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Fundamental solution in the theory of micropolar thermoelastic diffusion with voids

RAJNEESH KUMAR and TARUN KANSAL*
Department of Mathematics, Kurukshetra University, Kurukshetra-136 119, India
E-mails: rajneesh_kuk@rediffmail.com / tarun1_kansal@yahoo.co.in

Abstract. In the present article, we construct the fundamental solution of system of differential equations in the theory of micropolar thermoelastic diffusion with voids in case of steady oscillations in terms of elementary functions. Some basic properties of the fundamental solution are also established. Some special cases are also discussed.

Mathematical subject classification: 74Bxx, 74Fxx, 74Hxx.

Key words: fundamental solution, micropolar thermoelastic diffusion with voids, steady oscillations.

1 Introduction

The linear theory of elasticity is of paramount importance in the stress analysis of steel, which is the most common engineering structural material. To a lesser extent linear elasticity describes the mechanical behavior of other common solid materials, e.g., concrete, wood and coal. However, this theory does not apply to the behavior of many new synthetic materials of the elastomer and polymer type, e.g., polymethyl-methacrylate, polythylene, polyvinyl chloride.

Modern engineering structures are often made up of materials possessing an internal structure. Polycrystalline materials, materials with fibrous or coarse grain structure come in this category. Classical theory of elasticity is inadequate to represent the behavior of such materials. The micropolar elasticity theory

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*Corresponding author.

takes into consideration the granular character of the medium, and is intended to be applied to materials for which the ordinary classical theory of elasticity fails owing to the microstructure of the material. Within such a theory, solids can undergo macro-deformations and micro-rotations. The motion in this kind of solids is completely characterized by the displacement vector and the microrotation vector, whereas in case of classical elasticity, the motion is characterized by the displacement vector only. The micropolar theory have been extended to include thermal effects by Eringen (1970, 1999) and Nowacki (1966a,b,c). Boschi and Iesan (1973) extended a generalized theory of micropolar thermoelasticity.

Iesan (1986) established a linear theory of thermoelastic materials with voids. He presented the basic field equations and discussed the conditions of propagation of acceleration waves in a homogeneous isotropic thermoelastic material with voids. He showed that transverse wave propagates without effecting the temperature and the porosity of the material. Iesan (1987) extended the thermoelastic theory of elastic material with voids to include initial stress and the initial heat-flux effects. Dhaliwal and Wang (1995) also formulated a thermoelasticity theory for elastic material with voids to include heat flux among the consecutive variables and assumed an evolution equation for the heat-flux. Chirita and Scalia (2001) and Pompei and Scalia (2002) studied the spatial and temporial behavior of the transient solutions for the initial-boundary value problems associated with the linear theory of the thermoelastic materials with voids by using the time-weighted surface power function method. Scalia, Pompei and Chirita (2004) considered the steady time harmonic oscillations within the context of linear thermoelasticity for materials with voids and derived the spatial decay results for the amplitude of harmonic variations in a cylinder.

Scalia (1992) considered a grade consistent micropolar theory of thermoelasticity for materials with voids. Passarella (1996) introduced a theory of micropolar thermoelasticity for materials with voids based on the Lebon (1982) law for heat conduction.

Diffusion is defined as the spontaneous movement of the particles from a high concentration region to the low concentration region and it occurs in response to a concentration gradient expressed as the change in the concentration due to change in position. Thermal diffusion utilizes the transfer of heat across a thin liquid or gas to accomplish isotope separation. Today, thermal diffusion remains a practical process to separate isotopes of noble gases (e.g. xexon) and other light isotopes (e.g. carbon) for research purposes. In most of the applications, the concentration is calculated using what is known as Fick's law. This is a simple law which does not take into consideration the mutual interaction between the introduced substance and the medium into which it is introduced or the effect of temperature on this interaction. However, there is a certain degree of coupling with temperature and temperature gradients as temperature speeds up the diffusion process. The thermodiffusion in elastic solids is due to coupling of fields of temperature, mass diffusion and that of strain in addition to heat and mass exchange with the environment.

Nowacki (1974a,b,c, 1976) developed the theory of thermoelastic diffusion by using coupled thermoelastic model. Uniqueness and reciprocity theorems for the equations of generalized thermoelastic diffusion problem, in isotropic media, was proved by Sherief et al. (2004) on the basis of the variational principle equations, under restrictive assumptions on the elastic coefficients. Due to the inherit complexity of the derivation of the variational principle equations, Aouadi (2007) proved this theorem in the Laplace transform domain, under the assumption that the functions of the problem are continuous and the inverse Laplace transform of each is also unique. Aouadi (2008) derived the uniqueness and reciprocity theorems for the generalized problem in anisotropic media, under the restriction that the elastic, thermal conductivity and diffusion tensors are positive definite. Recently, Aouadi (2009) derived the uniqueness and reciprocity theorems for the generalized micropolar thermoelastic diffusion problem in anisotropic media. Also, Aouadi (2010) derived the uniqueness, reciprocity and existence theorems for the thermoelastic diffusion problem with voids in anisotropic media.

To investigate the boundary value problems of the theory of elasticity and thermoelasticity by potential method, it is necessary to construct a fundamental solution of systems of partial differential equations and to establish their basic properties respectively. Hetnarski (1964a,b) was the first to study the fundamental solutions in the classical theory of coupled thermoelasticity. The fundamental solutions in the theory of micropolar elasticity and thermoelasticity for materials with voids are presented by Scarpetta (1990) and Svanadze et al. (2007) respectively. The fundamental solutions in the microcontinuum fields theories have been constructed by Svanadze (1988, 1996, 2004) and Svanadze et al. (2006). The information related to fundamental solutions of differential equations is contained in the books of Hörmander (1963, 1983).

In this article, the fundamental solution of system of equations in the case of steady oscillations is considered in terms of elementary functions and basic properties of the fundamental solution are established. Some special cases of interest are also discussed.

2 Basic equations

Let $\mathbf{x} = (x_1, x_2, x_3)$ be the point of the Euclidean three-dimensional space E^3,

$$|\mathbf{x}| = \left(x_1^2 + x_2^2 + x_3^2\right)^{\frac{1}{2}}, \ \mathbf{D}_x = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right)$$

and let t denote the time variable.

Following Aouadi (2009, 2010), the basic equations for homogeneous isotropic generalized micropolar thermoelastic diffusion with voids in the absence of body forces, body couples, heat and mass diffusive sources are:

$$\begin{gathered}(\mu + K^*)\Delta\bar{\mathbf{u}} + (\lambda + \mu) \text{ grad div } \bar{\mathbf{u}} + K^* \text{ curl } \bar{\boldsymbol{\varphi}} + \gamma^* \text{ grad } \bar{\phi}^* \\ - \beta_1 \text{ grad } \bar{T} - \beta_2 \text{ grad } \bar{C} = \rho\ddot{\bar{\mathbf{u}}},\end{gathered} \tag{1}$$

$$(f^*\Delta - 2K^*)\bar{\boldsymbol{\varphi}} + (\alpha^* + \beta^*) \text{ grad div } \bar{\boldsymbol{\varphi}} + K^* \text{ curl } \bar{\mathbf{u}} = \rho j \ddot{\bar{\boldsymbol{\varphi}}}, \tag{2}$$

$$(a^*\Delta - d^*)\bar{\phi}^* - \gamma^* \text{div } \bar{\mathbf{u}} + \xi^* \bar{T} + \zeta^* \bar{C} = \rho\chi\ddot{\bar{\phi}}^*, \tag{3}$$

$$(1 + \tau_0 \frac{\partial}{\partial t})(\beta_1 T_0 \text{ div } \dot{\bar{\mathbf{u}}} + \xi^* T_0 \dot{\bar{\phi}}^* + \rho C_E \dot{\bar{T}} + a T_0 \dot{\bar{C}}) = K\Delta\bar{T}, \tag{4}$$

$$D\beta_2\Delta \text{ div } \bar{\mathbf{u}} + D\zeta^*\Delta\bar{\phi}^* + Da\Delta\bar{T} - Db\Delta\bar{C} + \dot{\bar{C}} + \tau^0\ddot{\bar{C}} = 0, \tag{5}$$

where

$$\beta_1 = (3\lambda + 2\mu + K^*)\alpha_t, \ \beta_2 = (3\lambda + 2\mu + K^*)\alpha_c .$$

