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A Note on the McCormick Second-Order Constraint Qualification

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ABSTRACT. The study of optimality conditions and constraint qualification is a key topic in nonlinear optimization. In this work, we present a reformulation of the well-known second-order constraint qualification described by McCormick in [17]. This reformulation is based on the use of feasible arcs, but is independent of Lagrange multipliers. Using such a reformulation, we can show that a local minimizer verifies the strong second-order necessary optimality condition. We can also prove that the reformulation is weaker than the known relaxed constant rank constraint qualification in [19]. Furthermore, we demonstrate that the condition is neither related to the $MFCQ+WCR$ in [8] nor to the $CCP2$ condition, the companion constraint qualification associated with the second-order sequential optimality condition $AKKT2$ in [5].

Keywords: Nonlinear programming, second-order optimality conditions, constraint qualification.

1 INTRODUCTION

In this paper, we consider the nonlinear optimization problem of the form:

$$\begin{array}{rl} \text{Minimize} & f(x) \\ \text{s. t.} & h_i(x)=0, \text{ for } i=1,\dots,m \\ & g_j(x)\le 0, \text{ for } j=1,\dots,p \end{array} \tag{1.1}$$

where the functions $f:\mathbb{R}^n\to\mathbb{R}, h_i:\mathbb{R}^n\to\mathbb{R},\ i=1,\dots,m, g_j:\mathbb{R}^n\to\mathbb{R},\ j=1,\dots,p$ are twice continuously differentiable on $\mathbb{R}^n$.

We denote by

$$\Omega=\{x\in\mathbb{R}^n : h(x)=0, g(x)\le 0\}$$

the feasible set of problem (1.1). For each $x\in\Omega$ we define as $A(x)=\{j\in\{1,\dots,p\}: g_j(x)=0\}$ the index set of active inequality constraints at x.

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The notion of optimality and, especially, of how to characterize an optimal solution is crucial for the study of nonlinear optimization problems due to its close relation to the construction of algorithms to find such points.

The best-known first-order analytical optimality condition for (1.1) is the Fritz-John property presented in [9]: given a feasible point x of (1.1), there exist multipliers $(\mu_0, \lambda, \mu) \in \mathbb{R}^{1+m+p}$ such that, $\mu_0 \geq 0$, and

$$\mu_0 \nabla f(x) + \sum_{i=1}^{m} \lambda_i \nabla h_i(x) + \sum_{j=1}^{p} \mu_j \nabla g_j(x) = 0,$$
$$\mu_j \geq 0, \mu_j g_j(x) = 0, j = 1, \ldots, p.$$

However, the Fritz-John optimality conditions can be satisfied by many points which are not local optimal solutions to the problem when $\mu_0 = 0$. Thus, when an additional regularity condition is assumed in the feasible set, the Fritz-John conditions become the most useful and important stationary properties for (1.1): the well-known Karush-Kuhn-Tucker conditions. We say that a feasible point x of problem (1.1) verifies the Karush-Kuhn-Tucker conditions (*KKT* conditions in [15]) if there exist multipliers $(\lambda, \mu) \in \mathbb{R}^{m+p}$ such that

$$\begin{aligned} \nabla f(x) + \sum_{i=1}^{m} \lambda_i \nabla h_i(x) + \sum_{j=1}^{p} \mu_j \nabla g_j(x) = 0, \\ \mu_j \geq 0, \mu_j g_j(x) = 0, j = 1, \ldots, p. \end{aligned} \tag{1.2}$$

The vectors λ and μ presented in (1.2) are known as Lagrange multipliers. The set of vectors (λ, μ) satisfying (1.2) at x is denoted by $\Delta(x)$.

A point that verifies (1.2) is a stationary point of the Lagrangian function associated to (1.1):

$$l(x, \lambda, \mu) = f(x) + \sum_{i=1}^{m} \lambda_i h_i(x) + \sum_{j=1}^{p} \mu_j g_j(x). \tag{1.3}$$

