

Index, Expressions and Properties of Interval-Valued Intuitionistic Fuzzy Implications*

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ABSTRACT. The interval-valued intuitionistic fuzzy logic, an extension of fuzzy logic, integrates the concepts of interval-valued fuzzy logic and intuitionistic fuzzy logic. The former, reflects the measure of vagueness and uncertainty in the diameter of an interval. The latter considers the hesitation related to the dual construction. This paper considers an expression to interval-valued intuitionistic fuzzy implications, which can be generated by idempotent interval-valued aggregation functions acting on a pair of mutual-dual functions, an interval-valued implication and its corresponding coimplication. Then, we show under which conditions interval-valued intuitionistic fuzzy implications are diagonal preserving operators. We study not only properties of such operators which were extended to intuitionistic fuzzy logic, but also analyse properties truly intuitionistic. The canonical representation in the class of such operators and an interval version of an intuitionistic fuzzy index conclude this study.

Keywords: interval-valued intuitionistic fuzzy connectives, interval-valued intuitionistic fuzzy implication, intuitionistic index.

1 INTRODUCTION

Among equivalent extensions of fuzzy sets (see, e.g., Vague Sets [16] and [6]), this paper considers the interval-valued fuzzy sets (shortly IvFSs), as introduced in [29] and [25], together with intuitionistic fuzzy sets (shortly IFSs), as presented in [2]. Both approaches are founded on the interval-valued fuzzy logic (shortly IvFL) and intuitionistic fuzzy logic (shortly IFL), respectively, which are important extensions of Fuzzy Logic (shortly FL). IFL and IvFL are closely related and both concepts are concerned with imprecision-handling strategies, see [12].

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More recently, other authors have sought to improve upon the deeper understanding of uncertainty and imprecision information [4, 8] and [27]. This paper considers the interval-valued intuitionistic fuzzy logic (shortly IvIFL) with contributions from Atanassov and Gargov [1] in the eighties, integrating the concepts of IvFSs and IFSs. By the former, the measure of vagueness and uncertainty in the diameter of an interval membership degree can be modelled, and by the latter, it can be also considered the hesitation related to the dual construction modelled by an inequality, i.e., the difference between an interval membership degree and its non-membership degree is less or equal to unit interval $U = [0, 1]$. This approach allows to relax the complementary operation stated over fuzzy sets in FL.

One motivation for our study has been the significant performance achieved by the use of interval-valued fuzzy implications (shortly IvFIs) in some applications, such as in the area of image processing, see [9] and [17]. The fuzzy implications are essential to deductive methods in approximate reasoning, interpreting the conditional fuzzy rule in systems based on FL. In both classical and fuzzy approaches, the occurrence of various expressions of implications is evident [20]. However, extensions of fuzzy implications are not the same as in the classical ones, resulting in distinct (S-implications, QL-implications, R-implications) classes of fuzzy implications. Additionally, there are different (explicit, implicit and axiomatic) representations of these connectives. Such diversity encourages the study of IvFIs and their dual constructions, the interval-valued fuzzy coimplications (shortly IvFCs), see [23].

This paper aims to explore interval-valued intuitionistic fuzzy implications (shortly IvIFIs), in the sense of [28], which are generated by interval-valued aggregation functions (shortly IvAs) acting on mutual-dual pairs of IvFIs and IvFCs. Such construction is analogous to [8]. In previous work [28], we studied properties of IvIFI operators extending analogous properties from IFL, which are also generalizations of properties from FL. Many properties truly intuitionistic of IvIFIs (e.g. correlation coefficient, degree of compatibility and entropy of IFSs) have been considered in the literature. Here, the Atanassov's intuitionistic fuzzy sets are studied based on the intuitionistic index of each element obtained by subtracting the sum of membership and non-membership from one. Thus, an interval-valued intuitionistic fuzzy index is introduced as a generalized Atanassov's intuitionistic fuzzy index, called interval degree of indeterminacy in [2].

We also study the action of canonical representation of such fuzzy connectives (aggregations, implications and coimplications) in order to characterize the class of IvIFIs. The authors follow the approach considered in [28], showing such IvIFIs as diagonal preserving operators over which it is possible to preserve main fuzzy implication properties. The results presented here extend the work introduced in [24].

The paper is organized as follows: Section 2 presents the definition and some properties of IvFIs and IvFCs. An expression for interval-valued intuitionistic version of the Atanassov's intuitionistic fuzzy index is introduced in Section 3, in order to study properties of IvIFIs, which are generated by idempotent interval-valued aggregation functions and a pair of mutual-dual interval-valued functions: IvFIs and their corresponding coimplications. In Section 4, main results, the ongoing work and final remarks are pointed out.

2 INTERVAL-VALUED FUZZY (CO)IMPLICATIONS

In IvFL, the membership degrees are subintervals of the unit interval $U = [0, 1]$ approximating the correct (but unknown) membership degree [14]. Thus, an interval-valued fuzzy negation \mathbb{N} is briefly described in this section founding the study of \mathbb{N} -dual (co)implications. And, interval aggregation functions and interval-valued fuzzy (co)implications are also presented. For additional information, see [13].

Let $\mathbb{U} = \{[x_1, x_2] | x_1, x_2 \in U \text{ and } 0 \leq x_1 \leq x_2 \leq 1\}$. The projections $l_{\mathbb{U}}, r_{\mathbb{U}} : \mathbb{U} \rightarrow U$ are defined by $l_{\mathbb{U}}([x_1, x_2]) = x_1$ and $r_{\mathbb{U}}([x_1, x_2]) = x_2$, respectively. For $X \in \mathbb{U}$, $l_{\mathbb{U}}(X)$ and $r_{\mathbb{U}}(X)$ are also denoted by \underline{X} and \overline{X} , respectively. For each $x \in U$, a degenerate interval $[x, x]$ will be denoted by \mathbf{x} and the set of all degenerate intervals will be denoted by \mathbb{D} . And, let $\leq_{\mathbb{U}} \subseteq \mathbb{U}^2$ be the Kulisch-Miranker (or product) order, such that for all $X, Y \in \mathbb{U}$, is given by ¹:

$$X \leq_{\mathbb{U}} Y \Leftrightarrow \underline{X} \leq \underline{Y} \text{ and } \overline{X} \leq \overline{Y}. \tag{2.1}$$

Thus, for all $X \in \mathbb{U}$, $X \geq_{\mathbb{U}} [0, 0] = \mathbf{0}$ and $X \leq [1, 1] = \mathbf{1}$. In addition, we also consider $\preceq_{\mathbb{U}} \subseteq \mathbb{U}^2$ as the binary relation² given by:

$$\forall X, Y \in \mathbb{U}, X \preceq_{\mathbb{U}} Y \Leftrightarrow \overline{X} \leq \underline{Y}. \tag{2.2}$$

Proposition 1. For all $X, Y \in \mathbb{U}$, if $X \preceq_{\mathbb{U}} Y$ then $X \leq_{\mathbb{U}} Y$.

Proof. It follows from Eqs. (2.1) and (2.2). □

Remark 1. Proposition 1 states that the partial order $\leq_{\mathbb{U}}$ extends the binary relation $\preceq_{\mathbb{U}}$.

For each real function $f : U^n \rightarrow U$, for all $\vec{X} = (X_1, X_2, \dots, X_n) \in \mathbb{U}^n$ and $\vec{x} = (x_1, x_2, \dots, x_n) \in U^n$ such that $x_i \in X_i$ for $1 \leq i \leq n$, the canonical representation of f [26] is the interval function $\widehat{f} : \mathbb{U}^n \rightarrow \mathbb{U}$, defined by

$$\widehat{f}(\vec{X}) = \left[\inf\{f(\vec{x}) \mid \vec{x} \in \vec{X}\}, \sup\{f(\vec{x}) \mid \vec{x} \in \vec{X}\} \right]. \tag{2.3}$$

The interval function $\widehat{f} : \mathbb{U}^n \rightarrow \mathbb{U}$ is well defined and called the best interval representation of a real function $f : U^n \rightarrow U$, meaning that for any other interval representation F of f , the inclusion $F(\vec{X}) \subseteq \widehat{f}(\vec{X})$ holds.