Here α_t, α_c are the coefficients of linear thermal expansion and diffusion expansion respectively; $\bar{\mathbf{u}} = (\bar{u_1}, \bar{u_2}, \bar{u_3})$ is the displacement vector; $\bar{\boldsymbol{\varphi}} = (\bar{\varphi_1}, \bar{\varphi_2}, \bar{\varphi_3})$ is the microrotation vector; ϕ^* is the volume fraction function; ρ, C_E are, respectively, the density and specific heat at constant strain; $\lambda, \mu, K, D, a, b, a^*, d^*$,

$f^*, \xi^*, \zeta^*, \alpha^*, \beta^*, K^*, \gamma^*$ are constitutive coefficients; j is microintertia density; χ is equilibrated inertia; $\bar{T} = \Theta - T_0$ is small temperature increment; Θ is the absolute temperature of the medium; T_0 is the reference temperature of the body chosen such that $|\frac{\bar{T}}{T_0}| \ll 1$; $\bar{C}$ is the concentration of the diffusive material in the elastic body; τ^0 is diffusion relaxation time and τ_0 is thermal relaxation time; Δ is the Laplacian operator. If $\tau_0 = \tau^0 = 0$, then from (1)-(5), we obtain the basic equations for micropolar thermoelastic diffusion with voids based upon the Fourier classical law of heat conduction.

We define the dimensionless quantities:

$$
\begin{gathered}
\mathbf{x}' = \frac{w_1^* \mathbf{x}}{c_1}, \quad \bar{\mathbf{u}}' = \frac{\rho w_1^* c_1 \bar{\mathbf{u}}}{\beta_1 T_0}, \quad \bar{\boldsymbol{\varphi}}' = \frac{\rho c_1^2 \bar{\boldsymbol{\varphi}}}{\beta_1 T_0}, \quad \bar{\phi^*}' = \frac{\rho \chi w_1^{*2} \bar{\phi^*}}{\beta_1 T_0}, \\
\bar{T}' = \frac{\bar{T}}{T_0}, \quad \bar{C}' = \frac{\beta_2 \bar{C}}{\beta_1 \bar{T}_0}, \quad t' = w_1^* t, \quad \tau_0' = w_1^* \tau_0, \quad \tau^{0'} = w_1^* \tau^0, \\
\delta_1 = \frac{\mu + K^*}{\lambda + 2\mu + K^*}, \quad \delta_2 = \frac{\lambda + \mu}{\lambda + 2\mu + K^*}, \quad \delta_3 = \frac{K^*}{\lambda + 2\mu + K^*}, \\
\delta_4 = \frac{\gamma^*}{\rho \chi w_1^{*2}}, \quad \delta_5 = \frac{f^* w_1^{*2}}{\rho c_1^4}, \quad \delta_6 = \frac{(\alpha^* + \beta^*) w_1^{*2}}{\rho c_1^4}, \\
\delta_7 = \frac{j w_1^{*2}}{c_1^2}, \quad \delta_8 = \frac{a^*}{\chi(\lambda + 2\mu + K^*)}, \quad \delta_9 = \frac{d^*}{\rho \chi w_1^{*2}}, \\
\delta_{10} = \frac{\gamma^*}{\lambda + 2\mu + K^*}, \quad \delta_{11} = \frac{\xi^*}{\beta_1}, \quad \delta_{12} = \frac{\zeta^*}{\beta_2}, \\
\zeta_1 = \frac{a T_0 c_1^2 \beta_1}{w_1^* K \beta_2}, \quad \zeta_2 = \frac{\beta_1^2 T_0}{\rho K w_1^*}, \quad \zeta_3 = \frac{\xi^* \beta_1 T_0 c_1^2}{\rho \chi K w_1^{*3}}, \\
q_1^* = \frac{D w_1^* \beta_2^2}{\rho c_1^4}, \quad q_2^* = \frac{D w_1^* \beta_2 a}{\beta_1 c_1^2}, \quad q_3^* = \frac{D w_1^* b}{c_1^2}, \quad q_4^* = \frac{D \zeta^* \beta_2}{\rho \chi w_1^* c_1^2},
\end{gathered}
\tag{6}
$$

where

$$
w_1^* = \frac{\rho C_E c_1^2}{K}, \quad c_1 = \sqrt{\frac{\lambda + 2\mu + K^*}{\rho}}.
$$

Upon introducing the quantities (6) in the basic equations (1)-(5), after suppressing the primes, we obtain

$$\delta_1 \Delta \bar{\mathbf{u}} + \delta_2 \text{ grad div } \bar{\mathbf{u}} + \delta_3 \text{ curl } \bar{\boldsymbol{\varphi}} + \delta_4 \text{ grad } \bar{\phi}^* - \text{grad } \bar{T} - \text{grad } \bar{C} = \ddot{\bar{\mathbf{u}}}, \quad (7)$$

$$(\delta_5 \Delta - 2\delta_3)\bar{\boldsymbol{\varphi}} + \delta_6 \text{ grad div } \bar{\boldsymbol{\varphi}} + \delta_3 \text{ curl } \bar{\mathbf{u}} = \delta_7 \ddot{\bar{\boldsymbol{\varphi}}}, \quad (8)$$

$$(\delta_8 \Delta - \delta_9)\bar{\phi}^* - \delta_{10} \text{ div } \bar{\mathbf{u}} + \delta_{11} \bar{T} + \delta_{12} \bar{C} = \ddot{\bar{\phi}}^*, \quad (9)$$

$$\tau_t^0(\zeta_2 \text{ div } \dot{\bar{\mathbf{u}}} + \zeta_3 \dot{\bar{\phi}}^* + \dot{\bar{T}} + \zeta_1 \dot{\bar{C}}) = \Delta \bar{T}, \quad (10)$$

$$q_1^* \Delta \text{ div } \bar{\mathbf{u}} + q_4^* \Delta \bar{\phi}^* + q_2^* \Delta \bar{T} - q_3^* \Delta \bar{C} + \tau_c^0 \dot{\bar{C}} = 0, \quad (11)$$

where

$$\tau_t^0 = 1 + \tau_0 \frac{\partial}{\partial t}, \quad \tau_c^0 = 1 + \tau^0 \frac{\partial}{\partial t}.$$

We assume the displacement vector, microrotation, volume fraction, temperature change and concentration functions as

$$\big(\bar{\mathbf{u}}(\mathbf{x}, t), \bar{\boldsymbol{\varphi}}(\mathbf{x}, t), \bar{\phi}^*(\mathbf{x}, t), \bar{T}(\mathbf{x}, t), \bar{C}(\mathbf{x}, t)\big) = \text{Re}\big[(\mathbf{u}, \boldsymbol{\varphi}, \phi^*, T, C)e^{-\iota\omega t}\big] \quad (12)$$

Using equation (12) in the equations (7)-(11), we obtain the system of equations of steady oscillations as

$$\begin{gathered}(\delta_1 \Delta + \omega^2)\mathbf{u} + \delta_2 \text{ grad div } \mathbf{u} + \delta_3 \text{ curl } \boldsymbol{\varphi} \\ + \delta_4 \text{ grad } \phi^* - \text{grad } T - \text{grad } C = \mathbf{0},\end{gathered} \quad (13)$$

$$(\delta_5 \Delta + \mu^*)\boldsymbol{\varphi} + \delta_6 \text{ grad div } \boldsymbol{\varphi} + \delta_3 \text{ curl } \mathbf{u} = \mathbf{0}, \quad (14)$$

$$-\delta_{10} \text{ div } \mathbf{u} + (\delta_8 \Delta + \chi^*)\phi^* + \delta_{11} T + \delta_{12} C = 0, \quad (15)$$

$$-\tau_t^{10}[\zeta_2 \text{ div } \mathbf{u} + \zeta_3 \phi^* + \zeta_1 C] + (\Delta - \tau_t^{10})T = 0, \quad (16)$$

$$q_1^* \Delta \text{ div } \mathbf{u} + q_4^* \Delta \phi^* + q_2^* \Delta T - q_3^* \Delta C + \tau_c^{10} C = 0, \quad (17)$$

where

$$\tau_t^{10} = -\iota\omega(1 - \iota\omega\tau_0), \; \tau_c^{10} = -\iota\omega(1 - \iota\omega\tau^0), \; \mu^* = \delta_7\omega^2 - 2\delta_3, \; \chi^* = \omega^2 - \delta_9.$$