Unfortunately, as we have already mentioned, (1.2) is not a first-order necessary optimality condition for a local minimizer. First-order constraint qualifications are conditions in the constraints under which it can be claimed that, if x is a local minimizer, then x is a stationary point of the Lagrangian function (1.3). The most widely used first-order constraint qualification is the linear independence of the gradients of equality and active inequality constraints at a given feasible point (*LICQ*). It is well-known that *LICQ* implies that $\Delta(x)$ is a singleton. There are other weaker first-order constraint qualifications in the literature which vary from easily verifiable but also somewhat restrictive in some situations to very abstract and difficult to check, but enjoyed by many feasible sets. On the one hand, among the easily verifiable conditions we can mention: the Mangasarian-Fromovitz condition (*MFCQ*) presented in [16]; the constant-rank constraint qualification (*CRCQ*) discussed in [13]; the relaxed constant-rank constraint qualification (*rCRCQ*) shown in [19]; the constant positive linear dependence condition (*CPLD*) described in [7, 20] and the relaxed constant positive linear dependence condition (*rCPLD*) given in [6]. On the other hand, among those more abstract and difficult to check but weaker first-order constraint qualifications we can mention: pseudonormality in [9]; quasinormality presented in [12]; the cone continuity property (*CCP*) described in [5]; Abadie's CQ shown in [1] and Guignard given in [11].

To check the local optimality in the candidates obtained using the *KKT* conditions, second-order necessary optimality conditions are studied and developed. These conditions take into account the curvature of the Lagrangian function over critical directions.

Given a *KKT* point x with multiplier $(\lambda,\mu)\in\Delta(x)$ we define as

$$C(x)=\left\{ d\in\mathbb{R}^n : \begin{array}{rl} \nabla h_i(x)^\intercal d=0, & i=1,\dots,m \\ \nabla g_j(x)^\intercal d=0, & j\in A(x):\mu_j>0 \\ \nabla g_j(x)^\intercal d\le 0, & j\in A(x):\mu_j=0 \end{array} \right\}$$

the critical cone or cone of critical directions at $x\in\Omega$. We are interested in the so-called strong second-order optimality condition (*SSOC*) described in [10, 17]: assume that x is a feasible point and $(\lambda,\mu)\in\Delta(x)$, then *SSOC* holds at x with multiplier (λ,μ) if

$$d^T\nabla^2 l(x,\lambda,\mu)d\ge 0, \tag{1.4}$$

for all directions $d\in C(x)$.

It is well established in the literature that, if a local minimizer of (1.1) verifies *LICQ*, then there is a unique *KKT* multiplier vector (λ,μ) for which *SSOC* holds (see [9]). Strong second-order constraint qualifications are conditions in the constraints under which it can be claimed that, if x is a local minimizer then x verifies the *KKT* condition and there is, at least, a *KKT* multiplier vector (λ,μ) that verifies *SSOC*.

In the last few years, weak constraint qualifications have been studied with the aim of obtaining theorems with strong results.

In [4], the authors have proved that if x is a local minimizer that verifies *CRCQ* defined in [13] then for all $(\lambda,\mu)\in\Delta(x)$, *SSOC* (1.4) holds. In [6], the authors observed that the same result proved with *CRCQ* can be demonstrated using *rCRCQ* presented in [8]: if x is a local minimizer that verifies *rCRCQ* then for all $(\lambda,\mu)\in\Delta(x)$, *SSOC* (1.4) holds.

Recently, in [2], the *SSOC* has been obtained by means of a "modified" Abadie constraint qualification (see Theorem 3.2 in [2]). However, we have noted that the "modified" Abadie constraint qualification introduced in [2] is not a proper constraint qualification as it involves the sign of the multipliers associated with the active inequality constraints in its definition.

In [18], the authors have introduced the notion of critical regularity condition (*CRC*) and have proved the validity of *SSOC* at a point of local minimum when *CRC* holds at this point. It is worth mentioning that even though *CRC* ensures the existence of Lagrange multipliers in a given solution, it is not a constraint qualification since its definition depends on the objective function.

Some of the second-order practical algorithms (see for example [3]) take into account the analysis of the Hessian of the Lagrangian function in the following tangent subspace:

$$C_0(x) = \left\{ d \in \mathbb{R}^n : \begin{array}{ll} \nabla h_i(x)^\intercal d = 0, & i = 1,\dots,m \\ \nabla g_j(x)^\intercal d = 0, & j \in A(x) \end{array} \right\}.$$

Clearly, for a feasible point x for which $\Delta(x) \neq \emptyset$, $C_0(x) \subseteq C(x)$ and $C_0(x)$ is independent of the Lagrange multiplier associated with a given *KKT* point. Considering $C_0(x)$, we can state the so-called weak second-order optimality condition: assume that x is a feasible point and $(\lambda,\mu) \in \Delta(x)$, then *WSOC* holds at x with multiplier (λ,μ) if

$$d^T \nabla^2 l(x,\lambda,\mu) d \geq 0, \tag{1.5}$$

for all $d \in \mathbb{R}^n$ such that $d \in C_0(x)$.