2.1 Interval-valued fuzzy negations and interval aggregations

An *interval-valued fuzzy negation* (shortly IvFN) is a function $\mathbb{N} : \mathbb{U} \rightarrow \mathbb{U}$ which verifies, for all $X, Y \in \mathbb{U}$, the following two properties:

¹See in [11], results extending the partial order $\leq_{\mathbb{U}}$ to a total order.

²Relation \preceq is not an order but its reflexive closure, it is a partial order.

N1 $\mathbb{N}(\mathbf{0}) = \mathbf{1}$ and $\mathbb{N}(\mathbf{1}) = \mathbf{0}$;

N2 If $X \leq_{\mathbb{U}} Y$ then $\mathbb{N}(Y) \leq_{\mathbb{U}} \mathbb{N}(X)$.

If \mathbb{N} also satisfies the involutive property ($\mathbb{N}(\mathbb{N}(X)) = X$) then \mathbb{N} is called a strong IvFN. See, e.g., the Zadeh’s IvFN which is given by

$$\mathbb{N}_S(X) = [N_S(\overline{X}), N_S(\underline{X})]. \tag{2.4}$$

Notice that, Eq. (2.4) provides a characterization of \mathbb{N}_S by the canonical representation of N_S , when $N_S(x) = 1 - x$, which means $\mathbb{N}_S = \widehat{N}_S$, by Eq.(2.3). In the following two propositions, some properties related to \mathbb{N}_S are presented, where the sum between $X, Y \in \mathbb{U}$ is defined as $X + Y = [\underline{X} + \underline{Y}, \overline{X} + \overline{Y}] \in \mathbb{U}$.

Proposition 2. For all $X, Y \in \mathbb{U}$, $X + Y \leq_{\mathbb{U}} \mathbf{1}$ if and only if $X \preceq_{\mathbb{U}} \mathbb{N}_S(Y)$.

Proof. (\Rightarrow) For all $X, Y \in \mathbb{U}$, $X + Y \leq_{\mathbb{U}} \mathbf{1}$ implies that $[\underline{X} + \underline{Y}, \overline{X} + \overline{Y}] \leq_{\mathbb{U}} \mathbf{1}$. It also means that $\underline{X} + \underline{Y} \leq 1$ and $\overline{X} + \overline{Y} \leq 1$. So it is equivalent to $\underline{X} \leq N_S(\underline{Y}) = \overline{\mathbb{N}_S(Y)}$ and $\overline{X} \leq N_S(\overline{Y}) = \underline{\mathbb{N}_S(Y)}$. Therefore we can conclude that $X \preceq_{\mathbb{U}} \mathbb{N}(Y)$. (\Leftarrow) Analogously done. \square

Corollary 1. For all $X, Y \in \mathbb{U}$, if $X + Y \leq_{\mathbb{U}} \mathbf{1}$ then $X \leq_{\mathbb{U}} \mathbb{N}_S(Y)$.

Proof. Straightforward from the Propositions 1 and 2. \square

An interval-valued aggregation $\mathbb{M}: \mathbb{U}^2 \rightarrow \mathbb{U}$ verifies the following properties:

A1 $\mathbb{M}(\mathbf{0}, \mathbf{0}) = \mathbf{0}$ and $\mathbb{M}(\mathbf{1}, \mathbf{1}) = \mathbf{1}$ (boundary conditions);

A2 If $X \leq_{\mathbb{U}} Z$ then $\mathbb{M}(X, Y) \leq_{\mathbb{U}} \mathbb{M}(Z, Y), \forall X, Y, Z \in \mathbb{U}$ (monotonicity property);

A3 $\mathbb{M}(X, Y) = \mathbb{M}(Y, X), \forall X, Y \in \mathbb{U}$ (commutativity property).

Additionally, an idempotent IvA function is an IvA also verifying

A4 $\mathbb{M}(X, X) = X, \forall X \in \mathbb{U}$ (idempotency property).

Remark 2. Consider $\vee, \wedge : U^2 \rightarrow U, \vee(x, y) = \max\{x, y\}$ and $\wedge(x, y) = \min\{x, y\}$ and their related interval extensions $\bigwedge, \bigvee : \mathbb{U}^2 \rightarrow \mathbb{U}$, which are idempotent IvAs given by:

$$\bigwedge(X, Y) = [\wedge(\underline{X}, \underline{Y}), \wedge(\overline{X}, \overline{Y})]; \quad \bigvee(X, Y) = [\vee(\underline{X}, \underline{Y}), \vee(\overline{X}, \overline{Y})]. \tag{2.5}$$

By the product order, for each IvA \mathbb{M} , $\bigwedge(X, Y) \leq_{\mathbb{U}} \mathbb{M}(X, Y) \leq_{\mathbb{U}} \bigvee(X, Y), \forall X, Y \in \mathbb{U}$. And, the canonical representation of an aggregation $M : U^2 \rightarrow U$ is characterized as:

$$\widehat{M}(X, Y) = [M(\underline{X}, \underline{Y}), M(\overline{X}, \overline{Y})], \quad \forall X, Y \in \mathbb{U}, \tag{2.6}$$

according to [14, Definition 4.3.].

2.2 Interval-valued fuzzy (co)implications

In this paper, the definition of an IvF(C)I is the following one, which is equivalent to the definition proposed by Kitainik [18] (see also Fodor and Roubens [15]). Such definition extends the one presented in [7, Definition 3], reproducing fuzzy (co)implications when degenerate intervals are considered.

Definition 1. [13, Definition 6] An IvF(C)I operator $(\mathbb{I}, \mathbb{J}) : \mathbb{U}^2 \rightarrow \mathbb{U}$ fulfills, for all $X, Y, Z \in \mathbb{U}$, the boundary condition stated by

$$\mathbb{I}\mathbf{0} : \mathbb{I}(\mathbf{1}, \mathbf{0}) = \mathbf{0}; \quad \mathbb{J}\mathbf{0} : \mathbb{J}(\mathbf{0}, \mathbf{1}) = \mathbf{1};$$

together with the following additional properties:

$$\mathbb{I}\mathbf{1} : X \leq_{\mathbb{U}} Z \Rightarrow \mathbb{I}(X, Y) \geq_{\mathbb{U}} \mathbb{I}(Z, Y); \quad \mathbb{J}\mathbf{1} : X \leq_{\mathbb{U}} Z \Rightarrow \mathbb{J}(X, Y) \leq_{\mathbb{U}} \mathbb{J}(Z, Y);$$

$$\mathbb{I}\mathbf{2} : Y \leq_{\mathbb{U}} Z \Rightarrow \mathbb{I}(X, Y) \leq_{\mathbb{U}} \mathbb{I}(X, Z); \quad \mathbb{J}\mathbf{2} : Y \leq_{\mathbb{U}} Z \Rightarrow \mathbb{J}(X, Y) \geq_{\mathbb{U}} \mathbb{J}(X, Z);$$

$$\mathbb{I}\mathbf{3} : \mathbb{I}(\mathbf{0}, Y) = \mathbf{1}; \quad \mathbb{J}\mathbf{3} : \mathbb{J}(\mathbf{1}, Y) = \mathbf{0};$$

$$\mathbb{I}\mathbf{4} : \mathbb{I}(X, \mathbf{1}) = \mathbf{1}; \quad \mathbb{J}\mathbf{4} : \mathbb{J}(X, \mathbf{0}) = \mathbf{0}.$$

Proposition 3. [23, Proposition 3.4] A function $\mathbb{I}_{\mathbb{N}}(\mathbb{J}_{\mathbb{N}}) : \mathbb{U}^2 \rightarrow \mathbb{U}$ is an IvF(C)I operator if and only if there exists an IvF(I)C operator $\mathbb{I}(\mathbb{J}) : \mathbb{U}^2 \rightarrow \mathbb{U}$ and an IvFN $\mathbb{N} : \mathbb{U} \rightarrow \mathbb{U}$ such that, $\forall X, Y \in \mathbb{U}$, the next equality holds:

$$\mathbb{I}_{\mathbb{N}}(X, Y) = \mathbb{N}(\mathbb{I}(\mathbb{N}(X), \mathbb{N}(Y))), \quad (\mathbb{J}_{\mathbb{N}}(X, Y) = \mathbb{N}(\mathbb{J}(\mathbb{N}(X), \mathbb{N}(Y))).$$

When \mathbb{N} is an involutive interval-valued negation, $(\mathbb{I}, \mathbb{I}_{\mathbb{N}})$ is said to be a pair of mutual \mathbb{N} -dual interval functions. In the following, IvF(C)Is are characterized as the canonical representability [26] of fuzzy (co)implications.