We introduce the matrix differential operator

$$\mathbf{F}(\mathbf{D}_x) = \|F_{gh}(\mathbf{D}_x)\|_{9\times 9}$$

where

$$F_{mn}(\mathbf{D}_x) = [\delta_1 \Delta + \omega^2]\delta_{mn} + \delta_2 \frac{\partial^2}{\partial x_m \partial x_n},$$

$$F_{m,n+3}(\mathbf{D}_x) = F_{m+3,n}(\mathbf{D}_x) = \delta_3 \sum_{r=1}^{3} \varepsilon_{mrn} \frac{\partial}{\partial x_r},$$

$$F_{m7}(\mathbf{D}_x) = \delta_4 \frac{\partial}{\partial x_m}, \; F_{m8}(\mathbf{D}_x) = F_{m9}(\mathbf{D}_x) = -\frac{\partial}{\partial x_m},$$

$$F_{m+3,n+3}(\mathbf{D}_x) = (\delta_5 \Delta + \mu^*)\delta_{mn} + \delta_6 \frac{\partial^2}{\partial x_m \partial x_n},$$

$$F_{m+3,7}(\mathbf{D}_x) = F_{7,n+3}(\mathbf{D}_x) = F_{m+3,8}(\mathbf{D}_x) = F_{8,n+3}(\mathbf{D}_x)$$

$$= F_{m+3,9}(\mathbf{D}_x) = F_{9,n+3}(\mathbf{D}_x) = 0,$$

$$F_{7n}(\mathbf{D}_x) = -\delta_{10} \frac{\partial}{\partial x_n}, \; F_{77}(\mathbf{D}_x) = \delta_8 \Delta + \chi^*, \; F_{78}(\mathbf{D}_x) = \delta_{11},$$

$$F_{79}(\mathbf{D}_x) = \delta_{12}, \; F_{8n}(\mathbf{D}_x) = -\zeta_2 \tau_t^{10} \frac{\partial}{\partial x_n},$$

$$F_{87}(\mathbf{D}_x) = -\zeta_3 \tau_t^{10}, \; F_{88}(\mathbf{D}_x) = \triangle - \tau_t^{10}, \; F_{89} = -\zeta_1 \tau_t^{10},$$

$$F_{9n}(\mathbf{D}_x) = q_1^* \Delta \frac{\partial}{\partial x_n}, \; F_{97}(\mathbf{D}_x) = q_4^* \Delta, \; F_{98}(\mathbf{D}_x) = q_2^* \Delta,$$

$$F_{99}(\mathbf{D}_x) = -q_3^* \Delta + \tau_c^{10}, \quad m, n = 1, 2, 3.$$

Here ε_{mrn} is alternating tensor and δ_{mn} is the Kronecker delta.

The system of equations (13)-(17) can be written as

$$\mathbf{F}(\mathbf{D}_x)\mathbf{U}(\mathbf{x}) = \mathbf{0},$$

where $\mathbf{U} = (\mathbf{u}, \boldsymbol{\varphi}, \phi^*, T, C)$ is a nine-component vector function on E^3.

We assume that

$$-\delta_1 q_3^* \delta_5(\delta_5 + \delta_6)\delta_8 \neq 0 \tag{18}$$

If the condition (18) is satisfied, then **F** is an elliptic differential operator (Hörmander, 1963).

Definition. The fundamental solution of the system of equations (13)-(17) (the fundamental matrix of operator $\mathbf{F}$) is the matrix $\mathbf{G}(\mathbf{x}) = \|G_{gh}(\mathbf{x})\|_{9\times 9}$ satisfying condition (Hörmander, 1963)

$$\mathbf{F}(\mathbf{D}_x)\mathbf{G}(\mathbf{x}) = \delta(\mathbf{x})\mathbf{I}(\mathbf{x}) \tag{19}$$

where δ is the Dirac delta, $\mathbf{I} = \|\delta_{gh}\|_{9\times 9}$ is the unit matrix and $\mathbf{x} \in \mathrm{E}^3$.

Now we construct $\mathbf{G}(\mathbf{x})$ in terms of elementary functions.

3 Fundamental solution of system of equations of steady oscillations

We consider the system of equations

$$\begin{gathered}\delta_1\Delta\mathbf{u} + \delta_2 \text{ grad div } \mathbf{u} + \delta_3 \text{ curl } \boldsymbol{\varphi} - \delta_{10} \text{ grad } \phi^* \\ -\zeta_2\tau_t^{10} \text{ grad } T + q_1^*\Delta \text{ grad } C + \omega^2\mathbf{u} = \mathbf{H}',\end{gathered} \tag{20}$$

$$(\delta_5\Delta + \mu^*)\boldsymbol{\varphi} + \delta_6 \text{ grad div } \boldsymbol{\varphi} + \delta_3 \text{ curl } \mathbf{u} = \mathbf{H}'', \tag{21}$$

$$\delta_4 \text{ div } \mathbf{u} + (\delta_8\Delta + \chi^*)\phi^* - \zeta_3\tau_t^{10}T + q_4^*\Delta C = Z, \tag{22}$$

$$-\text{ div } \mathbf{u} + \delta_{11}\phi^* + (\Delta - \tau_t^{10})T + q_2^*\Delta C = L, \tag{23}$$

$$-\text{div } \mathbf{u} + \delta_{12}\phi^* - \zeta_1\tau_t^{10}T - q_3^*\Delta C + \tau_c^{10}C = M, \tag{24}$$

where $\mathbf{H}'$ and $\mathbf{H}''$ are three-component vector functions on E^3; Z, L and M are scalar functions on E^3.

The system of equations (20)-(24) may be written in the form

$$\mathbf{F}^{tr}(\mathbf{D}_x)\mathbf{U}(\mathbf{x}) = \mathbf{Q}(\mathbf{x}), \tag{25}$$

where $\mathbf{F}^{tr}$ is the transpose of matrix $\mathbf{F}$, $\mathbf{Q} = (\mathbf{H}', \mathbf{H}'', Z, L, M)$ and $\mathbf{x} \in \mathrm{E}^3$.

Applying the operator div to the equations (20) and (21), we obtain

$$\begin{gathered}(\Delta + \omega^2) \text{ div } \mathbf{u} - \delta_{10}\Delta\phi^* - \zeta_2\tau_t^{10}\Delta T + q_1^*\Delta^2 C = \text{ div } \mathbf{H}', \\ (\nu^*\Delta + \mu^*) \text{ div } \boldsymbol{\varphi} = \text{ div } \mathbf{H}'', \\ \delta_4 \text{ div } \mathbf{u} + (\delta_8\Delta + \chi^*)\phi^* - \zeta_3\tau_t^{10}T + q_4^*\Delta C = Z, \\ -\text{ div } \mathbf{u} + \delta_{11}\phi^* + (\Delta - \tau_t^{10})T + q_2^*\Delta C = L, \\ -\text{div } \mathbf{u} + \delta_{12}\phi^* - \zeta_1\tau_t^{10}T - q_3^*\Delta C + \tau_c^{10}C = M,\end{gathered} \tag{26}$$

where $\nu^* = \delta_5 + \delta_6$.