In [8], the authors have proved that if x is a feasible point that satisfies *MFCQ* and the weak constant-rank condition (*WCR*, see Definition 3.3), then there exists $(\lambda,\mu) \in \Delta(x)$ such that *WSOC* (1.5) holds. Then in [4], the same results have been proved for all $(\lambda,\mu) \in \Delta(x)$.

In [5], the authors have introduced the second-order cone-continuity property (*CCP*2), a second-order constraint qualification strictly weaker than the joint condition $MFCQ + WCR$ [8], *CRCQ* [13], and *rCRCQ* [19], which can be used in the global convergence analysis of the second-order algorithms defined in [3, 8, 14]. The *CCP*2 condition is the companion second-order constraint qualification associated with the sequential second-order optimality condition called *AKKT*2 (see definition 3.1 in [5]) and the authors have proved that if x is a feasible point that satisfies *CCP*2, then there exists a multiplier $(\lambda,\mu) \in \Delta(x)$ for which *WSOC* holds.

In Theorem 4 of the original paper written by McCormick [17], it is shown that a local minimizer verifies *WSOC* using the following first and second-order constraint qualification based on arcs:

- A feasible point x verifies the McCormick first-order constraint qualification (*McCormick FOCQ*) presented in [17] if for any nonzero vector $d \in L(x)$ being $L(x)$ the linearized constraint set for Ω given by:

$$L(x) = \{d \in \mathbb{R}^n : \nabla h_i(x)^\intercal d = 0, i = 1,\dots,m; \nabla g_j(x)^\intercal d \leq 0, j \in A(x)\} \tag{1.6}$$

  there exists an arc $\alpha(t)$ contained in the feasible set such that $\alpha(0) = x, \alpha^{'}(0) = d$, and α is differentiable $\forall t \in [0,\delta]$.

- A feasible point x verifies the McCormick second-order constraint qualification (*McCormick SOCQ*) in [17] if for any non-zero vector $d \in C_0(x)$ there exists a twice differentiable arc $\alpha(t)$ such that $\alpha(0) = x, \alpha^{'}(0) = d$ and $\forall t \in [0,\delta]$

$$\begin{aligned} &h_i(\alpha(t)) = 0, i = 1\dots,m; \\ &g_j(\alpha(t)) = 0, \forall\, j \in A(x). \end{aligned}$$

It is worth mentioning that although $C(x)$ is the cone that explicitly appears in *SSOC*, the first-order cone of feasible variations $L(x)$ (1.6) is a more natural approximation of the tangent directions of the feasible set Ω and $C_0(x) \subset C(x) \subset L(x)$. The set $L(x)$ is essential and has to be taken into account to demonstrate the existence of the multipliers at a local minimizer. In fact, it appears in *McCormick FOCQ* because it demonstrates the existence of the Lagrange multipliers in a given solution, but it does not ensure that (1.4) holds. At the same time, McCormick shows that *McCormick SOCQ* is a second-order CQ which does not imply *McCormick FOCQ*.

In this work, we present a strong second-order CQ (Definition 2.1) which is a reformulation of the *McCormick FOCQ* and *McCormick SOCQ* and is independent of the sign of the Lagrange multiplier associated with a KKT point. Using such a reformulation, we can show that a local minimizer verifies *SSOC*. We also show that the reformulation is weaker than the *rCRCQ* described in [19] and is neither equivalent to $MFCQ + WCR$ in [8] nor to $CCP2$ presented in [5].

The rest of this paper is organized as follows. In Section 2 we describe the formal definition of the reformulation of *McCormick FOCQ* and *McCormick SOCQ*. In Section 3, we show the relationships between other strong second-order CQs. In Section 4, we present some concluding remarks.

2 THE REFORMULATION OF MCCORMICK FOCQ AND MCCORMICK SOCQ

Definition 2.1. *Let $x \in \Omega$ be a feasible point. We say that x verifies the reformulation of the McCormick FOCQ and McCormick SOCQ (REF-McCormick) if for any nonzero vector $d \in L(x)$ there exists a twice differentiable arc $\alpha(t), \forall t \in [0,\delta]$, such that $\alpha(0) = x, \alpha^{'}(0) = d$ and $\forall t \in (0,\delta]$*

$$\begin{aligned} &h_i(\alpha(t)) = 0, i = 1,\ldots,m;\\ &g_j(\alpha(t)) = 0, \forall j \in A(x) : \nabla g_j(x)^{\intercal} d = 0;\\ &g_j(\alpha(t)) < 0, \forall j \in A(x) : \nabla g_j(x)^{\intercal} d < 0. \end{aligned} \tag{2.1}$$

The following theorem establishes that *REF-McCormick* is a strong second-order constraint qualification. We include the proof here for completeness.

