$$

then

$$\lim_{n\to\infty} n^{k-1}\lambda_n = \frac{l}{k-1} \qquad (k>1)\,.$$

Theorem 1. *Let the sequence* $(v_n)_{n\in\mathbb{N}}$ *be defined by* (6). *Then for*

$$\begin{cases} p = -\dfrac{1}{2} - \dfrac{1}{6}\sqrt{9+6\sqrt{3}} \\ q = -\dfrac{1}{6}\sqrt{9+6\sqrt{3}} \\ r = -\dfrac{1}{2} - \dfrac{1}{18}\sqrt{3}\sqrt{9+6\sqrt{3}}\,, \end{cases} \tag{8}$$

or

$$\begin{cases} p = -\dfrac{1}{2} + \dfrac{1}{6}\sqrt{9+6\sqrt{3}} \\ q = \dfrac{1}{6}\sqrt{9+6\sqrt{3}} \\ r = -\dfrac{1}{2} + \dfrac{1}{18}\sqrt{3}\sqrt{9+6\sqrt{3}}\,, \end{cases} \tag{9}$$

we have

$$\lim_{n\to\infty} n^5(v_n - v_{n+1}) = \frac{1}{180} + \frac{\sqrt{3}}{54} \quad \textit{and} \quad \lim_{n\to\infty} n^4 v_n = \frac{1}{720} + \frac{\sqrt{3}}{216}\,.$$

The speed of convergence of the sequence $(v_n)_{n\in\mathbb{N}}$ *is given by the order estimate* $O\left(n^{-4}\right)$ *as* $n \to \infty$.

Proof. First of all, we write the difference $v_n - v_{n+1}$ as the following power series in n^{-1}:

$$\begin{aligned} & v_n - v_{n+1} \\ = {} & \frac{2q-2p-1}{2n^2} + \frac{-3q-6qr+3p+3p^2+1}{3n^3} \\ & + \frac{4q+12qr+12qr^2-1-4p-6p^2-4p^3}{4n^4} \\ & + \frac{-5q-20qr-30qr^2-20qr^3+1+5p+10p^2+10p^3+5p^4}{5n^5} \\ & + O\left(\frac{1}{n^6}\right) \qquad (n\to\infty)\,. \end{aligned} \tag{10}$$

According to Lemma 1, the three parameters p, q and r, which produce the fastest convergence of the sequence $(v_n)_{n\in\mathbb{N}}$ are given by (10)

$$\begin{cases} 2q-2p-1=0 \\ -3q-6qr+3p+3p^2+1=0 \\ 4q+12qr+12qr^2-1-4p-6p^2-4p^3=0\,, \end{cases}$$

that is, by (8) and (9). We thus find that

$$v_n - v_{n+1} = \left(\frac{1}{180}+\frac{\sqrt{3}}{54}\right)\frac{1}{n^5}+O\left(\frac{1}{n^6}\right) \qquad (n\to\infty).$$

Finally, by using Lemma 1, we obtain the assertion (1) of Theorem 1. □

Solutions (8) and (9) provide the best approximations of type (7):

$$\psi(n) \approx \ln\left(n-\frac{1}{2}-\frac{1}{6}\sqrt{9+6\sqrt{3}}\right)+\frac{3\sqrt{9+6\sqrt{3}}}{18n-9-\sqrt{3}\sqrt{9+6\sqrt{3}}} \tag{11}$$

and

$$\psi(n) \approx \ln\left(n-\frac{1}{2}+\frac{1}{6}\sqrt{9+6\sqrt{3}}\right)-\frac{3\sqrt{9+6\sqrt{3}}}{18n-9+\sqrt{3}\sqrt{9+6\sqrt{3}}}\,. \tag{12}$$

Theorem 2 below presents closer bounds for psi function.

Theorem 2. *For* $x > \frac{1}{2}+\frac{1}{6}\sqrt{9+6\sqrt{3}} = 1.23394491\ldots$, *then*

$$\begin{aligned} &\ln\left(x-\frac{1}{2}-\frac{1}{6}\sqrt{9+6\sqrt{3}}\right)+\frac{3\sqrt{9+6\sqrt{3}}}{18x-9-\sqrt{3}\sqrt{9+6\sqrt{3}}} \\ &\quad+\left(\frac{1}{720}+\frac{\sqrt{3}}{216}\right)\frac{1}{x^4} < \psi(x) < \ln\left(x-\frac{1}{2}+\frac{1}{6}\sqrt{9+6\sqrt{3}}\right) \\ &\quad-\frac{3\sqrt{9+6\sqrt{3}}}{18x-9+\sqrt{3}\sqrt{9+6\sqrt{3}}}+\left(\frac{1}{720}+\frac{\sqrt{3}}{216}\right)\frac{1}{x^4}\,. \end{aligned} \tag{13}$$