The equations $(26)_1$, $(26)_3$, $(26)_4$ and $(26)_5$ may be expressed in the following form

$$\mathbf{N}(\Delta)\mathbf{S} = \bar{\mathbf{Q}}, \tag{27}$$

where $\mathbf{S} = (\text{div}\,\mathbf{u}, \phi^*, T, C)$, $\bar{\mathbf{Q}} = (d_1, d_2, d_3, d_4) = (\text{div}\mathbf{H}', Z, L, M)$ and

$$\begin{aligned}\mathbf{N}(\Delta) &= \left\| N_{mn}(\Delta) \right\|_{4\times4} \\ &= \left\| \begin{array}{cccc} \Delta + \omega^2 & -\delta_{10}\Delta & -\zeta_2\tau_t^{10}\Delta & q_1^*\Delta^2 \\ \delta_4 & \delta_8\Delta + \chi^* & -\zeta_3\tau_t^{10} & q_4^*\Delta \\ -1 & \delta_{11} & \Delta - \tau_t^{10} & q_2^*\Delta \\ -1 & \delta_{12} & -\zeta_1\tau_t^{10} & -q_3^*\Delta + \tau_c^{10} \end{array} \right\|_{4\times4}\end{aligned} \tag{28}$$

The equations $(26)_1$, $(26)_3$, $(26)_4$ and $(26)_5$ can be also written as

$$\Gamma_1(\Delta)\mathbf{S} = \mathbf{\Psi}, \tag{29}$$

where

$$\mathbf{\Psi} = (\Psi_1, \Psi_2, \Psi_3, \Psi_4),\ \Psi_n = e^* \sum_{m=1}^{4} N_{mn}^* d_m,$$

$$\Gamma_1(\Delta) = e^* \det \mathbf{N}(\Delta),\ \ e^* = -\frac{1}{q_3^*\delta_8}\ \ n = 1, 2, 3, 4 \tag{30}$$

and N_{mn}^* is the cofactor of the elements N_{mn} of the matrix $\mathbf{N}$.

From equations (28) and (30), we see that

$$\Gamma_1(\Delta) = \prod_{m=1}^{4} (\Delta + \lambda_m^2)$$

where λ_m^2, $m = 1, 2, 3, 4$ are the roots of the equation $\Gamma_1(-\kappa) = 0$ (with respect to κ).

From equation $(26)_2$, it follows that

$$(\Delta + \lambda_7^2)\ \text{div}\,\boldsymbol{\varphi} = \frac{1}{\nu^*}\ \text{div}\,\mathbf{H}'', \tag{31}$$

where $\lambda_7^2 = \dfrac{\mu^*}{\nu^*}$.

Applying the operators $\delta_5\Delta + \mu^*$ and δ_3 curl to the equations (20) and (21), respectively, we obtain

$$\begin{aligned}&(\delta_5\Delta + \mu^*)\big[\delta_1\Delta\mathbf{u} + \delta_2 \text{ grad div } \mathbf{u} + \omega^2\mathbf{u}\big] + \delta_3(\delta_5\Delta + \mu^*) \text{ curl } \boldsymbol{\varphi}\\ &= (\delta_5\Delta + \mu^*)\big[\mathbf{H}' + \delta_{10} \text{ grad } \phi^* + \zeta_2\tau_t^{10} \text{ grad } T - q_1^*\Delta \text{ grad } C\big]\end{aligned} \tag{32}$$

and

$$\delta_3(\delta_5\Delta + \mu^*) \text{ curl } \boldsymbol{\varphi} = -\delta_3^2 \text{ curl curl } \mathbf{u} + \delta_3 \text{ curl } \mathbf{H}'' \tag{33}$$

Now

$$\text{curl curl } \mathbf{u} = \text{ grad div } \mathbf{u} - \Delta\mathbf{u} \tag{34}$$

Using equations (33) and (34) in equation (32), we obtain

$$\begin{aligned}&(\delta_5\Delta + \mu^*)\big[\delta_1\Delta\mathbf{u} + \delta_2 \text{ grad div } \mathbf{u} + \omega^2\mathbf{u}\big]\\ &+ \delta_3^2\Delta\mathbf{u} - \delta_3^2 \text{ grad div } \mathbf{u} = (\delta_5\Delta + \mu^*)\\ &\times\big[\mathbf{H}' + \delta_{10} \text{ grad } \phi^* + \zeta_2\tau_t^{10} \text{ grad } T - q_1^*\Delta \text{ grad } C\big] - \delta_3 \text{ curl } \mathbf{H}''\end{aligned} \tag{35}$$

The above equation can also be written as

$$\begin{aligned}&\big\{\big[(\delta_5\Delta + \mu^*)\delta_1 + \delta_3^2\big]\Delta + (\delta_5\Delta + \mu^*)\omega^2\big\}\mathbf{u}\\ &= -\big[\delta_2(\delta_5\Delta + \mu^*) - \delta_3^2\big] \text{ grad div } \mathbf{u} + (\delta_5\Delta + \mu^*)\\ &\times\big[\mathbf{H}' + \delta_{10} \text{ grad } \phi^* + \zeta_2\tau_t^{10} \text{ grad } T - q_1^*\Delta \text{ grad } C\big] - \delta_3 \text{ curl } \mathbf{H}''\end{aligned} \tag{36}$$

Applying the operator $\Gamma_1(\Delta)$ to the equation (36) and using equation (29), we get

$$\begin{aligned}&\Gamma_1(\Delta)\big[\delta_5\delta_1\Delta^2 + (\mu^*\delta_1 + \delta_5\omega^2 + \delta_3^2)\Delta + \mu^*\omega^2\big]\mathbf{u}\\ &= -\big[\delta_2(\delta_5\Delta + \mu^*) - \delta_3^2\big] \text{ grad } \Psi_1 + (\delta_5\Delta + \mu^*)\\ &\times\big[\Gamma_1(\Delta)\mathbf{H}' + \delta_{10} \text{ grad } \Psi_2 + \zeta_2\tau_t^{10} \text{ grad } \Psi_3 - q_1^*\Delta \text{ grad } \Psi_4\big]\\ &- \delta_3\Gamma_1(\Delta) \text{ curl } \mathbf{H}''\end{aligned} \tag{37}$$

The above equation may also be written in the following form

$$\Gamma_1(\Delta)\Gamma_2(\Delta)\mathbf{u} = \boldsymbol{\Psi}', \tag{38}$$

where

$$\Gamma_2(\Delta) = f^* \det \begin{Vmatrix} \delta_1\Delta + \omega^2 & \delta_3\Delta \\ -\delta_3 & \delta_5\Delta + \mu^* \end{Vmatrix}_{2\times 2}, \; f^* = \frac{1}{\delta_1\delta_5}$$

and

$$\begin{aligned} \mathbf{\Psi}' = f^*\Big\{ &-\big[\delta_2(\delta_5\Delta + \mu^*) - \delta_3^2\big] \text{ grad } \Psi_1 + (\delta_5\Delta + \mu^*) \\ &\times \big[\Gamma_1(\Delta)\mathbf{H}' + \delta_{10} \text{ grad } \Psi_2 + \zeta_2\tau_t^{10} \text{ grad } \Psi_3 - q_1^*\Delta \text{ grad } \Psi_4\big] \\ &- \delta_3\Gamma_1(\Delta) \text{ curl } \mathbf{H}''\Big\} \end{aligned} \tag{39}$$

It can be seen that

$$\Gamma_2(\Delta) = \big(\Delta + \lambda_5^2\big)\big(\Delta + \lambda_6^2\big),$$

where λ_5^2, λ_6^2 are the roots of the equation $\Gamma_2(-\kappa) = 0$ (with respect to κ).

Applying the operators δ_3 curl and $\delta_1\Delta + \omega^2$ to the equations (20) and (21), respectively, we obtain

$$\delta_3(\delta_1\Delta + \omega^2) \text{ curl } \mathbf{u} = \delta_3 \text{ curl } \mathbf{H}' - \delta_3^2 \text{ curl curl } \boldsymbol{\varphi} \tag{40}$$

and

$$\begin{aligned} (\delta_1\Delta + \omega^2)(\delta_5\Delta + \mu^*)\boldsymbol{\varphi} + \delta_6(\delta_1\Delta + \omega^2) \text{ grad div } \boldsymbol{\varphi} \\ + \delta_3(\delta_1\Delta + \omega^2) \text{ curl } \mathbf{u} = (\delta_1\Delta + \omega^2)\mathbf{H}'' \end{aligned} \tag{41}$$

Now

$$\text{curl curl } \boldsymbol{\varphi} = \text{ grad div } \boldsymbol{\varphi} - \Delta\boldsymbol{\varphi} \tag{42}$$