Teorema 2.1. *Suppose that $x^* \in \Omega$ is a local minimum of (1.1) and that REF-McCormick holds at x^*. Then x^* is a KKT point and for every $(\lambda,\mu) \in \Delta(x^*)$, (1.4) holds.*

Proof. Let us suppose that x^* is a local minimizer of (1.1). Then, $\forall d \in L(x^*)$, by the reformulated condition *REF-McCormick*, there exists a twice differentiable arc $\alpha(t)$ such that $\alpha(0) = x^*, \alpha^{'}(0) = d$ and $\forall t \in [0,\delta]$, (2.1) holds. Thus, $f(\alpha(t)) \geq f(x^*)$ and we have that $\forall d \in L(x^*)$: $\nabla f(x^*)^{\intercal} d \geq 0$. Then,

$$-\nabla f(x^*) \in (L(x^*))^{\circ} = \left\{ z \in \mathbb{R}^n : z = \sum_{i=1}^{m} \lambda_i \nabla h_i(x^*) + \sum_{j=1}^{p} \mu_j \nabla g_j(x^*), \mu_j \geq 0 \ \forall j \in A(x^*) \right\}.$$

The notation C° indicates the polar cone of C. Therefore, x^* is a KKT point.

Let us consider a Lagrange multiplier vector $(\lambda^*, \mu^*) \in \Delta(x^*)$ and a nonzero direction $d \in C(x^*)$. Then, by *REF-McCormick*, d is tangent of a twice differentiable arc $\alpha(t)$ (where $t \geqslant 0$) along which $\alpha(0) = x^*$, $\alpha'(0) = d$ and (2.1) holds.

Let us define $\ell(t) = l(\alpha(t), \lambda^*, \mu^*)$. Following the feasibility and complementarity property we have that $\ell(0) = f(x^*) + \sum_{i=1}^{m} \lambda_i^* h_i(x^*) + \sum_{j=1}^{p} \mu_j^* g_j(x^*) = f(x^*)$. By the KKT condition:

$$\ell'(0) = \nabla_x l(x^*, \lambda^*, \mu^*)^T d = 0$$

and

$$\ell''(0) = d^T \nabla_x^2 l(x^*, \lambda^*, \mu^*) d.$$

Thus, using Taylor on ℓ around $t = 0$, we obtain that

$$\ell(t) = f(x^*) + \frac{t^2}{2} d^T \nabla_x^2 l(x^*, \lambda^*, \mu^*) d + o(t^2).$$

Then, from the definition of ℓ and (2.1) we have that

$$f(\alpha(t)) = f(x^*) + \frac{t^2}{2} d^T \nabla_x^2 l(x^*, \lambda^*, \mu^*) d + o(t^2).$$

Since x^* is a local minimizer, $\forall t$ small enough

$$0 \leq f(\alpha(t)) - f(x^*) \leq \frac{t^2}{2} d^T \nabla_x^2 l(x^*, \lambda^*, \mu^*) d + o(t^2).$$

Dividing by t^2 the last inequality and taking limits when $t \to 0$, we obtain the proof. $\square$

3 RELATIONS

In this section we present the relationship between *REF-McCormick* and other well-known second-order CQs.

Definition 3.2. *(Ref. [19]) Let $x \in \Omega$. We say that the relaxed constant rank constraint qualification rCRCQ holds if there exists a neighbourhood V of x such that for every index set $J \subset A(x)$, the set*

$$\{\nabla h_i(y)\}_{i=1,\ldots,m} \cup \{\nabla g_j(y)\}_{j \in J}$$

has the same rank for all $y \in V \cap \Omega$.

Teorema 3.2. *Suppose that $x^* \in \Omega$ and that rCRCQ holds at x^*. Then, REF-McCormick holds at x^*.*

Proof. Let us consider a direction $d \in L(x^*)$. Without loss of generality we rename the equality constraints as $c_i(x) = h_i(x), i = 1, \ldots, m$ and the inequality constraints as $c_{m+j}(x) = g_j(x), j = 1, \ldots, p$.

Define the index set $I_0(x^*,d) = \{j \in \{1,\dots,m+p\} : \nabla c_j(x^*)^\intercal d = 0\}$.

According to this hypothesis, the family of gradients $\{\nabla c_i(y)\}_{i \in I_0(x^*,d)}$ has the same rank for every y in a neighbourhood N of x^*. Let us suppose that the rank of the family $\{\nabla c_i(y)\}_{i \in I_0(x^*,d)}$ is l, and we denote as I_{rCR} the index of the linearly independent vectors, then $l = |I_{rCR}|$.