Proposition 4. [21, Proposition 6][5, Proposition 10] A function $\widehat{I}(\widehat{J})$, called canonical representation of $I(J) : U^2 \rightarrow U$, is given as

$$\widehat{I}(X, Y) = [I(\overline{X}, \underline{Y}), I(\underline{X}, \overline{Y})], \quad (\widehat{J}(X, Y) = [J(\overline{X}, \underline{Y}), J(\underline{X}, \overline{Y})]). \quad (2.7)$$

3 INTERVAL-VALUED INTUITIONISTIC FUZZY IMPLICATIONS

As a generalization of an IFS, an interval-valued intuitionistic fuzzy set (shortly IvIFS) is given as

$$\mathbf{A} = \{(x, M_{\mathbf{A}}(x), N_{\mathbf{A}}(x)) \mid x \in X, M_{\mathbf{A}}(x) + N_{\mathbf{A}}(x) \leq_{\mathbb{U}} \mathbf{1}\},$$

where $N_{\mathbf{A}}(x)$ is the interval degree of non-membership of x in set \mathbf{A} , which is less, at most equal to the complement of the related interval degree of membership $M_{\mathbf{A}}(x)$, and therefore does not necessarily equal to one.

The set $\tilde{U} = \{\tilde{X} = (X_1, X_2) \in \mathbb{U}^2 | X_1 + X_2 \leq_{\mathbb{U}} \mathbf{1}\}$ of interval-valued intuitionistic fuzzy membership and non-membership degrees has two projections: $l_{\tilde{U}}, r_{\tilde{U}} : \tilde{U} \rightarrow \mathbb{U}$, which are defined by $l_{\tilde{U}}(\tilde{X}) = l_{\tilde{U}}(X_1, X_2) = X_1$ and $r_{\tilde{U}}(\tilde{X}) = r_{\tilde{U}}(X_1, X_2) = X_2$, respectively. According to [10], we also consider the binary relations in \tilde{U} :

$$\begin{aligned} \mathbb{O}_{\tilde{U}}1 : \tilde{X} \leq_{\tilde{U}} \tilde{Y} &\Leftrightarrow X_1 \leq_{\mathbb{U}} Y_1 \text{ and } X_2 \geq_{\mathbb{U}} Y_2; \text{ or} \\ \mathbb{O}_{\tilde{U}}2 : \tilde{X} \leq_{\tilde{U}} \tilde{Y} &\Leftrightarrow X_1 \leq_{\mathbb{U}} Y_1 \text{ and } X_2 \leq_{\mathbb{U}} Y_2; \end{aligned}$$

also $\tilde{\mathbf{0}} = (\mathbf{0}, \mathbf{1}) \leq_{\tilde{U}} \tilde{X}$ and $\tilde{\mathbf{1}} = (\mathbf{1}, \mathbf{0}) \geq_{\tilde{U}} \tilde{X}$, for all $\tilde{X}, \tilde{Y} \in \tilde{U}$.

Additionally, let $\tilde{D} = \{\tilde{X} = (X_1, X_2) | X_1 + X_2 = \mathbf{1}\} \subseteq \tilde{U}$ be the set of diagonal elements on \tilde{U} defined by pairs of complementary degenerate intervals.

3.1 Interval intuitionistic fuzzy index

Consider χ as an ordinary finite non-empty set. Let M_A and N_A be the interval-valued degrees of membership and non-membership related to an element $x \in \chi$ in an IvIFS A . A function $\Pi_A : \chi \rightarrow \mathbb{U}$, called an interval intuitionistic fuzzy index (IvIFix) of an element x in an IvIFS A , is given as

$$\Pi_A(x) = \mathbb{N}_S(M_A(x) + N_A(x)) \tag{3.1}$$

Thus, it follows that $\Pi_A(x) \leq \mathbb{N}_S(M_A(x)) = \bigvee (\mathbb{N}_S(M_A(x)), N_A(x))$. And, when $\tilde{X} = (M_A(x), N_A(x)) = (X_1, X_2) \in \tilde{U}$, we denote $\Pi_A(x)$ by $\Pi_{\tilde{X}}$. Thus, Eq. (3.1) can be expressed as $\Pi_{\tilde{X}} = \mathbb{N}_S(X_1 + X_2)$, with $\mathbf{0} \leq_{\mathbb{U}} \Pi_{\tilde{X}} \leq_{\mathbb{U}} \mathbf{1}$.

Proposition 5. *The IvIFix of $\tilde{X} = (X_1, X_2) \in \tilde{U}$ can be expressed as:*

$$\Pi_{\tilde{X}} = [\mathbb{N}_S(\overline{X_1 + X_2}), \mathbb{N}_S(\underline{X_1 + X_2})]. \tag{3.2}$$

Proof. When $\tilde{X} = (X_1, X_2)$, by Eqs. (2.4) and (3.1), $\Pi_{\tilde{X}} = \mathbb{N}_S(X_1 + X_2)$. Then $\Pi_{\tilde{X}} = \mathbb{N}_S[\underline{X_1 + X_2}, \overline{X_1 + X_2}] = [\mathbb{N}_S(\overline{X_1 + X_2}), \mathbb{N}_S(\underline{X_1 + X_2})]$ and Eq. (3.2) is verified. \square

Proposition 6. *Consider $\tilde{U} = \{(x_1, x_2) \in \mathbb{U}^2 | x_1 \leq \mathbb{N}_S(x_2)\}$ and $\pi_{\tilde{X}}$ as the intuitionistic fuzzy index of $\tilde{x} = (x_1, x_2) \in \tilde{U}$, given by $\pi_{\tilde{x}} = \mathbb{N}_S(x_1 + x_2)$ (see [7, Section 1.2]). Then, for all $\tilde{X} = (X_1, X_2) \in \tilde{U}$, $\Pi_{\tilde{X}}$ is given as $\Pi_{\tilde{X}} = [\pi_{(\overline{X_1}, \overline{X_2})}, \pi_{(\underline{X_1}, \underline{X_2})}]$.*

Proof. Since $X_1 + X_2 \leq_{\mathbb{U}} \mathbf{1}$, we obtain both inequalities $\underline{X_1} + \underline{X_2} \leq 1$ and $\overline{X_1} + \overline{X_2} \leq 1$. Then we have that $(\overline{X_1}, \overline{X_2}), (\underline{X_1}, \underline{X_2}) \in \tilde{U}$. Moreover, these imply that $\Pi_{(X_1, X_2)} = \mathbb{N}_S(X_1 + X_2) = [\mathbb{N}_S(\overline{X_1 + X_2}), \mathbb{N}_S(\underline{X_1 + X_2})]$. Therefore, the expression $\Pi_{(X_1, X_2)} = [\mathbb{N}_S(\overline{X_1 + X_2}), \mathbb{N}_S(\underline{X_1 + X_2})] = [\pi_{(\overline{X_1}, \overline{X_2})}, \pi_{(\underline{X_1}, \underline{X_2})}]$ is obtained. \square

Definition 2. *An IvIFI $\mathbb{I}_I : \tilde{U}^2 \rightarrow \tilde{U}$ is a function satisfying the boundary condition $\mathbb{I}_I(\tilde{\mathbf{1}}, \tilde{\mathbf{0}}) = \tilde{\mathbf{0}}$ and, for all $\tilde{X}, \tilde{Y}, \tilde{Z} \in \tilde{U}$, the following properties hold:*

$\mathbb{I}_I \mathbf{1} \tilde{X} \leq_{\tilde{\mathbb{U}}} \tilde{Z} \Rightarrow \mathbb{I}_I(\tilde{X}, \tilde{Y}) \geq_{\tilde{\mathbb{U}}} \mathbb{I}_I(\tilde{Z}, \tilde{Y})$ (right antitonicity property);

$\mathbb{I}_I \mathbf{2} \tilde{Y} \leq_{\tilde{\mathbb{U}}} \tilde{Z} \Rightarrow \mathbb{I}_I(\tilde{X}, \tilde{Y}) \leq_{\tilde{\mathbb{U}}} \mathbb{I}_I(\tilde{X}, \tilde{Z})$ (left isotonicity property);

$\mathbb{I}_I \mathbf{3} \mathbb{I}_I(\tilde{\mathbf{0}}, \tilde{Y}) = \tilde{\mathbf{1}}$ (falsity principle);

$\mathbb{I}_I \mathbf{4} \mathbb{I}_I(\tilde{X}, \tilde{\mathbf{1}}) = \tilde{\mathbf{1}}$ (neutrality principle).