Using equations (40) and (42) in equation (41), we obtain

$$\begin{aligned} (\delta_1\Delta + \omega^2)(\delta_5\Delta + \mu^*)\boldsymbol{\varphi} + \delta_6(\delta_1\Delta + \omega^2) \text{ grad div } \boldsymbol{\varphi} + \delta_3^2\Delta\boldsymbol{\varphi} \\ - \delta_3^2 \text{ grad div } \boldsymbol{\varphi} = (\delta_1\Delta + \omega^2)\mathbf{H}'' - \delta_3 \text{ curl } \mathbf{H}' \end{aligned} \tag{43}$$

The above equation may also be rewritten as

$$\begin{aligned} \{[(\delta_5\Delta + \mu^*)\delta_1 + \delta_3^2]\Delta + (\delta_5\Delta + \mu^*)\omega^2\}\boldsymbol{\varphi} \\ = -[\delta_6(\delta_1\Delta + \omega^2) - \delta_3^2] \text{ grad div } \boldsymbol{\varphi} + (\delta_1\Delta + \omega^2)\mathbf{H}'' - \delta_3 \text{ curl } \mathbf{H}' \end{aligned} \tag{44}$$

Applying the operator $\Delta + \lambda_7^2$ to the equation (44) and using equation (31), we get

$$\begin{aligned} (\Delta + \lambda_7^2)[\delta_5\delta_1\Delta^2 + (\mu^*\delta_1 + \delta_5\omega^2 + \delta_3^2)\Delta + \mu^*\omega^2]\boldsymbol{\varphi} = -\delta_3(\Delta + \lambda_7^2) \text{ curl } \mathbf{H}' \\ + (\delta_1\Delta + \omega^2)(\Delta + \lambda_7^2)\mathbf{H}'' - [\delta_6(\delta_1\Delta + \omega^2) - \delta_3^2] \text{ grad } \Psi_5 \end{aligned}$$

The above equation may also be rewritten in the form

$$\Gamma_2(\Delta)(\Delta + \lambda_7^2)\boldsymbol{\varphi} = \boldsymbol{\Psi}'', \tag{45}$$

where

$$\begin{aligned}\boldsymbol{\Psi}'' = f^*\big\{ -\delta_3(\Delta + \lambda_7^2)\ \mathrm{curl}\ \mathbf{H}' + (\delta_1\Delta + \omega^2)(\Delta + \lambda_7^2)\mathbf{H}'' \\ -\big[\delta_6(\delta_1\Delta + \omega^2) - \delta_3^2\big]\ \mathrm{grad}\ \Psi_5\big\}\end{aligned} \tag{46}$$

From equations (29), (38) and (45), we obtain

$$\boldsymbol{\Theta}(\Delta)\mathbf{U}(\mathbf{x}) = \hat{\boldsymbol{\Psi}}(\mathbf{x}) \tag{47}$$

where $\hat{\boldsymbol{\Psi}} = (\boldsymbol{\Psi}', \boldsymbol{\Psi}'', \Psi_3, \Psi_4)$ and

$$\boldsymbol{\Theta}(\Delta) = \|\Theta_{gh}(\Delta)\|_{9\times 9}$$

$$\Theta_{mm}(\Delta) = \Gamma_1(\Delta)\Gamma_2(\Delta) = \prod_{q=1}^{6}(\Delta + \lambda_q^2)$$

$$\Theta_{m+3,m+3}(\Delta) = \Gamma_2(\Delta)(\Delta + \lambda_7^2) = \prod_{q=5}^{7}(\Delta + \lambda_q^2)$$

$$\Theta_{gh}(\Delta) = 0,\ \Theta_{77}(\Delta) = \Theta_{88}(\Delta) = \Theta_{99}(\Delta) = \Gamma_1(\Delta),$$

$$m = 1, 2, 3\ \ g, h = 1, \ldots.., 9\ \ g \neq h$$

The equations (30), (39) and (46) can be rewritten in the form

$$\begin{aligned}\boldsymbol{\Psi}' = \big[f^*(\delta_5\Delta + \mu^*)\Gamma_1(\Delta)\mathbf{J} + q_{11}(\Delta)\ \mathrm{grad\ div}\big]\mathbf{H}' + q_{21}(\Delta)\ \mathrm{curl}\ \mathbf{H}'' \\ + q_{31}(\Delta)\ \mathrm{grad}\ Z + q_{41}(\Delta)\ \mathrm{grad}\ L + q_{51}(\Delta)\ \mathrm{grad}\ M,\end{aligned} \tag{48}$$

$$\boldsymbol{\Psi}'' = q_{12}(\Delta)\ \mathrm{curl}\ \mathbf{H}' + \big\{f^*(\Delta + \lambda_7^2)(\delta_1\Delta + \omega^2)\mathbf{J} + q_{22}(\Delta)\ \mathrm{grad\ div}\big\}\mathbf{H}'', \tag{49}$$

$$\Psi_2 = q_{13}(\Delta)\ \mathrm{div}\ \mathbf{H}' + q_{33}(\Delta)Z + q_{43}(\Delta)L + q_{53}(\Delta)M, \tag{50}$$

$$\Psi_3 = q_{14}(\Delta)\ \mathrm{div}\ \mathbf{H}' + q_{34}(\Delta)Z + q_{44}(\Delta)L + q_{54}(\Delta)M, \tag{51}$$

$$\Psi_4 = q_{15}(\Delta)\ \mathrm{div}\ \mathbf{H}' + q_{35}(\Delta)Z + q_{45}(\Delta)L + q_{55}(\Delta)M, \tag{52}$$

where $\mathbf{J} = \|\delta_{gh}\|_{3\times 3}$ is the unit matrix.

In the equations (48)-(52), we have used the following notations:

$$q_{11}(\Delta) = f^* e^* \{(\delta_5 \Delta + \mu^*)[\delta_{10} N^*_{12} + \zeta_2 \tau_t^{10} N^*_{13} - q^*_1 \Delta N^*_{14}] - (\delta_2(\delta_5 \Delta + \mu^*) - \delta_3^2) N^*_{11}\},$$

$$q_{21}(\Delta) = -f^* \delta_3 \Gamma_1(\Delta),\ q_{12}(\Delta) = -f^* \delta_3(\Delta + \lambda_7^2),\ q_{22}(\Delta) = -\frac{f^*}{\nu^*}[\delta_6(\delta_1 \Delta + \omega^2) - \delta_3^2],$$

$$q_{31}(\Delta) = f^* e^* \{(\delta_5 \Delta + \mu^*)[\delta_{10} N^*_{22} + \zeta_2 \tau_t^{10} N^*_{23} - q^*_1 \Delta N^*_{24}] - (\delta_2(\delta_5 \Delta + \mu^*) - \delta_3^2) N^*_{21}\},$$

$$q_{41}(\Delta) = f^* e^* \{(\delta_5 \Delta + \mu^*)[\delta_{10} N^*_{32} + \zeta_2 \tau_t^{10} N^*_{33} - q^*_1 \Delta N^*_{34}] - (\delta_2(\delta_5 \Delta + \mu^*) - \delta_3^2) N^*_{31}\},$$

$$q_{51}(\Delta) = f^* e^* \{(\delta_5 \Delta + \mu^*)[\delta_{10} N^*_{42} + \zeta_2 \tau_t^{10} N^*_{43} - q^*_1 \Delta N^*_{44}] - (\delta_2(\delta_5 \Delta + \mu^*) - \delta_3^2) N^*_{41}\},$$

$$q_{13}(\Delta) = e^* N^*_{12},\ q_{14}(\Delta) = e^* N^*_{13},\ q_{15}(\Delta) = e^* N^*_{14},\ q_{33}(\Delta) = e^* N^*_{22},$$

$$q_{34}(\Delta) = e^* N^*_{23},\ q_{35}(\Delta) = e^* N^*_{24},\ q_{43}(\Delta) = e^* N^*_{32},\ q_{44}(\Delta) = e^* N^*_{33},$$

$$q_{45}(\Delta) = e^* N^*_{34},\ q_{53}(\Delta) = e^* N^*_{42},\ q_{54}(\Delta) = e^* N^*_{43},\ q_{55}(\Delta) = e^* N^*_{44},$$