This means that, in N, l functions of the family $\{\nabla c_i(y)\}_{i \in I_{rCR}}$ are independent. Without loss of generality we can assume that the first l functions $c_1,\dots,c_l$ are independent and the other functions (if they exist) depend on $c_1,\dots,c_l$.

Define the vector function as $c : \mathbb{R}^n \to \mathbb{R}^l$ by $c(x) = (c_1(x)\dots c_l(x))^\intercal$ and consider $C : \mathbb{R}^{n+1} \to \mathbb{R}^l$ given by

$$C(y,t) := c(x^* + td + y). \tag{3.1}$$

Thus,

$$C(0,0) = c(x^*) = 0$$

Moreover, using the chain rule, the Jacobian of C with respect to y is the matrix

$$C_y(y,t) = Jc(x^* + td + y)$$

and, in particular,

$$C_y(0,0) = Jc(x^*).$$

Since the gradients $\{\nabla c_i(x^*)\}_{i \in I_{rCR}}$ are linearly independent, the matrix $C_y(0,0)$ has rank l.

Without loss of generality we can assume that the rank of $C_y(0,0)$ is equal to l with respect to the first l coordinates of vector y. Denote $y = (y^1,y^2)$ where $y^1 = (y_1,\dots,y_l), y^2 = (y_{l+1},\dots,y_n)$.

The Implicit Function Theorem (Theorem 2.13 described in Spivak [22]) ensures that near $(y,t) = (0,0)$ there exists an implicit continuously differentiable function $\bar{r} : \mathbb{R}^{n-l+1} \to \mathbb{R}^l, \bar{r}(y^2,t) = y^1$ such that $C((\bar{r}(y^2,t),y^2),t) = 0$ and $\bar{r}(0,0) = 0$. Let us define the function $r : (-\varepsilon_0,\varepsilon_0) \to \mathbb{R}^l$ as $r(t) = \bar{r}(0,t)$. Then the curve $r(t)$ is a differentiable arc for which

$$C((r(t),0),t) = 0 \tag{3.2}$$

and $r(0) = 0$ hold.

Let us show that $r'(0) = 0$. By (3.2), we have that $C_{y^1}((r(t),0),t)r'(t) + C_t((r(t),0),t) = 0$ and, taking $t = 0$

$$C_{y^1}(0,0)r'(0) + C_t(0,0) = 0. \tag{3.3}$$

By (3.1), $C_t(y,t) = Jc(x^* + td + y)d$. Then, since $d \in L(x^*)$,

$$C_t(0,0) = Jc(x^*)d = 0.$$

Therefore, using the matrix $C_y(0,0)$ which has rank l and (3.3), we obtain that $r'(0) = 0$.

Using $r(t)$, we define, on a suitable open interval containing $t = 0$, the differentiable arc

$$\alpha(t) = x^* + td + r(t).$$

By construction, $\alpha(0) = x^*$, $\alpha'(0) = d$ and we have that

$$c(\alpha(t)) = c(x^* + td + r(t)) = C((r(t), 0), t) = 0$$

on $(-\varepsilon, \varepsilon)$.

If $i \in A(x^*)$ but $i \notin I_0(x^*, d)$, we have that $\nabla c_i(x^*)^\intercal d < 0$. In this case, consider the auxiliary function $\phi(t) = c_i(\alpha(t))$ which satisfies

$$\phi(0) = c_i(\alpha(0)) = c_i(x^*) = 0$$

and

$$\phi'(0) = \nabla c_i(x^*)^\intercal \alpha'(0) = \nabla c_i(x^*)^\intercal d < 0.$$

Then, using Taylor's theorem, we obtain that there exists $\varepsilon_i > 0$ such that $\phi(t) < 0$ for all $t \in (0, \varepsilon_i)$. Taking $\varepsilon = min\{\varepsilon_i\}$, we finish the proof. $\square$

Counterexample 1. *rCRCQ* is strictly stronger than *REF-McCormick*.

In $\mathbb{R}^2$, consider $(x_1^*, x_2^*) = (0,0)$ and the following inequality constraints

$$\begin{aligned} g_1(x_1,x_2) &= x_1; \\ g_2(x_1,x_2) &= x_1 e^{x_2}. \end{aligned}$$

Then,

$$\begin{aligned} \nabla g_1(x_1,x_2) &= (1,0), & \nabla g_1(0,0) &= (1,0); \\ \nabla g_2(x_1,x_2) &= (e^{x_2}, x_1 e^{x_2}), & \nabla g_2(0,0) &= (1,0). \end{aligned}$$

Hence, *rCRCQ* fails.