Definition 3. An IvIFI $\mathbb{I}_I : \tilde{\mathbb{U}}^2 \rightarrow \tilde{\mathbb{U}}$ is called a **diagonal preserving IvIFI** if, for all $\tilde{X}, \tilde{Y} \in \tilde{\mathbb{D}}$, it verifies the additional property:

$\mathbb{I}_I \mathbf{5} \Pi_{\mathbb{I}_I(\tilde{X}, \tilde{Y})} = \mathbf{0}$;

Lemma 1. $\Pi_{\tilde{X}} = \mathbf{0}$ if and only if $\tilde{X} \in \tilde{\mathbb{D}}$.

Proof. $\Pi_{\tilde{X}} = \mathbf{0} \Leftrightarrow \mathbb{N}_S(X_1 + X_2) = \mathbf{0}$. So, we have $\Pi_{\tilde{X}} = \mathbf{1} - (X_1 + X_2) = \mathbf{0} \Leftrightarrow [1 - X_1 + X_2; 1 - X_1 + X_2] = \mathbf{0}$. In addition, it also means that $\Pi_{\tilde{X}} = \overline{X_1 + X_2} = 1$ and $\underline{X_1 + X_2} = 1 \Leftrightarrow \overline{X_1} = N_S(\overline{X_2})$ and $\underline{X_1} = N_S(\underline{X_2})$. Therefore, $\Pi_{\tilde{X}} = \tilde{X} \in \tilde{\mathbb{D}}$. \square

Proposition 7. An IvIFI operator $\mathbb{I}_I : \tilde{\mathbb{U}}^2 \rightarrow \tilde{\mathbb{U}}$ is a diagonal preserving operator if and only if $\mathbb{I}_I(\tilde{X}, \tilde{Y}) \in \tilde{\mathbb{D}}$ when $\tilde{X}, \tilde{Y} \in \tilde{\mathbb{D}} \subseteq \tilde{\mathbb{U}}$.

Proof. (\Rightarrow) If \mathbb{I}_I is diagonal preserving IvIFI then by Definition 3, $\Pi_{\mathbb{I}_I(\tilde{X}, \tilde{Y})} = \mathbf{0}$ when $\tilde{X}, \tilde{Y} \in \tilde{\mathbb{D}}$. Therefore, by Lemma 1, $\mathbb{I}_I(\tilde{X}, \tilde{Y}) \in \tilde{\mathbb{D}}$. (\Leftarrow) Let $\tilde{X}, \tilde{Y} \in \tilde{\mathbb{D}}$. If $\mathbb{I}_I(\tilde{X}, \tilde{Y}) \in \tilde{\mathbb{D}}$, then there exists $\tilde{Z} = (Z_1, Z_2) \in \tilde{\mathbb{D}}$, which means that $Z_2 = \mathbf{1} - Z_1$, such that $\mathbb{I}_I(\tilde{X}, \tilde{Y}) = \tilde{Z}$. Therefore, $\Pi_{\mathbb{I}_I(\tilde{X}, \tilde{Y})} = \Pi_{\tilde{Z}} = \mathbb{N}_S(Z_1 + Z_2) = \mathbb{N}_S(\mathbf{1}) = \mathbf{0}$. Hence, $\Pi_{\mathbb{I}_I(\tilde{X}, \tilde{Y})} = \mathbf{0}$ and, we conclude, by Lemma 1, \mathbb{I}_I is a diagonal preserving IvIFI. \square

3.2 IvIFIs generated from IvAs, IvFIs and IvFCs operators

Based on [7, Proposition 3], we present an expression for an interval intuitionistic fuzzy implication \mathbb{I}_I generated by a finite set of interval aggregation functions \mathcal{M}_i and a pair of interval mutually dual functions $(\mathbb{I}, \mathbb{I}_N)$. Such pair consists of an IvFI operator \mathbb{I} and its corresponding IvIFC operator \mathbb{I}_N , obtained by a strong IvFN \mathbb{N} .

Definition 4. Let \mathbb{I} be an IvFI, \mathbb{I}_N be its corresponding \mathbb{N}_S -dual construction, and $\mathcal{M} = \{\mathbb{M}_i : \mathbb{U}^2 \rightarrow \mathbb{U} \mid i \in \{1, 2, 3, 4\}\}$ be a family of idempotent IvA verifying the next conditions:

$$\mathbb{M}_1(X, Y) + \mathbb{M}_3(\mathbb{N}_S(X), \mathbb{N}_S(Y)) \geq_{\mathbb{U}} \mathbf{1}; \tag{3.3}$$

$$\mathbb{M}_2(X, Y) + \mathbb{M}_4(\mathbb{N}_S(X), \mathbb{N}_S(Y)) \leq_{\mathbb{U}} \mathbf{1}. \tag{3.4}$$

For all $\tilde{X} = (X_1, X_2), \tilde{Y} = (Y_1, Y_2) \in \tilde{U}, \mathbb{I}_I: \tilde{U}^2 \rightarrow \tilde{U}$ is defined by the projection-functions as:

$$(l_{\tilde{U}} \circ \mathbb{I}_I)(\tilde{X}, \tilde{Y}) = \mathbb{I}(\mathbb{M}_1(X_1, \mathbb{N}_S(X_2)), \mathbb{M}_2(Y_1, \mathbb{N}_S(Y_2))); \tag{3.5}$$

$$(r_{\tilde{U}} \circ \mathbb{I}_I)(\tilde{X}, \tilde{Y}) = \mathbb{I}_N(\mathbb{M}_3(\mathbb{N}_S(X_1), X_2), \mathbb{M}_4(\mathbb{N}_S(Y_1), Y_2)). \tag{3.6}$$

Theorem 1. The binary function $\mathbb{I}_I: \tilde{U}^2 \rightarrow \tilde{U}$ given in Definition 4 is an IvIFI operator.

Proof. It is immediate that \mathbb{I}_I is well defined, satisfying the boundary conditions:

$$\mathbb{I}_I(\tilde{\mathbf{1}}, \tilde{\mathbf{0}}) = (\mathbb{I}(\mathbf{1}, \mathbf{0}), \mathbb{I}_N(\mathbf{0}, \mathbf{1})) = (\mathbf{0}, \mathbf{1}) = \tilde{\mathbf{0}}.$$

For all \tilde{X}, \tilde{Y} and $\tilde{Z} \in \tilde{U}$, we have that

[\mathbb{I}_I 1]

Assuming that $\tilde{X} \leq_{\tilde{U}} \tilde{Z}$, which means $X_1 \leq Z_1$ and $X_2 \geq Z_2$ and equivalently, $N(X_2) \leq N(Z_2)$. Thus, by the isotonic functions \mathbb{M}_1 and \mathbb{M}_3 , both inequations are verified: $\mathbb{M}_1(X_1, \mathbb{N}_S(X_2)) \leq_{\tilde{U}} \mathbb{M}_1(Z_1, \mathbb{N}_S(Z_2))$ and $\mathbb{M}_3(\mathbb{N}_S(X_1), X_2) \geq_{\tilde{U}} \mathbb{M}_3(\mathbb{N}_S(Z_1), Z_2)$. So, for Definition 1, the inequations related to \mathbb{I}_I -projections are given as:

$$\begin{aligned} (l_{\tilde{U}} \circ \mathbb{I}_I)(\tilde{X}, \tilde{Y}) &= \mathbb{I}(\mathbb{M}_1(X_1, \mathbb{N}_S(X_2)), \mathbb{M}_2(Y_1, \mathbb{N}_S(Y_2))) \text{ by Property } \mathbb{I}1 \\ &\geq_{\tilde{U}} \mathbb{I}(\mathbb{M}_1(Z_1, \mathbb{N}_S(Z_2)), \mathbb{M}_2(Y_1, \mathbb{N}_S(Y_2))) = (l_{\tilde{U}} \circ \mathbb{I}_I)(\tilde{Z}, \tilde{Y}); \\ (r_{\tilde{U}} \circ \mathbb{I}_I)(\tilde{X}, \tilde{Y}) &= \mathbb{I}_N(\mathbb{M}_3(\mathbb{N}_S(X_1), X_2), \mathbb{M}_4(\mathbb{N}_S(Y_1), Y_2)) \text{ by Property } \mathbb{J}1 \\ &\leq_{\tilde{U}} \mathbb{I}_N(\mathbb{M}_3(\mathbb{N}_S(Z_1), Z_2), \mathbb{M}_4(\mathbb{N}_S(Y_1), Y_2)) = (r_{\tilde{U}} \circ \mathbb{I}_I)(\tilde{Z}, \tilde{Y}). \end{aligned}$$

Therefore, \mathbb{I}_I verifies the first place antitonicity.