Proof. The lower bound of (13) is obtained by considering the function F defined by

$$\begin{aligned} F(x) &= \psi(x) - \ln\left(x - \frac{1}{2} - \frac{1}{6}\sqrt{9+6\sqrt{3}}\right) - \frac{3\sqrt{9+6\sqrt{3}}}{18x - 9 - \sqrt{3}\sqrt{9+6\sqrt{3}}} \\ &\quad - \left(\frac{1}{720} + \frac{\sqrt{3}}{216}\right)\frac{1}{x^4}, \quad x > \frac{1}{2} + \frac{1}{6}\sqrt{9+6\sqrt{3}}\,. \end{aligned}$$

We conclude from the asymptotic formula (1) that

$$\lim_{x\to\infty} F(x) = 0\,.$$

It follows form [2, Theorem 9] that

$$\begin{aligned} \frac{1}{x} + \frac{1}{2x^2} + \frac{1}{6x^3} - \frac{1}{30x^5} < \psi'(x) < \frac{1}{x} + \frac{1}{2x^2} + \frac{1}{6x^3} \\ -\frac{1}{30x^5} + \frac{1}{42x^7}, \quad x > 0\,. \end{aligned} \tag{14}$$

Differentiating $F(x)$ with respect to x and applying the second inequality in (14) yields, for $x > \frac{1}{2} + \frac{1}{6}\sqrt{9+6\sqrt{3}}$,

$$\begin{aligned} F'(x) &= \psi'(x) - \frac{6}{6x - 3 - \sqrt{9+6\sqrt{3}}} + \frac{54\sqrt{9+6\sqrt{3}}}{\left(18x - 9 - \sqrt{3}\sqrt{9+6\sqrt{3}}\right)^2} \\ &\quad + \frac{10\sqrt{3}+3}{540x^5} < \frac{1}{x} + \frac{1}{2x^2} + \frac{1}{6x^3} - \frac{1}{30x^5} + \frac{1}{42x^7} \\ &\quad - \frac{6}{6x - 3 - \sqrt{9+6\sqrt{3}}} + \frac{54\sqrt{9+6\sqrt{3}}}{\left(18x - 9 - \sqrt{3}\sqrt{9+6\sqrt{3}}\right)^2} + \frac{10\sqrt{3}+3}{540x^5} \\ &= -\frac{P(x)}{7x^7\left(6x - 3 - \sqrt{9+6\sqrt{3}}\right)\left(18x - 9 - \sqrt{3}\sqrt{9+6\sqrt{3}}\right)^2}\,, \end{aligned}$$

where

$$\begin{aligned} P(x) &= 312 + 96\sqrt{9+6\sqrt{3}}\sqrt{3} + 426\sqrt{3} + 151\sqrt{9+6\sqrt{3}} \\ &\quad + \left(593\sqrt{9+6\sqrt{3}} + 1902\sqrt{3} + 1113 + 372\sqrt{9+6\sqrt{3}}\sqrt{3}\right)(x-1) \end{aligned}$$

$$
\begin{aligned}
&+\Big(540\sqrt{9+6\sqrt{3}}\sqrt{3}+880\sqrt{9+6\sqrt{3}}+1425+3192\sqrt{3}\Big)(x-1)^2\\
&+\Big(567\sqrt{9+6\sqrt{3}}+336\sqrt{9+6\sqrt{3}}\sqrt{3}+705+2310\sqrt{3}\Big)(x-1)^3\\
&+\Big(630\sqrt{3}+84\sqrt{9+6\sqrt{3}}\sqrt{3}+189+147\sqrt{9+6\sqrt{3}}\Big)(x-1)^4\\
&> 0 \quad \text{for} \quad x > \frac{1}{2}+\frac{1}{6}\sqrt{9+6\sqrt{3}}\,.
\end{aligned}
$$

Therefore, $F'(x) < 0$ for $x > \frac{1}{2}+\frac{1}{6}\sqrt{9+6\sqrt{3}}$. This leads to

$$
F(x) > \lim_{x\to\infty} F(x) = 0\,.
$$

This means that the first inequality in (13) holds for $x > \frac{1}{2}+\frac{1}{6}\sqrt{9+6\sqrt{3}}$.