For any non-zero vector $d \in L(0,0) = \{(d_1,d_2) : d_1 \le 0, d_2 \in \mathbb{R}\}$, we consider the following two cases:

In the first one, we take into account the directions $d = (0, d_2)$. We propose the curve $\alpha(t) \in C^2$ given by $\alpha(t) = (0, d_2 t)$ $\forall t \in [0, \delta]$ such that $\alpha(0) = (0,0), \alpha'(0) = d$. Then $g_1(\alpha(t)) = 0$ and $\nabla g_1(0,0)^\intercal d = 0$ $\forall t \in (0, \delta]$.

For the second constraint, we have $\nabla g_2(0,0)^\intercal d = 0$ and $g_2(\alpha(t)) = 0$ $\forall t \in (0, \delta]$.

In the second case, we consider $d = (d_1, d_2)$, $d_1 < 0$. Then, there exists a curve $\alpha(t) \in C^2$ given by $\alpha(t) = (d_1 t, d_2 t)$ $\forall t \in [0, \delta]$ such that $\alpha(0) = (0,0)$, $\alpha'(0) = d$. Furthermore, since $g_1(\alpha(t)) = d_1 t$ we obtain that $\nabla g_1(0,0)^\intercal d = d_1 < 0$ and $g_1(\alpha(t)) < 0$ $\forall t \in (0, \delta]$.

For g_2, we have $g_2(\alpha(t)) = d_1 t e^{d_2 t}$. Then, $\nabla g_2(0,0)^\intercal d = d_1 < 0$ and $g_2(\alpha(t)) = d_1 t e^{d_2 t} < 0$, $\forall t \in (0, \delta]$.

Hence, *REF-McCormick* holds.

Definition 3.3. *(Ref. [8]) Let $x \in \Omega$. We say that the weak constant rank condition WCR holds if there is a neighbourhood V of x such that the matrix made of the gradients*

$$\{\nabla h_i(y)\}_{i=1,\dots,m} \cup \{\nabla g_j(y)\}_{j \in A(x)}$$

has the same rank for all $y \in V$.

Definition 3.4. *(Ref. [16]) We say that $x \in \Omega$ satisfies the Mangasarian-Fromovitz constraint qualification if the gradients $\{\nabla h_i(x)\}_{i=1,\dots,m}$ are linearly independent and there exists a vector $d \in \mathbb{R}^n$ such that $\nabla h_i(x)^\intercal d = 0$, $i = 1,\dots,m$ and $\nabla g_j(x)^\intercal d < 0$, $j \in A(x)$.*

The following counterexamples show that $MFCQ + WCR$ and *REF-McCormick* are independent.

Counterexample 2. *REF-McCormick* does not imply $WCR + MFCQ$.

In $\mathbb{R}^2$, consider $(x_1^*, x_2^*) = (0,0)$ and the inequality constraints defined by

$$\begin{aligned} g_1(x_1,x_2) &= x_1; \\ g_2(x_1,x_2) &= x_1 - x_2^2; \\ g_3(x_1,x_2) &= x_1 + x_2; \\ g_4(x_1,x_2) &= -x_1 - x_2. \end{aligned}$$

Then,

$$\begin{aligned} \nabla g_1(x_1,x_2) &= (1,0); \\ \nabla g_2(x_1,x_2) &= (1,-2x_2); \\ \nabla g_3(x_1,x_2) &= (1,1); \\ \nabla g_4(x_1,x_2) &= (-1,-1); \end{aligned}$$

and $L(0,0) = \{(d_1,-d_1) : d_1 \le 0\}$. For any non-zero vector $d \in L(0,0)$ consider the curve $\alpha(t) \in C^2$, $\alpha(t) = (td_1, -td_1), \forall t \in [0,\delta]$ which verifies $\alpha(0) = (0,0), \alpha'(0) = d$ and, $\forall t \in (0,\delta]$

$$\begin{array}{ll} g_1(\alpha(t)) = td_1 < 0, & \nabla g_1(0,0)^\intercal d < 0; \\ g_2(\alpha(t)) = td_1(1 - td_1) < 0, & \nabla g_2(0,0)^\intercal d < 0; \\ g_3(\alpha(t)) = 0, & \nabla g_3(0,0)^\intercal d = 0; \\ g_4(\alpha(t)) = 0, & \nabla g_4(0,0)^\intercal d = 0. \end{array}$$

Clearly, *REF-McCormick* holds. And, it is trivial that *MFCQ* does not hold.