[\mathbb{I}_I 2]

Assuming that $\tilde{Y} \leq_{\tilde{U}} \tilde{Z}$. By the isotonicity of \mathbb{M}_2 and \mathbb{M}_4 , we have the following:

$$\mathbb{M}_2(Y_1, \mathbb{N}_S(Y_2)) \leq_{\tilde{U}} \mathbb{M}_2(Z_1, \mathbb{N}_S(Z_2)) \text{ and } \mathbb{M}_4(\mathbb{N}_S(Y_1), Y_2) \geq_{\tilde{U}} \mathbb{M}_4(\mathbb{N}_S(Z_1), Z_2).$$

For Definition 1, we have that:

$$\begin{aligned} (l_{\tilde{U}} \circ \mathbb{I}_I)(\tilde{X}, \tilde{Y}) &= \mathbb{I}(\mathbb{M}_1(X_1, \mathbb{N}_S(X_2)), \mathbb{M}_2(Y_1, \mathbb{N}_S(Y_2))) \text{ by Property } \mathbb{I}2 \\ &\leq_{\tilde{U}} \mathbb{I}(\mathbb{M}_1(X_1, \mathbb{N}_S(X_2)), \mathbb{M}_2(Z_1, \mathbb{N}_S(Z_2))) = (l_{\tilde{U}} \circ \mathbb{I}_I)(\tilde{X}, \tilde{Z}); \\ (r_{\tilde{U}} \circ \mathbb{I}_I)(\tilde{X}, \tilde{Y}) &= \mathbb{I}_N(\mathbb{M}_3(\mathbb{N}_S(X_1), X_2), \mathbb{M}_4(\mathbb{N}_S(Y_1), Y_2)) \text{ by Property } \mathbb{J}2 \\ &\geq_{\tilde{U}} \mathbb{I}_N(\mathbb{M}_3(\mathbb{N}_S(X_1), X_2), \mathbb{M}_4(\mathbb{N}_S(Z_1), Z_2)) = (r_{\tilde{U}} \circ \mathbb{I}_I)(\tilde{X}, \tilde{Z}). \end{aligned}$$

Therefore, \mathbb{I}_I verifies the second place isotonicity.

[I_I3]

By Definition 1, \mathbb{I} and $\mathbb{I}_{\mathbb{N}}$ fulfills $\mathbb{I}3$ and $\mathbb{J}3$, respectively. Then the projections hold:

$$\begin{aligned} (l_{\mathbb{U}} \circ \mathbb{I}_I)(\tilde{\mathbf{0}}, \tilde{Y}) &= \mathbb{I}(\mathbb{M}_1(\mathbf{0}, \mathbf{0}), \mathbb{M}_2(Y_1, \mathbb{N}_S(Y_2))) = \mathbb{I}(\mathbf{0}, \mathbb{M}_2(Y_1, \mathbb{N}_S(Y_2))) = \mathbf{1}; \\ (r_{\mathbb{U}} \circ \mathbb{I}_I)(\tilde{\mathbf{0}}, \tilde{Y}) &= \mathbb{I}_{\mathbb{N}}(\mathbb{M}_3(\mathbf{1}, \mathbf{1}), \mathbb{M}_4(\mathbb{N}_S(Y_1), Y_2)) = \mathbb{I}_{\mathbb{N}}(\mathbf{1}, \mathbb{M}_4(\mathbb{N}_S(Y_1), Y_2)) = \mathbf{0}. \end{aligned}$$

Therefore $\mathbb{I}_I(\tilde{\mathbf{0}}, Y) = \tilde{\mathbf{1}}$.

[I_I4]

By Definition 1, \mathbb{I} and $\mathbb{I}_{\mathbb{N}}$ fulfills $\mathbb{I}4$ and $\mathbb{J}4$, respectively. Thus, the equations hold:

$$\begin{aligned} (l_{\mathbb{U}} \circ \mathbb{I}_I)(X, \tilde{\mathbf{1}}) &= \mathbb{I}(\mathbb{M}_1(X_1, \mathbb{N}_S(X_2)), \mathbb{M}_2(\mathbf{1}, \mathbf{1})) = \mathbb{I}(\mathbb{M}_1(X_1, \mathbb{N}_S(X_2)), \mathbf{1}) = \mathbf{1} \\ (r_{\mathbb{U}} \circ \mathbb{I}_I)(X, \tilde{\mathbf{1}}) &= \mathbb{I}_{\mathbb{N}}(\mathbb{M}_3(\mathbb{N}_S(X_1), X_2), \mathbb{M}_4(\mathbf{0}, \mathbf{0})) = \mathbb{I}_{\mathbb{N}}(\mathbb{M}_3(\mathbb{N}_S(X_1), X_2), \mathbf{0}) = \mathbf{0}. \end{aligned}$$

Therefore $\mathbb{I}_I(X, \tilde{\mathbf{1}}) = \tilde{\mathbf{1}}$.

[I_I5]

If $X_1 = \mathbb{N}_S(X_2)$ and $Y_1 = \mathbb{N}_S(Y_2)$ then we have the following:

$$\begin{aligned} \Pi_{\mathbb{I}_I}(\tilde{X}, \tilde{Y}) &= \mathbb{N}_S(\mathbb{I}(\mathbb{M}_1(X_1, \mathbb{N}_S(X_2)), \mathbb{M}_2(Y_1, \mathbb{N}_S(Y_2))) \\ &\quad + \mathbb{I}_{\mathbb{N}}(\mathbb{M}_3(X_2, \mathbb{N}_S(X_1)), \mathbb{M}_4(Y_2, \mathbb{N}_S(Y_1)))) \\ &= \mathbb{N}_S(\mathbb{I}(\mathbb{M}_1(X_1, X_1), \mathbb{M}_2(Y_1, Y_1)) + \mathbb{I}_{\mathbb{N}}(\mathbb{M}_3(X_2, X_2), \mathbb{M}_4(Y_2, Y_2))) \\ &= \mathbb{N}_S(\mathbb{I}(X_1, Y_1) + \mathbb{I}_{\mathbb{N}}(X_2, Y_2)) = \mathbb{N}_S(\mathbb{I}(X_1, Y_1) + \mathbb{I}_{\mathbb{N}}(\mathbb{N}_S(X_1), \mathbb{N}_S(Y_1))) \\ &= \mathbb{N}_S(\mathbb{I}(X_1, Y_1)) + \mathbb{N}_S(\mathbb{I}(X_1, Y_1)) = \mathbb{N}_S(\mathbf{1}). \end{aligned}$$

Therefore, $\Pi_{\mathbb{I}_I((X_1, X_2), (Y_1, Y_2))} = \mathbf{0}$. □

3.3 Extending fuzzy implications from IvFL to IvIFL

See, in Table 1, some examples of intuitionistic fuzzy implication (shortly IFI) I_I which are generated by action of aggregators in \mathcal{M}_i , with $i \in \{1, 2, 3, 4\}$, such that $M_1 = M_3 = \vee$ and $M_2 = M_4 = \wedge$, as presented in [19]. For details see [19] and [3]. In Table 2 we present the corresponding generalized IvIFIs (\mathbb{I}_I), based on Definition 4.

In the following example, an interval-valued extension the of Łukasiewicz’s implication is considered according to Definition 4. It is obtained based on the Łukasiewicz’s implication and its N_S -dual coimplication.

Example 1. For $i \in \{1, 2, 3, 4\}$, let $\mathbb{M}_i : \mathbb{U}^2 \rightarrow \mathbb{U}$ be idempotent IvA operator given by:

- (i) $\mathbb{M}_1(X, Y) = \mathbb{M}_3(X, Y) = \vee(X, Y) = [\vee(\underline{X}, \underline{Y}), \vee(\overline{X}, \overline{Y})];$
- (ii) $\mathbb{M}_2(X, Y) = \mathbb{M}_4(X, Y) = \wedge(X, Y) = [\wedge(\underline{X}, \underline{Y}), \wedge(\overline{X}, \overline{Y})].$

Table 1: IFIs generated by aggregations and pairs of mutual dual functions.