Now from equations (48)-(52), we have

$$\hat{\mathbf{\Psi}}(\mathbf{x}) = \mathbf{R}^{tr}(\mathbf{D}_x)\mathbf{Q}(\mathbf{x}) \tag{53}$$

where

$$\mathbf{R} = \|R_{gh}\|_{9\times 9}$$

$$R_{mn}(\mathbf{D}_x) = f^*(\delta_5 \Delta + \mu^*)\Gamma_1(\Delta)\delta_{mn} + q_{11}(\Delta)\frac{\partial^2}{\partial x_m \partial x_n},$$

$$R_{m,n+3}(\mathbf{D}_x) = q_{12}(\Delta)\sum_{r=1}^{3}\varepsilon_{mrn}\frac{\partial}{\partial x_r},\quad R_{mp}(\mathbf{D}_x) = q_{1,p-4}(\Delta)\frac{\partial}{\partial x_m},$$

$$R_{m+3,n}(\mathbf{D}_x) = q_{21}(\Delta)\sum_{r=1}^{3}\varepsilon_{mrn}\frac{\partial}{\partial x_r}, \tag{54}$$

$$R_{m+3,n+3}(\mathbf{D}_x) = f^*(\Delta + \lambda_7^2)(\delta_1 \Delta + \omega^2)\delta_{mn} + q_{22}(\Delta)\frac{\partial^2}{\partial x_m \partial x_n},$$

$$R_{m+3,p}(\mathbf{D}_x) = R_{p,m+3}(\mathbf{D}_x) = 0,\ R_{pm}(\mathbf{D}_x) = q_{p-4,1}(\Delta)\frac{\partial}{\partial x_m},$$

$$R_{ps}(\mathbf{D}_x) = q_{p-4,s-4}(\Delta)\ m = 1, 2, 3\ p, s = 7, 8, 9.$$

From equations (25), (47) and (53), we obtain

$$\mathbf{\Theta U} = \mathbf{R}^{tr}\mathbf{F}^{tr}\mathbf{U}$$

The above relation implies

$$\mathbf{R}^{tr}\mathbf{F}^{tr} = \mathbf{\Theta}$$

Therefore, we obtain

$$\mathbf{F}(\mathbf{D}_x)\mathbf{R}(\mathbf{D}_x) = \mathbf{\Theta}(\Delta) \tag{55}$$

We assume that

$$\lambda_m^2 \neq \lambda_n^2 \neq 0,\ \ m, n = 1, 2, 3, 4, 5, 6, 7\ \ m \neq n$$

Let

$$\mathbf{Y}(\mathbf{x}) = \|Y_{rs}(\mathbf{x})\|_{9\times 9},\ \ Y_{mm}(\mathbf{x}) = \sum_{n=1}^{6} r_{1n}\varsigma_n(\mathbf{x}),\ \ Y_{m+3,m+3}(\mathbf{x}) = \sum_{n=5}^{7} r_{2n}\varsigma_n(\mathbf{x}),$$

$$Y_{77}(\mathbf{x}) = Y_{88}(\mathbf{x}) = Y_{99}(\mathbf{x}) = \sum_{n=1}^{4} r_{3n}\varsigma_n(\mathbf{x})\,,$$

$$Y_{vw}(\mathbf{x}) = 0,\ \ m = 1, 2, 3\ \ v, w = 1, 2, \ldots\ldots, 9\ \ v \neq w$$

where

$$\varsigma_n(\mathbf{x}) = -\frac{1}{4\pi|\mathbf{x}|}\exp\left(\iota\lambda_n|\mathbf{x}|\right),\ \ n = 1, 2, \ldots, 7$$

$$r_{1l} = \prod_{m=1, m\neq l}^{6}\left(\lambda_m^2 - \lambda_l^2\right)^{-1},\ \ l = 1, 2, 3, 4, 5, 6$$

$$r_{2v} = \prod_{m=5, m\neq v}^{7}\left(\lambda_m^2 - \lambda_v^2\right)^{-1},\ \ v = 5, 6, 7$$

$$r_{3w} = \prod_{m=1, m\neq w}^{4}\left(\lambda_m^2 - \lambda_w^2\right)^{-1},\ \ w = 1, 2, 3, 4$$

We will prove the following Lemma:

Lemma. The matrix $\mathbf{Y}$ defined above is the fundamental matrix of operator $\mathbf{\Theta}(\Delta)$, that is

$$\mathbf{\Theta}(\Delta)\mathbf{Y}(\mathbf{x}) = \delta(\mathbf{x})\mathbf{I}(\mathbf{x}) \tag{56}$$

Proof. To prove the Lemma, it is sufficient to prove that

$$\begin{aligned}\Gamma_1(\Delta)\Gamma_2(\Delta)Y_{11}(\mathbf{x}) &= \delta(\mathbf{x}),\, \Gamma_2(\Delta)(\Delta+\lambda_7^2)Y_{44}(\mathbf{x})\\ &= \delta(\mathbf{x}),\, \Gamma_1(\Delta)Y_{77}(\mathbf{x}) = \delta(\mathbf{x}).\end{aligned} \tag{57}$$

Consider

$$r_{31}+r_{32}+r_{33}+r_{34} = \frac{-f_1+f_2-f_3+f_4}{f_5},$$

where

$$f_1 = \big(\lambda_2^2-\lambda_3^2\big)\big(\lambda_2^2-\lambda_4^2\big)\big(\lambda_3^2-\lambda_4^2\big),\;\; f_2 = \big(\lambda_1^2-\lambda_3^2\big)\big(\lambda_1^2-\lambda_4^2\big)\big(\lambda_3^2-\lambda_4^2\big),$$

$$f_3 = \big(\lambda_1^2-\lambda_2^2\big)\big(\lambda_1^2-\lambda_4^2\big)\big(\lambda_2^2-\lambda_4^2\big),\;\; f_4 = \big(\lambda_1^2-\lambda_2^2\big)\big(\lambda_1^2-\lambda_3^2\big)\big(\lambda_2^2-\lambda_3^2\big),$$

$$f_5 = \big(\lambda_1^2-\lambda_2^2\big)\big(\lambda_1^2-\lambda_3^2\big)\big(\lambda_1^2-\lambda_4^2\big)\big(\lambda_2^2-\lambda_3^2\big)\big(\lambda_2^2-\lambda_4^2\big)\big(\lambda_3^2-\lambda_4^2\big).$$

On simplifying the right hand side of above relation, we obtain

$$r_{31}+r_{32}+r_{33}+r_{34} = 0, \tag{58}$$

Similarly, we find that

$$r_{32}\big(\lambda_1^2-\lambda_2^2\big)+r_{33}\big(\lambda_1^2-\lambda_3^2\big)+r_{34}\big(\lambda_1^2-\lambda_4^2\big) = 0, \tag{59}$$

$$r_{33}\big(\lambda_1^2-\lambda_3^2\big)\big(\lambda_2^2-\lambda_3^2\big)+r_{34}\big(\lambda_1^2-\lambda_4^2\big)\big(\lambda_2^2-\lambda_4^2\big) = 0, \tag{60}$$

Also,

$$r_{34}\big(\lambda_1^2-\lambda_4^2\big)\big(\lambda_2^2-\lambda_4^2\big)\big(\lambda_3^2-\lambda_4^2\big) = \frac{\big(\lambda_1^2-\lambda_4^2\big)\big(\lambda_2^2-\lambda_4^2\big)\big(\lambda_3^2-\lambda_4^2\big)}{\big(\lambda_1^2-\lambda_4^2\big)\big(\lambda_2^2-\lambda_4^2\big)\big(\lambda_3^2-\lambda_4^2\big)} = 1, \tag{61}$$

$$\big(\Delta+\lambda_m^2\big)\varsigma_n(\mathbf{x}) = \delta(\mathbf{x})+\big(\lambda_m^2-\lambda_n^2\big)\varsigma_n(\mathbf{x}),\;\; m,n = 1,2,3,4. \tag{62}$$