Counterexample 3. $MFCQ + WCR$ does not imply *REF-McCormick*.

Consider the following example given in [18]. In $\mathbb{R}^3$, consider $(x_1^*, x_2^*, x_3^*) = (0,0,0)$ and the following inequality constraints

$$\begin{aligned} g_1(x_1,x_2,x_3) &= -x_1 + x_2 - x_3^2; \\ g_2(x_1,x_2,x_3) &= -x_1 - x_2; \\ g_3(x_1,x_2,x_3) &= -x_1; \\ g_4(x_1,x_2,x_3) &= -x_1^2 - x_2^2 - x_3^2 + x_3. \end{aligned}$$

Then,

$$\begin{array}{ll}\nabla g_1(x_1,x_2,x_3) = (-1,1,-2x_3), & \nabla g_1(0,0,0) = (-1,1,0);\\ \nabla g_2(x_1,x_2,x_3) = (-1,-1,0), & \nabla g_2(0,0,0) = (-1,-1,0);\\ \nabla g_3(x_1,x_2,x_3) = (-1,0,0), & \nabla g_3(0,0,0) = (-1,0,0);\\ \nabla g_4(x_1,x_2,x_3) = (-2x_1,-2x_2,-2x_3+1), & \nabla g_4(0,0,0) = (0,0,1).\end{array}$$

It is easy to see that $WCR+MFCQ$ holds.

We will see that *REF-McCormick* does not hold.

We have that $L(0,0,0) = \{(d_1,d_2,d_3) : d_1 \geq 0, -d_1 \leq d_2 \leq d_1, d_3 \leq 0\}$. Let us take a generic arc $\alpha(t) = (\alpha_1(t), \alpha_2(t), \alpha_3(t))$ for $t \in [0,\delta]$.

Let us consider the direction $d = (0,0,d_3)$, $d_3 < 0$. Then, $\nabla g_i(0,0,0)^\intercal d = 0$ for $i = 1,2,3$ and $\nabla g_4(0,0,0)^\intercal d = d_3 < 0$.

As the arc has to be feasible for $t \in [0,\delta]$ and (2.1) must hold, the equalities: $g_1(\alpha(t)) = g_2(\alpha(t)) = g_3(\alpha(t)) = 0$ must be verified, $\forall t \in (0,\delta]$.

However, this is a contradiction since the unique arc which can be considered is the null one. Therefore, *REF-McCormick* fails.

In [5] the authors have presented the $CCP2$ condition defined below.

Let us consider $x^* \in \Omega$. For $x \in \mathbb{R}^n$ define the cone

$$C_W(x,x^*) = \left\{d \in \mathbb{R}^n : \nabla h_i(x)^\intercal d = 0, i = 1,\ldots,m; \nabla g_j(x)^\intercal d = 0, j \in A(x^*)\right\}.$$

and denote by $K_2^W(x)$ the following set:

$$\bigcup_{\substack{(\lambda,\mu)\in\mathbb{R}^m\times\mathbb{R}^p_+\\ \mu_j=0 \text{ for } j\notin A(x^*)}} \left\{ \begin{array}{c} \left(\sum_{i=1}^m \lambda_i \nabla h_i(x) + \sum_{j\in A(x^*)} \mu_j \nabla g_j(x), H\right), \\ \text{such that } H \preceq \sum_{i=1}^m \lambda_i \nabla^2 h_i(x) + \sum_{j\in A(x^*)} \mu_j \nabla^2 g_j(x) \text{ on } C_W(x,x^*) \end{array} \right\}$$

where we write $A \succeq B$ if $d^T A d \geq d^T B d$ for all $d \in \mathbb{R}^n$. The set $K_2^W(x)$ is a convex cone included in $\mathbb{R}^n \times Sym(n)$ where $Sym(n)$ denotes the set of symmetric matrices of order n, [5].

Definition 3.5. *(Ref. [5]) We say that $x^* \in \Omega$ satisfies the second-order cone-continuity property $CCP2$ if the set-valued mapping (multifunction) $x \longmapsto K_2^W(x)$, is outer semicontinuous at x^*, that is,*

$$\limsup_{x\to x^*} K_2^W(x) \subset K_2^W(x^*).$$

The authors proved that $CCP2$ is less stringent than $MFCQ+WCR$ and $rCRCQ$. In the following counterexamples we show that *REF-McCormick* does neither imply nor is implied by $CCP2$.

Counterexample 4. *REF-McCormick* does not imply *CCP*2.