IFI (Name)	IFI (Expressions)
Łukasiewicz (LK)	$I_{LZ}(\tilde{x}, \tilde{y}) = \wedge(1, (x_2 + y_1)), \vee(0, (x_1 + y_2 - 1))$
Kleene-Dienes (KD)	$I_{KD}(\tilde{x}, \tilde{y}) = (\vee(x_2, y_1), \wedge(x_1, y_2))$
Reichenbach (RC)	$I_{RC}(\tilde{x}, \tilde{y}) = (x_2 + y_1 - x_2 y_1, x_1 y_2)$
Zadeh (ZH)	$I_{ZH}(\tilde{x}, \tilde{y}) = (\vee(x_2, \wedge(N_S(x_2), y_1)), \wedge(x_1, \vee(N_S(x_1), y_2)))$
Probabilistic Conjunction (PC)	$I_{PC}(\tilde{x}, \tilde{y}) = (y_1 - y_1 x_2, N_S(x_1) + x_1 y_2)$
Bounded Conjunction (BC)	$I_{BC}(\tilde{x}, \tilde{y}) = (\vee(0, y_1 - x_2), \wedge(1, N_S(x_1) + y_2))$
Probabilistic Disjunction (PD)	$I_{PD}(\tilde{x}, \tilde{y}) = (N_S(x_2) + x_2 y_1, y_2 - y_2 x_1)$
Bounded Disjunction (BD)	$I_{BD}(\tilde{x}, \tilde{y}) = (\wedge(1, N_S(x_2) + y_1), \vee(0, y_2 - x_1))$

Table 2: IvIFIs generated by IvAs and pairs of mutual dual interval functions.

IvIFI (Expressions)
$\mathbb{I}_{LZ}(\tilde{X}, \tilde{Y}) = \wedge(\mathbf{1}, X_2 + Y_1), \vee(\mathbf{0}, Y_2 - N_S(X_1))$
$\mathbb{I}_{KD}(\tilde{X}, \tilde{Y}) = (\vee(X_2, Y_1), \wedge(X_1, Y_2))$
$\mathbb{I}_{RC}(\tilde{X}, \tilde{Y}) = (X_2 + Y_1 - X_2 \cdot Y_1, X_1 \cdot Y_2)$
$\mathbb{I}_{ZH}(\tilde{X}, \tilde{Y}) = (\vee(X_2, \wedge(N_S(X_2), Y_1)), \wedge(X_1, \vee(N_S(X_1), Y_2)))$
$\mathbb{I}_{PC}(\tilde{X}, \tilde{Y}) = (Y_1 - Y_1 \cdot X_2, N_S(X_1) - X_1 \cdot Y_2)$
$\mathbb{I}_{BC}(\tilde{X}, \tilde{Y}) = (\vee(\mathbf{0}, Y_1 - X_2), \wedge(\mathbf{1}, N_S(X_1) + Y_2))$
$\mathbb{I}_{PD}(\tilde{X}, \tilde{Y}) = (N_S(X_2) - X_2 \cdot Y_1, Y_2 - Y_2 \cdot X_1)$
$\mathbb{I}_{BD}(\tilde{X}, \tilde{Y}) = (\wedge(\mathbf{1}, N_S(X_2) + Y_1), \vee(\mathbf{0}, Y_2 - X_1))$

It is clear, by Definition 4, for each $i \in \{1, 2, 3, 4\}$, \mathbb{M}_i satisfies the conditions stated by Eqs. (3.3) and (3.4), assuring that, by Theorem 1, IvIFIs can be generated by IvAs and pairs of implications and coimplications.

Let $(\mathbb{I}_{LK}, \mathbb{J}_{LK})$ be the pair of N_S -dual functions given as:

(i) $\mathbb{I}_{LK}(X, Y) = [\wedge(1, N_S(\overline{X}) + \underline{Y}), \wedge(1, N_S(\underline{X}) + \overline{Y})]$; and

(ii) $\mathbb{J}_{LK}(X, Y) = [\vee(0, \underline{Y} - \overline{X}), \vee(0, \overline{Y} - \underline{X})]$.

By Theorem 1, the \mathbb{I}_{LK} -projections are given as: $(l_{\mathbb{I}} \circ \mathbb{I}_{LK})(\tilde{X}, \tilde{Y}) = \wedge(\mathbf{1}, X_2 + Y_1)$; and $(r_{\mathbb{I}} \circ \mathbb{I}_{LK})(\tilde{X}, \tilde{Y}) = \vee(\mathbf{0}, Y_2 - N_S(X_1))$. So, \mathbb{I}_{LK} is a natural interval extension of $I_{LK}(\tilde{x}, \tilde{y}) = \wedge(1, (x_2 + y_1)), \vee(0, (x_1 + y_2 - 1))$ as seen in the first lines of Tables 1 and 2. Its dual construction can be analogously obtained. The other IvIFIs of Table 1 can be analogously constructed, see these in Table 2.

Table 3 reports the index expression related to IvIFIs presented in Table 2. See, for instance, the respective expressions of the fuzzy index of I_{ILZ} and I_{IRC} as follows:

$$\begin{aligned} \pi_{I_{ILZ}} &= 1 - \wedge(1, (x_2 + y_1)) - \vee(0, (x_1 + y_2 - 1)) \\ &= \begin{cases} (N_S(x_2 + y_1) + N_S(x_1 + y_2)), & x_1 + x_2 \geq 1; \\ \vee(0, N_S(x_2 + y_1)), & \text{otherwise.} \end{cases} \\ \pi_{I_{IRC}} &= 1 - \vee(x_2, y_1) \wedge (x_1, y_2) = N_S(x_2) \cdot N_S(y_1) - x_1 \cdot y_2. \end{aligned}$$

And, the interval expressions of these fuzzy indexes, which were obtained by considering the corresponding IvIFIs, as presented in the two first lines in Table 3:

$$\begin{aligned} \Pi_{\mathbb{I}_{ILZ}} &= \bigvee(0, \mathbb{N}_S(X_2 + Y_1)) - \bigvee(0, X_1 + Y_2 - 1); \\ \Pi_{\mathbb{I}_{IKD}} &= \bigwedge(\mathbb{N}_S(X_2), \mathbb{N}_S(Y_1)) - \bigwedge(X_1, Y_2). \end{aligned}$$

The other indexes can be analogously obtained.

Table 3: IvIF index

IvIFI index
$\Pi_{\mathbb{I}_{ILZ}} = \bigvee(0, \mathbb{N}_S(X_2 + Y_1)) - \bigvee(0, X_1 + Y_2 - 1)$
$\Pi_{\mathbb{I}_{IKD}} = \bigwedge(\mathbb{N}_S(X_2), \mathbb{N}_S(Y_1)) - \bigwedge(X_1, Y_2)$
$\Pi_{\mathbb{I}_{IRC}} = \mathbb{N}_S(X_2 + Y_1 + X_1 \cdot Y_2) + X_2 \cdot Y_1$
$\Pi_{\mathbb{I}_{IZH}} = \bigwedge(\mathbb{N}_S(X_2), \bigvee(X_2, \mathbb{N}_S(Y_1))) - \bigwedge(X_1, \bigvee(\mathbb{N}_S(X_1), Y_2))$
$\Pi_{\mathbb{I}_{IPC}} = \mathbb{N}_S(Y_1 + X_1 \cdot Y_2) + X_1 + X_2 \cdot Y_1$
$\Pi_{\mathbb{I}_{IRC}} = \bigwedge(1, \mathbb{N}_S(Y_1 - X_2)) - \bigwedge(1, \mathbb{N}_S(X_1 - Y_2))$
$\Pi_{\mathbb{I}_{IPD}} = X_2 - Y_2 + X_1 \cdot Y_2 - X_2 \cdot Y_1$
$\Pi_{\mathbb{I}_{IRD}} = \bigvee(0, X_2 - Y_1) - \bigvee(0, Y_2 - X_1)$

Now, based on the canonical representability, we present the condition under which an IvIFI is representable. For that, let I_I be an IFI generated by a set of idempotent aggregations M_i , for $i \in \{1, 2, 3, 4\}$, together with an implication I and its corresponding intuitionistic fuzzy coimplication (IFC) I_N , based on [7, Proposition 3].