Now consider

$$\begin{aligned}&\Gamma_1(\Delta)Y_{77}(\mathbf{x})\\ &= \big(\Delta+\lambda_1^2\big)\big(\Delta+\lambda_2^2\big)\big(\Delta+\lambda_3^2\big)\big(\Delta+\lambda_4^2\big)\sum_{n=1}^{4} r_{3n}\varsigma_n(\mathbf{x})\\ &= \big(\Delta+\lambda_2^2\big)\big(\Delta+\lambda_3^2\big)\big(\Delta+\lambda_4^2\big)\sum_{n=1}^{4} r_{3n}\big[\delta(\mathbf{x})+\big(\lambda_1^2-\lambda_n^2\big)\varsigma_n(\mathbf{x})\big]\\ &= \big(\Delta+\lambda_2^2\big)\big(\Delta+\lambda_3^2\big)\big(\Delta+\lambda_4^2\big)\left[\delta(\mathbf{x})\sum_{n=1}^{4} r_{3n}+\sum_{n=2}^{4} r_{3n}(\lambda_1^2-\lambda_n^2)\varsigma_n(\mathbf{x})\right]\end{aligned}$$

Using equation (58) in the above relation, we obtain

$$\begin{aligned}
\Gamma_1(\Delta)Y_{77}(\mathbf{x}) &= \big(\Delta+\lambda_2^2\big)\big(\Delta+\lambda_3^2\big)\big(\Delta+\lambda_4^2\big)\sum_{n=2}^{4} r_{3n}\big(\lambda_1^2-\lambda_n^2\big)\varsigma_n(\mathbf{x}) \\
&= \big(\Delta+\lambda_3^2\big)\big(\Delta+\lambda_4^2\big)\sum_{n=2}^{4} r_{3n}\big(\lambda_1^2-\lambda_n^2\big)\big[\delta(\mathbf{x})+\big(\lambda_2^2-\lambda_n^2\big)\varsigma_n(\mathbf{x})\big] \\
&= \big(\Delta+\lambda_3^2\big)\big(\Delta+\lambda_4^2\big)\sum_{n=3}^{4} r_{3n}\big(\lambda_1^2-\lambda_n^2\big)\big(\lambda_2^2-\lambda_n^2\big)\varsigma_n(\mathbf{x}) \\
&= \big(\Delta+\lambda_4^2\big)\sum_{n=3}^{4} r_{3n}\big(\lambda_1^2-\lambda_n^2\big)\big(\lambda_2^2-\lambda_n^2\big)\big[\delta(\mathbf{x})+\big(\lambda_3^2-\lambda_n^2\big)\varsigma_n(\mathbf{x})\big] \\
&= \big(\Delta+\lambda_4^2\big)\sum_{n=4}^{4} r_{3n}\big(\lambda_1^2-\lambda_n^2\big)\big(\lambda_2^2-\lambda_n^2\big)\big(\lambda_3^2-\lambda_n^2\big)\varsigma_n(\mathbf{x}) \\
&= \big(\Delta+\lambda_4^2\big)\varsigma_4(\mathbf{x}) = \delta(\mathbf{x})
\end{aligned}$$

Similarly, the equations $(57)_1$ and $(57)_2$ can be proved.

We introduce the matrix

$$\mathbf{G}(\mathbf{x}) = \mathbf{R}(\mathbf{D}_x)\mathbf{Y}(\mathbf{x}) \tag{63}$$

From equations (55), (56) and (63), we obtain

$$\mathbf{F}(\mathbf{D}_x)\mathbf{G}(\mathbf{x}) = \mathbf{F}(\mathbf{D}_x)\mathbf{R}(\mathbf{D}_x)\mathbf{Y}(\mathbf{x}) = \boldsymbol{\Theta}(\Delta)\mathbf{Y}(\mathbf{x}) = \delta(\mathbf{x})\mathbf{I}(\mathbf{x})$$

Hence, $\mathbf{G}(\mathbf{x})$ is a solution to equation (19).

Therefore we have proved the following Theorem:

Theorem. The matrix $\mathbf{G}(\mathbf{x})$ defined by equation (63) is the fundamental solution of system of equations (13)-(17).

4 Basic properties of the matrix G (x)

Property 1. Each column of the matrix $\mathbf{G}(\mathbf{x})$ is the solution of the system of equations (13)-(17) at every point $\mathbf{x} \in \mathrm{E}^3$ except the origin.

Property 2. The matrix $\mathbf{G}(\mathbf{x})$ can be written in the form

$$\mathbf{G} = \|G_{gh}\|_{9\times 9}$$
$$\mathbf{G}_{mn}(\mathbf{x}) = \mathbf{R}_{mn}(\mathbf{D}_x)Y_{11}(\mathbf{x}),$$
$$\mathbf{G}_{m,n+3}(\mathbf{x}) = \mathbf{R}_{m,n+3}(\mathbf{D}_x)Y_{44}(\mathbf{x}),$$
$$\mathbf{G}_{mp}(\mathbf{x}) = \mathbf{R}_{mp}(\mathbf{D}_x)Y_{77}(\mathbf{x}) \; m = 1, 2....., 9 \; n = 1, 2, 3 \; p = 7, 8, 9.$$

5 Special cases

(i) Neglecting the diffusion effect in the equations (13)-(17), we obtain the system of equations of steady oscillations for homogeneous isotropic generalized micropolar thermoelasticity with voids as:

$$\big(\delta_1\Delta + \omega^2\big)\mathbf{u} + \delta_2 \text{ grad div } \mathbf{u} + \delta_3 \text{ curl } \boldsymbol{\varphi} + \delta_4 \text{ grad } \phi^* - \text{grad } T = \mathbf{0}, \quad (64)$$

$$\big(\delta_5\Delta + \mu^*\big)\boldsymbol{\varphi} + \delta_6 \text{ grad div } \boldsymbol{\varphi} + \delta_3 \text{ curl } \mathbf{u} = \mathbf{0}, \quad (65)$$

$$-\delta_{10} \text{ div } \mathbf{u} + \big(\delta_8\Delta + \chi^*\big)\phi^* + \delta_{11}T = 0, \quad (66)$$

$$-\tau_t^{10}\big[\zeta_2 \text{ div } \mathbf{u} + \zeta_3\phi^*\big] + \big(\Delta - \tau_t^{10}\big)T = 0. \quad (67)$$

The fundamental solution of the system of equations (64)-(67) is similar as obtained by Svanadze et al. (2007) by changing the dimensionless quantities into physical quantities.

(ii) If we neglect the void effect in the equations (13)-(17), we obtain the system of equations of steady oscillations for homogeneous isotropic generalized micropolar thermoelastic diffusion as:

$$\big(\delta_1\Delta + \omega^2\big)\mathbf{u} + \delta_2 \text{ grad div } \mathbf{u} + \delta_3 \text{ curl } \boldsymbol{\varphi} - \text{grad } T - \text{grad } C = \mathbf{0}, \quad (68)$$

$$\big(\delta_5\Delta + \mu^*\big)\boldsymbol{\varphi} + \delta_6 \text{ grad div } \boldsymbol{\varphi} + \delta_3 \text{ curl } \mathbf{u} = \mathbf{0}, \quad (69)$$

$$-\tau_t^{10}\big[\zeta_2 \text{ div } \mathbf{u} + \zeta_1 C\big] + \big(\Delta - \tau_t^{10}\big)T = 0, \quad (70)$$

$$q_1^*\Delta \text{ div } \mathbf{u} + q_2^*\Delta T - q_3^*\Delta C + \tau_c^{10}C = 0. \quad (71)$$

The fundamental solution of the system of equations (68)-(71) is similar as obtained by Kumar and Kansal (2012).

(iii) If we neglect the micropolar effect in the equations (13)-(17), we obtain the system of equations of steady oscillations for homogeneous isotropic generalized thermoelastic diffusion with voids as:

$$\left(\delta_1 \Delta + \omega^2\right)\mathbf{u} + \delta_2 \text{ grad div } \mathbf{u} + \delta_4 \text{ grad } \phi^* - \text{grad } T - \text{grad } C = \mathbf{0}, \tag{72}$$

$$-\delta_{10} \text{ div } \mathbf{u} + \left(\delta_8 \Delta + \chi^*\right)\phi^* + \delta_{11} T + \delta_{12} C = 0, \tag{73}$$

$$-\tau_t^{10}\left[\zeta_2 \text{ div } \mathbf{u} + \zeta_3 \phi^* + \zeta_1 C\right] + \left(\Delta - \tau_t^{10}\right) T = 0, \tag{74}$$

$$q_1^* \Delta \text{ div } \mathbf{u} + q_4^* \Delta \phi^* + q_2^* \Delta T - q_3^* \Delta C + \tau_c^{10} C = 0. \tag{75}$$

The fundamental solution of the system of equations (72)-(75) is similar as obtained by Kumar and Kansal (2012) based upon Lord-Shulman theory of thermoelastic diffusion with voids.