The upper bound of (13) is obtained by considering the function G defined by

$$
\begin{aligned}
G(x) = \psi(x) - \ln\left(x-\frac{1}{2}+\frac{1}{6}\sqrt{9+6\sqrt{3}}\right) + \frac{3\sqrt{9+6\sqrt{3}}}{18x-9+\sqrt{3}\sqrt{9+6\sqrt{3}}}\\
-\left(\frac{1}{720}+\frac{\sqrt{3}}{216}\right)\frac{1}{x^4}\,, \qquad x > 0.
\end{aligned}
$$

We conclude from the asymptotic formula (1) that

$$
\lim_{x\to\infty} G(x) = 0\,.
$$

Differentiating $G(x)$ with respect to x and applying the first inequality in (14) yields, for $x > 0$,

$$
\begin{aligned}
G'(x) &= \psi'(x) - \frac{6}{6x-3+\sqrt{9+6\sqrt{3}}} - \frac{54\sqrt{9+6\sqrt{3}}}{\big(18x-9+\sqrt{3}\sqrt{9+6\sqrt{3}}\big)^2}\\
&\quad + \frac{10\sqrt{3}+3}{540x^5} > \frac{1}{x}+\frac{1}{2x^2}+\frac{1}{6x^3}-\frac{1}{30x^5}-\frac{6}{6x-3+\sqrt{9+6\sqrt{3}}}\\
&\quad - \frac{54\sqrt{9+6\sqrt{3}}}{\big(18x-9+\sqrt{3}\sqrt{9+6\sqrt{3}}\big)^2} + \frac{10\sqrt{3}+3}{540x^5}\\
&= \frac{Q(x)}{x^5\big(6x-3+\sqrt{9+6\sqrt{3}}\big)\big(18x-9+\sqrt{3}\sqrt{9+6\sqrt{3}}\big)^2}\,,
\end{aligned}
$$

where

$$
\begin{aligned}
Q(x) &= \Big(-90\sqrt{3}+21\sqrt{9+6\sqrt{3}}-27+12\sqrt{3}\sqrt{9+6\sqrt{3}}\Big)x^2 \\
&\quad +\Big(30\sqrt{3}-39-3\sqrt{9+6\sqrt{3}}\Big)x+\sqrt{9+6\sqrt{3}}-6\sqrt{3}+6 \\
&= \Big(-90\sqrt{3}+21\sqrt{9+6\sqrt{3}}-27+12\sqrt{3}\sqrt{9+6\sqrt{3}}\Big) \\
&\quad \times (x-x_1)(x-x_2)
\end{aligned}
$$

with

$$
\begin{aligned}
x_1 &= \frac{\begin{matrix}13-10\sqrt{3}+\sqrt{9+6\sqrt{3}} \\ -\sqrt{-350-190\sqrt{3}+78\sqrt{9+6\sqrt{3}}+44\sqrt{3}\sqrt{9+6\sqrt{3}}}\end{matrix}}{2\Big(-30\sqrt{3}+7\sqrt{9+6\sqrt{3}}-9+4\sqrt{3}\sqrt{9+6\sqrt{3}}\Big)} \\
&= 0.0638967475\ldots,
\end{aligned}
$$

$$
\begin{aligned}
x_2 &= \frac{\begin{matrix}13-10\sqrt{3}+\sqrt{9+6\sqrt{3}} \\ +\sqrt{-350-190\sqrt{3}+78\sqrt{9+6\sqrt{3}}+44\sqrt{3}\sqrt{9+6\sqrt{3}}}\end{matrix}}{2\Big(-30\sqrt{3}+7\sqrt{9+6\sqrt{3}}-9+4\sqrt{3}\sqrt{9+6\sqrt{3}}\Big)} \\
&= 0.158650823\ldots.
\end{aligned}
$$

Therefore, $Q(x) > 0$ and $G'(x) > 0$ for $x > x_2$. This leads to

$$G(x) < \lim_{x\to\infty} G(x) = 0 \quad x > x_2\,.$$

This means that the second inequality in (13) holds for $x > 0.158650823\ldots.$ $\square$

Some computer experiments indicate that for $x > 2.30488055$, the lower bound in (13) is sharper than one in (5). For $x > 0.5690291018$, the upper bound in (13) is sharper than one in (5).

The inequality (13) provides the best approximations:

$$\psi(x) \approx \ln\left(x-\frac{1}{2}-\frac{1}{6}\sqrt{9+6\sqrt{3}}\right)+\frac{3\sqrt{9+6\sqrt{3}}}{18x-9-\sqrt{3}\sqrt{9+6\sqrt{3}}} \tag{15}$$

and

$$\psi(x) \approx \ln\left(x - \frac{1}{2} + \frac{1}{6}\sqrt{9+6\sqrt{3}}\right) - \frac{3\sqrt{9+6\sqrt{3}}}{18x - 9 + \sqrt{3}\sqrt{9+6\sqrt{3}}}. \tag{16}$$

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