Proposition 8. Consider $\mathbb{I}_I : \tilde{\mathbb{U}}^2 \rightarrow \tilde{\mathbb{U}}$ an IvIFI given as in Definition 4 by the following representable fuzzy connectives: $\mathbb{M}_i = \hat{M}_i$ for $i \in \{1, 2, 3, 4\}$ and where each M_i is an aggregation function; $\mathbb{I} = \hat{T}$ and $\mathbb{I}_N = \hat{T}_N$ whenever (I, I_N) is a pair of N_S -dual functions given as a fuzzy implication and its corresponding fuzzy coimplication. For all $\tilde{X} = (X_1, X_2), \tilde{Y} = (Y_1, Y_2) \in \tilde{\mathbb{U}}$, \mathbb{I}_I can be characterized as:

$$(l_{\tilde{\mathbb{U}}} \circ \mathbb{I}_I)(\tilde{X}, \tilde{Y}) = [(l_{\tilde{\mathbb{U}}} \circ I)((\bar{X}_1, \underline{X}_2), (\underline{Y}_1, \bar{Y}_2)), (l_{\tilde{\mathbb{U}}} \circ I)(\underline{X}_1, \bar{X}_2), (\bar{Y}_1, \underline{Y}_2)]; \tag{3.7}$$

$$(r_{\tilde{\mathbb{U}}} \circ \mathbb{I}_I)(\tilde{X}, \tilde{Y}) = [(r_{\tilde{\mathbb{U}}} \circ I)((\bar{X}_1, \underline{X}_2), (\underline{Y}_1, \bar{Y}_2)), (r_{\tilde{\mathbb{U}}} \circ I)(\underline{X}_1, \bar{X}_2), (\bar{Y}_1, \underline{Y}_2)]; \tag{3.8}$$

when $I_I : \tilde{U}^2 \rightarrow \tilde{U}$ is the IFI obtained from M_i and the pair (I, I_N) , see [7, Section 3.2].

Proof. Firstly, let $\tilde{X} = (X_1, X_2), \tilde{Y} = (Y_1, Y_2) \in \tilde{U}$. Then, $\overline{X_1} + \overline{X_2} \leq 1$ and therefore $\underline{X_1} + \underline{X_2} \leq 1$ which means $(\overline{X_1}, \underline{X_2}) \in \tilde{U}$. Analogously, $(\underline{Y_1}, \overline{Y_2}), (\underline{X_1}, \overline{X_2}), (\overline{Y_1}, \underline{Y_2}) \in \tilde{U}$. Now, by Definition 4, it follows that:

$$\begin{aligned} (l_{\tilde{U}} \circ I_I)(\tilde{X}, \tilde{Y}) &= \mathbb{I}(\mathbb{M}_1(X_1, \mathbb{N}_S(X_2)), (\mathbb{M}_2(Y_1, \mathbb{N}_S(Y_2)))) \text{ by Eq. (3.5)} \\ &= \mathbb{I}([M_1(\underline{X_1}, N_S(\overline{X_2})), M_1(\overline{X_1}, N_S(\underline{X_2}))], \\ &\quad [M_2(\underline{Y_1}, N_S(\overline{Y_2})), M_2(\overline{Y_1}, N_S(\underline{Y_2}))]) \text{ by Eqs. (2.4) and (2.6)} \\ &= [I(M_1(\overline{X_1}, N_S(\underline{X_2})), M_2(\underline{Y_1}, N_S(\overline{Y_2}))), \\ &\quad I(M_1(\underline{X_1}, N_S(\overline{X_2})), M_2(\overline{Y_1}, N_S(\underline{Y_2})))] \text{ by Eq. (2.7)} \\ &= [(l_{\tilde{U}} \circ I_I)((\overline{X_1}, \underline{X_2}), (\underline{Y_1}, \overline{Y_2})), (l_{\tilde{U}} \circ I_I)((\underline{X_1}, \overline{X_2}), (\overline{Y_1}, \underline{Y_2}))]. \end{aligned}$$

Therefore, we conclude that Eq. (3.7) is held. Analogously, Eq. (3.8) is proved. □

In the following, Corollary 2 presents the conditions under which projection functions over pairs of degenerate intervals are preserved by an IvIFI \mathbb{I}_I , as given in Definition 4.

Corollary 2. *Under the conditions of Proposition 8, the IvIFI operator $\mathbb{I}_I : \tilde{U}^2 \rightarrow \tilde{U}$ is a diagonal preserving operator.*

Proof. By Proposition 7, it is sufficient to show that, for all $\tilde{X}, \tilde{Y} \in \tilde{\mathbb{D}}$ it holds that $\mathbb{I}_I(\tilde{X}, \tilde{Y}) \in \tilde{\mathbb{D}}$. By Proposition 8, it follows that $\mathbb{I}_I(\tilde{X}, \tilde{Y}) = \mathbb{I}_I((X_1, X_2), (Y_1, Y_2)) = (\mathbb{I}(X_1, Y_1), \mathbb{I}_N(X_2, Y_2))$. So, in accordance with the characterization stated by Eqs. (3.7) and (3.8), we have the equations:

$$\begin{aligned} \mathbb{I}(X_1, Y_1) &= l_{\tilde{U}} \circ \mathbb{I}_I(\tilde{X}, \tilde{Y}) = [(l_{\tilde{U}} \circ I_I)((\overline{X_1}, \underline{X_2}), (\underline{Y_1}, \overline{Y_2})), (l_{\tilde{U}} \circ I_I)((\underline{X_1}, \overline{X_2}), (\overline{Y_1}, \underline{Y_2}))]; \\ \mathbb{I}_N(X_2, Y_2) &= r_{\tilde{U}} \circ \mathbb{I}_I(\tilde{X}, \tilde{Y}) = [(r_{\tilde{U}} \circ I_I)((\overline{X_1}, \underline{X_2}), (\underline{Y_1}, \overline{Y_2})), (r_{\tilde{U}} \circ I_I)((\underline{X_1}, \overline{X_2}), (\overline{Y_1}, \underline{Y_2}))]. \end{aligned}$$

Since, $\tilde{X}, \tilde{Y} \in \tilde{\mathbb{D}}$, then $\underline{X_1} = \overline{X_1}, \underline{X_2} = \overline{X_2}, \underline{Y_1} = \overline{Y_1}$ and $\underline{Y_2} = \overline{Y_2}$. Therefore both are degenerate intervals. In addition, since $(\mathbb{I}, \mathbb{I}_N)$ is a pair of \mathbb{N}_S -dual functions, we have that $\mathbb{I}_N(X_2, Y_2) = \mathbb{N}_S(\mathbb{I}(\mathbb{N}_S(X_2), \mathbb{N}_S(Y_2))) = \mathbb{N}_S(\mathbb{I}(X_1, Y_1))$. Therefore, $\mathbb{I}(X_1, Y_1) + \mathbb{I}_N(X_2, Y_2) = \mathbf{1}$, concluding that $\mathbb{I}_I(\tilde{X}, \tilde{Y}) \in \tilde{\mathbb{D}}$. □

3.4 Properties of IvIFIs

An IvIFI operator can be demanded to verify other properties which are inherited when the fuzzy implication is generalized to the intuitionistic fuzzy case, see [7, 12] and [13]. In such context, we consider the next truly intuitionistic fuzzy properties:

\mathbb{I}_I 6 $\Pi_{\mathbb{I}_I(\tilde{X}, \tilde{Y})} \leq \Pi(\mathbb{I}(\mathbb{N}_S(X_2), Y_1), \mathbb{I}_N(\mathbb{N}_S(X_1), Y_2));$

$$\mathbb{I}_I 7 \quad \mathbb{I}_I(\tilde{X}, \tilde{Y}) \succeq_{\tilde{U}} (\mathbb{I}(\mathbb{N}_S(X_2), Y_1), \mathbb{I}_{\mathbb{N}}(\mathbb{N}_S(X_1), Y_2));$$

$$\mathbb{I}_I 8 \quad \Pi_{\mathbb{I}_I(\tilde{X}, \tilde{Y})} \leq \bigvee(\mathbb{N}_S(X_1), \mathbb{N}_S(Y_1));$$

$$\mathbb{I}_I 9 \quad \text{If } \tilde{X} = \tilde{Y} \text{ then } \Pi_{\mathbb{I}_I(\tilde{X}, \tilde{Y})} = \Pi_{\tilde{X}};$$

$$\mathbb{I}_I 10 \quad \text{If } \Pi_{\tilde{X}} = \Pi_{\tilde{Y}} \text{ then } \Pi_{\mathbb{I}_I(\tilde{X}, \tilde{Y})} = \Pi_{\tilde{X}}.$$