(iv) If we neglect the micropolar and void effects in the equations (13)-(17), we obtain the system of equations of steady oscillations for homogeneous isotropic generalized thermoelastic diffusion as:

$$\left(\delta_1 \Delta + \omega^2\right)\mathbf{u} + \delta_2 \text{ grad div } \mathbf{u} - \text{grad } T - \text{grad } C = \mathbf{0}, \tag{76}$$

$$-\tau_t^{10}\left[\zeta_2 \text{ div } \mathbf{u} + \zeta_1 C\right] + \left(\Delta - \tau_t^{10}\right) T = 0, \tag{77}$$

$$q_1^* \Delta \text{ div } \mathbf{u} + q_2^* \Delta T - q_3^* \Delta C + \tau_c^{10} C = 0. \tag{78}$$

The fundamental solution of the system of equations (74)-(76) is similar as obtained by Kumar and Kansal (2012) based upon Lord-Shulman theory of thermoelastic diffusion.

6 Conclusions

The fundamental solution $\mathbf{G}(\mathbf{x})$ of the system of equations (13)-(17) makes it possible to investigate three-dimensional boundary value problems of generalized theory of micropolar thermoelastic diffusion with voids by potential method (Kupradze et al., 1979).

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REFERENCES

[1] M. Aouadi. *Uniqueness and reciprocity theorems in the theory of generalized thermoelastic diffusion.* Journal of Thermal Stresses, **30** (2007), 665–678.

[2] M. Aouadi. *Generalized theory of thermoelastic diffusion for anisotropic media.* Journal of Thermal Stresses, **31** (2008), 270–285.

[3] M. Aouadi. *Theory of generalized micropolar thermoelastic diffusion under Lord-Shulman model.* Journal of Thermal Stresses, **32** (2009), 923–942.

[4] M. Aouadi. *A theory of thermoelastic diffusion materials with voids.* ZAMP, **61** (2010), 357–379.

[5] E. Boschi and D. Iesan. *A generalized theory of linear micropolar thermoelasticity.* Meccanica, **8** (1973), 154–157.

[6] S. Chirita and A. Scalia. *On the spatial and temporial behavior in linear thermo-elasticity of materials with voids.* Journal of Thermal Stresses, **24** (2001), 433–455.

[7] M. Svanadze, M. Ciarletta and A. Scalia. *Fundamental solution in the theory of micropolar thermoelasticity for materials with voids.* Journal of Thermal Stresses, **30** (2007), 213–229.

[8] R.S. Dhaliwal and J. Wang. *A heat-flux dependent theory of thermoelasticity with voids.* Acta Mechanica, **110**(1-4) (1995), 33–39.

[9] A.C. Eringen. *Foundations of micropolar thermoelasticity.* International Center for Mechanical Science, Courses and Lectures, no. 23, Springer, Berlin (1970).

[10] A.C. Eringen. *Microcontinuum field theory I: Foundations and solids.* Springer-Verlag, Berlin (1999).

[11] R.B. Hetnarski. *The fundamental solution of the coupled thermoelastic problem for small times.* Archwm. Mech. Stosow., **16** (1964a), 23–31.

[12] R.B. Hetnarski. *Solution of the coupled problem of thermoelasticity in form of a series of functions.* Archwm. Mech. Stosow., **16** (1964b), 919–941.

[13] L. Hörmander. *Linear Partial Differential operators.* Springer-Verlag: Berlin (1963).

[14] L. Hörmander. *The analysis of linear partial differential Operators II: Differential operators with constant coefficients.* Springer-Verlag: Berlin (1983).

[15] D. Iesan. *A theory of thermoelastic materials with voids.* Acta Mechanica, **60** (1986), 67–89.

[16] D. Iesan. *A theory of initially stressed thermoelastic material with voids.* An. Stiint. Univ. Ai. I. Cuza Lasi Sect. I a Mat, **33** (1987), 167–184.

[17] R. Kumar and T. Kansal. *Plane waves and fundamental solution in the generalized theories of thermoelastic diffusion.* Int. J. Appl. Math. Mech., **8** (2012), 1–20.

[18] R. Kumar and T. Kansal. *Fundamental solution in the theory of micropolar thermoelastic diffusion.* Int. J. Appl. Math. Mech., **8** (2012), 21–34.

[19] R. Kumar and T. Kansal. *Propagation of plane waves and fundamental solution in thermoelastic diffusive materials with voids.* Int. J. Appl. Math. Mech. (Accepted) (2012).

[20] V.D. Kupradze, T.G. Gegelia, M.O. Basheleishvili and T.V. Burchuladze. *Three dimensional problems of the mathematical theory of elasticity and thermoelasticity.* North-Holland Pub. Company: Amsterdam, New York, Oxford (1979).

[21] G. Lebon. *A Generalized theory of thermoelasticity.* Journal of Technical Physics, **23** (1982), 37–46.

[22] W. Nowacki. *Couple stresses in the theory of thermoelasticity I.* Bulletin of Polish Academy of Sciences Series, Science and Technology, **14** (1966a), 129–138.

[23] W. Nowacki. *Couple stresses in the theory of thermoelasticity II.* Bulletin of Polish Academy of Sciences Series, Science and Technology, **14** (1966b), 263–272.

[24] W. Nowacki. *Couple stresses in the theory of thermoelasticity III.* Bulletin of Polish Academy of Sciences Series, Science and Technology, **14** (1966c), 801–809.

[25] W. Nowacki. *Dynamical problems of thermodiffusion in solids – I.* Bulletin of Polish Academy of Sciences Series, Science and Technology, **22** (1974a), 55–64.

[26] W. Nowacki. *Dynamical problems of thermodiffusion in solids – II.* Bulletin of Polish Academy of Sciences Series, Science and Technology, **22** (1974b), 205–211.

[27] W. Nowacki. *Dynamical problems of thermodiffusion in solids – III.* Bulletin of Polish Academy of Sciences Series, Science and Technology, **22** (1974c), 257–266.

[28] W. Nowacki. *Dynamical problems of diffusion in solids.* Engineering Fracture Mechanics, **8** (1976), 261–266.

[29] F. Passarella. *Some results in micropolar thermoelasticity.* Mechanics Research Communication, **23** (1996), 349–357.

[30] A. Pompei and A. Scalia. *On the asymptotic spatial behavior in the linear thermoelasticity of materials with voids.* Journal of Thermal Stresses, **25** (2002), 183–193.

[31] A. Scalia. *A grade consistent micropolar theory of thermoelastic materials with voids.* ZAMM, **72** (1992), 133–140.

[32] A. Scalia, A. Pompei and S. Chirita. *On the behavior of steady time harmonic oscillations thermoelastic materials with voids.* Journal of Thermal Stresses, **27** (2004), 209–226.

[33] E. Scarpetta. *On the fundamental solution in micropolar elasticity with voids.* Acta Mechanica, **82** (1990), 151–158.

[34] H.H. Sherief, F.A. Hamza and H.A. Saleh. *The theory of generalized thermoelastic diffusion.* International Journal of Engineering Science, **42** (2004), 591–608.

[35] M. Svanadze. *The fundamental matrix of the linearized equations of the theory of elastic mixtures.* Proc. I. Vekua Institute of Applied Mathematics, Tbilisi State University, **23** (1988), 133–148.

[36] M. Svanadze. *The fundamental solution of the oscillation equations of thermoelasticity theory of mixtures of two solids.* Journal of Thermal Stresses, **19** (1996), 633–648.

[37] M. Svanadze. *Fundamental solutions of the equations of the theory of thermoelasticity with microtemperatures.* Journal of Thermal Stresses, **27** (2004), 151–170.

[38] M. Svanadze, V. Tibullo and V. Zampoli. *Fundamental solution in the theory of micropolar thermoelasticity without energy dissipation.* Journal of Thermal Stresses, **29** (2006), 57–66.