Consider the following example given in [21] where the authors proved that *CCP*2 fails. In $\mathbb{R}^2$, consider $(x_1^*, x_2^*) = (0,0)$ and the inequality constraints given by

$$g_1(x_1,x_2) = -x_1;$$

$$g_2(x_1,x_2) = x_1 + \max\{x_1,0\}^2 e^{x_2^2}.$$

Then

$$\begin{aligned}
&\nabla g_1(x_1,x_2) = (-1,0),\\
&\nabla g_2(x_1,x_2) = (1+2\max\{x_1,0\}e^{x_2^2}, 2x_2\max\{0,x_1\}^2 e^{x_2^2}), \quad \nabla g_2(0,0) = (1,0);
\end{aligned}$$

and $L(0,0) = \{(0,d_2) : d_2 \in \mathbb{R}\}$. For any nonzero vector $d \in L(0,0)$ there exists the curve $\alpha(t) \in C^2$, $\alpha(t) = (0,d_2 t)$, $\forall t \in [0,\delta]$ such that $\alpha(0) = 0, \alpha'(0) = d$ and, $\forall t \in (0,\delta]$

$$\begin{aligned}
g_1(\alpha(t)) = 0, &\quad \nabla g_1(0,0)^\intercal d = 0;\\
g_2(\alpha(t)) = 0, &\quad \nabla g_2(0,0)^\intercal d = 0.
\end{aligned}$$

Clearly, *REF-McCormick* holds.

Counterexample 5. *CCP*2 does not imply *REF-McCormick*. In $\mathbb{R}^2$, consider the following example given in [5]: $(x_1^*, x_2^*) = (0,0)$ and the equality and inequality constraints

$$\begin{aligned}
h_1(x_1,x_2) &= x_1;\\
g_1(x_1,x_2) &= -x_1^2 + x_2;\\
g_2(x_1,x_2) &= -x_1^2 + x_2^3.
\end{aligned}$$

We have

$$\begin{aligned}
\nabla h_1(x_1,x_2) &= (1,0), & \nabla h_1(0,0) &= (1,0);\\
\nabla g_1(x_1,x_2) &= (-2x_1,1), & \nabla g_1(0,0) &= (0,1);\\
\nabla g_2(x_1,x_2) &= (-2x_1,3x_2^2), & \nabla g_2(0,0) &= (0,0).
\end{aligned}$$

As a result, we see that *rCRCQ* fails at (x_1^*, x_2^*). Now, since $C_W((x_1,x_2),(0,0)) = \{(0,0)\}$, we get $K_2^W(x_1,x_2) = \mathbb{R} \times \mathbb{R}_+ \times Sym(2)$. Clearly, $K_2^W(x_1,x_2)$ is outer semicontinuous on $\mathbb{R}^2$ and *CCP*2 holds.

We will see that *REF-McCormick* does not hold. We have that $L(0,0) = \{(0,d_2) : d_2 \le 0\}$. Let us take $d = (0,d_2), d_2 < 0$, and a generic arc $\alpha(t) = (\alpha_1(t), \alpha_2(t))$ for $t \in [0,\delta]$. As the arc has to be feasible and (2.1) must hold, the equalities: $h_1(\alpha(t)) = g_2(\alpha(t)) = 0$ must be verified, $\forall t \in (0,\delta]$.

But, this is a contradiction. Therefore, *REF-McCormick* fails.

In Figure 1, we show the relationship between the CQs discussed in this article.

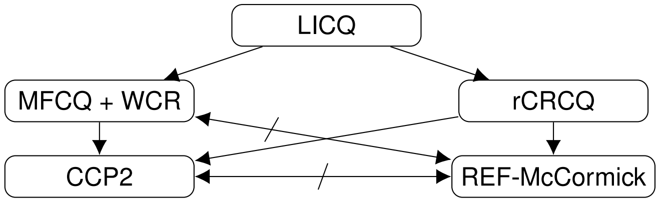

Figure 1: Relationship of second-order CQs. An arrow between two CQs means that one is strictly stronger than the other.

4 FINAL REMARKS

In the present paper, we have presented the condition called *REF-McCormick*, a second-order constraint qualification which unifies *McCormick FOCQ* and *McCormick SOCQ* conditions presented in [17]. Using *REF-McCormick* we have proved that a local minimizer verifies *SSOC*. We have also shown that *REF-McCormick* is weaker than the strong second-order condition *rCRCQ* described in [19]. Furthermore, we have demonstrated that *REF-McCormick* is independent of $CCP2$ and $MFCQ+WCR$ conditions, which imply $WSOC$.

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