Proposition 9. *Let $\mathbb{I}_I : \tilde{U}^2 \rightarrow \tilde{U}$ be an IvIFI operator whose projections are given by Eqs. (3.5) and (3.6). Therefore \mathbb{I}_I verifies $\mathbb{I}_I 6$ and $\mathbb{I}_I 7$.*

Proof. Let $\mathbb{M}_{i \in \{1,2,3,4\}}$ be an idempotent IvA operators. Then, it follows that:

$$X_1 \leq \mathbb{M}_1(X_1, \mathbb{N}_S(X_2)) \leq \mathbb{N}_S(X_2); \quad \text{and} \quad Y_1 \leq \mathbb{M}_2(Y_1, \mathbb{N}_S(Y_2)) \leq \mathbb{N}_S(Y_2);$$

$$X_2 \leq \mathbb{M}_3(X_2, \mathbb{N}_S(X_1)) \leq \mathbb{N}_S(X_1); \quad \text{and} \quad Y_2 \leq \mathbb{M}_4(Y_2, \mathbb{N}_S(Y_1)) \leq \mathbb{N}_S(Y_1).$$

By $\mathbb{I}_I 1$ and $\mathbb{I}_I 2$, we have that: (i) $\mathbb{I}(\mathbb{M}_1(X_1, \mathbb{N}_S(X_2)), \mathbb{M}_2(Y_1, \mathbb{N}_S(Y_2))) \geq \mathbb{I}(\mathbb{N}_S(X_2), Y_1)$; and (ii) $\mathbb{I}_{\mathbb{N}}(\mathbb{M}_3(X_2, \mathbb{N}_S(X_1)), \mathbb{M}_4(Y_2, \mathbb{N}_S(Y_1))) \geq \mathbb{I}_{\mathbb{N}}(\mathbb{N}_S(X_1), Y_2)$. So, \mathbb{I}_I is well defined.

[$\mathbb{I}_I 6$]

By (i) and (ii), $\underline{\mathbb{I}_I}(\tilde{X}, \tilde{Y}) + \overline{\mathbb{I}_I}(\tilde{X}, \tilde{Y}) \geq \mathbb{I}(\mathbb{N}_S(X_2), Y_1) + \mathbb{I}_{\mathbb{N}}(\mathbb{N}_S(X_1), Y_2)$ and we have:
 $\Pi_{\mathbb{I}_I((X_1, X_2), (Y_1, Y_2))} \leq \Pi_{(\mathbb{I}(\mathbb{N}_S(X_2), Y_1), \mathbb{I}_{\mathbb{N}}(\mathbb{N}_S(X_1), Y_2))}$. Then, $\mathbb{I}_I 6$ is verified.

[$\mathbb{I}_I 7$]

Based on (i) and (ii), $\mathbb{I}_I((X_1, X_2), (Y_1, Y_2)) \succeq (\mathbb{I}(\mathbb{N}_S(X_2), Y_1), \mathbb{I}_{\mathbb{N}}(\mathbb{N}_S(X_1), Y_2))$. So, it means that \mathbb{I}_I satisfies $\mathbb{I}_I 7$.

Therefore, Proposition 9 holds. □

Proposition 10. *Let \mathbb{I}_I be an IvIFI operator whose projections are given by Eqs. (3.5), (3.6). When $\mathbb{I}(X, Y) \geq \bigwedge(X, Y)$ \mathbb{I}_I verifies $\mathbb{I}_I 8$.*

Proof. For all $\tilde{X}, \tilde{Y} \in \tilde{U}$, the following holds.

$$\begin{aligned} l_{\tilde{U}}(\mathbb{I}_I(\tilde{X}, \tilde{Y})) + r_{\tilde{U}}(\mathbb{I}_I(\tilde{X}, \tilde{Y})) &\geq l_{\tilde{U}}(\mathbb{I}_I(\tilde{X}, \tilde{Y})) = \mathbb{I}(\mathbb{M}_1(X_1, \mathbb{N}_S(X_2)), \mathbb{M}_2(Y_1, \mathbb{N}_S(Y_2))) \\ &\geq \mathbb{I}(\bigwedge(X_1, \mathbb{N}_S(X_2)), \bigvee(Y_1, \mathbb{N}_S(Y_2))) \text{ by Eq. (2.5)} \\ &\geq \mathbb{I}(X_1, Y_1) \geq \bigwedge(X_1, Y_1) \end{aligned}$$

By hypothesis $\mathbb{I}(X, Y) \geq \bigwedge(X, Y)$ and then $l_{\tilde{U}}(\mathbb{I}_I(\tilde{X}, \tilde{Y})) + r_{\tilde{U}}(\mathbb{I}_I(\tilde{X}, \tilde{Y})) \geq \bigwedge(X_1, Y_1)$. Moreover, it holds that

$$\Pi_{\mathbb{I}_I(\tilde{X}, \tilde{Y})} = \mathbb{N}_S(l_{\tilde{U}}(\mathbb{I}_I(\tilde{X}, \tilde{Y})) + r_{\tilde{U}}(\mathbb{I}_I(\tilde{X}, \tilde{Y}))) \leq \mathbb{N}_S(\bigwedge(X_1, Y_1)).$$

Therefore, $\Pi_{\mathbb{I}_I(\tilde{X}, \tilde{Y})} \leq \bigvee(\mathbb{N}_S(X_1), \mathbb{N}_S(Y_2))$, meaning that \mathbb{I}_I satisfies $\mathbb{I}_I 8$. □

Proposition 11. *Let \mathbb{I}_I be an IvIFI operator whose projections are given by Eqs. (3.5), (3.6). If \mathbb{I}_I verifies $\mathbb{I}_I 9$ then also verifies $\mathbb{I}_I 10$.*

Proof. Straightforward. □

4 CONCLUSION

IvIFL and IFL have been studied as a natural generalization of FL taking into account: (i) in the interval approach, the uncertainty due to the lack of information in the definition of a membership function; and (ii) in the intuitionistic approach, the non-complementary idea of the dual construction of membership functions.

Thus, it seems interesting to move on to another challenging approach such as IvIFL integrating both approaches. So, this work studies IvIFIs in the sense of [28] and provides interpretation for interval-valued intuitionistic fuzzy index. It has shown that the IvFIs obtained by action of canonical representation of aggregation functions and mutual dual functions also extends the work introduced by [7].

Further work considers applying the generalization introduced in Theorem 1 in order to analyse the main classes of IvIFIs and other constructions based on interval automorphisms, based on our previous works [5, 21, 22] and [23].

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RESUMO. A Lógica Fuzzy Intuicionista Valorada Intervalarmente consiste em uma extensão da lógica fuzzy, integrando conceitos da lógica fuzzy valorada intervalarmente e da lógica fuzzy intuicionista. A primeira considera a hesitação relacionada com a construção dual de conjuntos fuzzy intuicionistas, enquanto a segunda reflete a medida da imprecisão e incerteza no diâmetro dos intervalos associados aos elementos de um conjunto fuzzy. Este artigo mostra que uma implicação fuzzy valorada intervalarmente (IvIFI) pode ser obtida pela ação de funções de agregação valoradas intervalarmente sobre pares de funções mutuamente duais, ou seja uma implicação e sua correspondente coimplicação valorada intervalarmente. Na construção de IvIFIs, são discutidas as condições que preservam operadores diagonais e funções duais. As principais propriedades destes operadores são estudadas, incluindo tanto as propriedades preservadas da lógica fuzzy quanto propriedades inerentes da abordagem fuzzy intuicionista. A representação canônica para a classe de IvIFIs e a versão intervalar para o operador denominado índice fuzzy intuicionista são apresentados e exemplos destes operadores concluem o estudo descrito neste trabalho.

Palavras-chave: lógica fuzzy intuicionista valorada intervalarmente, implicação fuzzy valorada intuicionista, índice intuicionista.

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