Bernstein-type Theorems in Hypersurfaces with Constant Mean Curvature

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ABSTRACT

By using the nodal domains of some natural function arising in the study of hypersurfaces with constant mean curvature we obtain some Bernstein-type theorems.

Key words: Riemannian manifold, eigenvalue, hypersurface, mean curvature.

0. INTRODUCTION

The Bernstein theorem on minimal surfaces $x: M^2 \to R^3$ in the Euclidean space R^3 states that if $x(M^2)$ is a graph over a plane P of R^3 which is defined for all points of P then M^2 is itself a plane. This beautiful result has been the basis of a large number of investigations on minimal surfaces. Among its generalizations is a theorem proved independently by (do Carmo & Peng 1979) and (Fischer-Colbrie & Schoen 1980) which states that if M^2 is complete and stable then it is a plane.

A generalization of this theorem for higher dimensions was obtained by (do Carmo-Peng 1980) as follows:

THEOREM A. *Let $x : M^n \to R^{n+1}$ be a minimal hypersurface. Assume that M^n is stable, complete and that*

$$\lim_{R\to+\infty} \frac{\int_{B(R)} \|A\|^2 dM}{R^{2+2q}} = 0, \quad q < \sqrt{2/n}.$$

Then M^n is a hyperplane in R^{n+1}.

Here $\|A\|$ is the second fundamental form and $B(R)$ is a geodesic ball of radius ball R centered at some fixed point in M.

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Theorem A has been recently extended to hypersurfaces with constant mean curvature. A crucial point is to replace A by the traceless second fundamental form $\phi = -A + HI$; here H is the mean curvature of $x : M^n \to R^{n+1}$. The precise statement is as follows:

THEOREM B. (Alencar & do Carmo 1994a). *Let $x : M \to R^{n+1}$, $n \le 5$ be a complete noncompact hypersurface with constant mean curvature H. Assume that M is strongly stable (see definition in Section 1), and that*

$$\lim_{R\to+\infty} \frac{\int_{B(R)} \|\phi\|^2 dM}{R^{2+2q}} = 0, \quad q < \frac{1}{6n+1}. \tag{0.1}$$

Then M is a hyperplane in R^{n+1}.

In the present paper, we extend Theorem B in two directions. First we relax the growth condition on $P(R) = \int_{B(R)} |\phi|^2 dM$ and extend Theorem B to this weaker condition. More precisely, we prove

THEOREM 1. *Let M^n be a strongly stable complete noncompact hypersurface of R^{n+1} ($n \le 5$) with constant mean curvature H. If $P(r) \le Ce^{\alpha H r}$, for some positive constants C, and α, where α depends on n given in the proof, then M is a hyperplane.*

Next we improve the dimension condition from $n \le 5$ to $n \le 6$ and prove

THEOREM 2. *Let M be a strongly stable complete noncompact hypersurface of R^{n+1} ($n \le 6$) with constant mean curvature H. Assume that*

$$\lim_{R\to+\infty} \frac{\int_{B(R)} \|\phi\|^2 dM}{R^{2-2/n}} = 0.$$

Then M is a hyperplane.

Theorem 1 is the main theorem of this paper and goes a long way towards getting rid of condition (0.1) in Theorem B. For its proof we need an auxiliary proposition that might be interesting by itself and states that the function $|\phi|$ on a hypersurface M^n with constant mean curvature in R^{n+1} has no bounded nodal domain.

1. NOTATIONS AND PRELIMINARIES

Let M^n be a complete noncompact hypersurface in R^{n+1}. Fix $p \in M$ and choose a local unit normal field N. Define a linear map A: $T_pM \to T_pM$ by

$$\langle AX, Y\rangle = \langle \bar{\nabla}_X Y, N\rangle$$

where X, Y are the tangent vector fields and $\bar{\nabla}$ is the standard connection on R^{n+1}. The map A can be diagonalized, i.e., there exists a tangent basis $\{e_1, e_2, \cdots, e_n\}$ such that $Ae_i = k_i e_i$, $i = 1, 2, \cdots, n$. We then define the mean curvature $H := \frac{1}{n}\sum_{i=1}^n k_i$ and the square of the second fundamental form $|A|^2 := \sum_{i=1}^n k_i^2$. It is well known that the above objects are independent of the choices made.

If M is minimal($H = 0$), we say M is stable if for all piecewise smooth functions $f : M \to R$ with compact support, we have that

$$\int_M |\nabla f|^2 dM \geq \int_M |A|^2 f^2 dM; \tag{1.1}$$

here ∇f is the gradient of f in the induced metric.

The notion of stability has been extended to hypersurfaces with constant mean curvature as follows: M is said to be strongly stable if (1.1) holds for all piecewise smooth functions $f : M \to R$ with compact support. M is said to be weakly stable if (1.1) holds for all piecewise smooth functions $f : M \to R$ with compact support and $\int_M f = 0$.

Let $x : M^n \to \overline{M}^{n+1}$ be an isometric immersion of a complete, noncompact Riemannian n-dimensional manifold M^n into an oriented, complete, Riemannian $(n+1)$-dimensional manifold, N a smooth unit normal field along M, and $\overline{\mathrm{Ric}}(N)$ the value of the Ricci curvature of $\overline{M}^{n+1}$ in the vector N. Here $\overline{\mathrm{Ric}}(N) = \sum_{i=1}^n K(e_i \wedge N)$ (this is different from the normalized one). The Morse index $\mathrm{ind}\, M$ of M is defined as follows. Let L be the second order differential operator on M given by

$$L = \Delta + |A|^2 + \overline{\mathrm{Ric}}(N). \tag{1.2}$$

Associated to L is the quadratic form

$$I(f) = -\int_M fLf dM, \tag{1.3}$$

defined on the vector space of functions f on M that have support on a compact domain $K \subset M$. For each such K, define the index $\mathrm{ind}_L\, K$ of L in K as the maximal dimension of a subspace where I is negative definite. The index $\mathrm{ind}\, M$ of L in M is the number defined by

$$\mathrm{ind}\, M = \sup_{K \subset M} \mathrm{ind}_L\, K \tag{1.4}$$

where the supremum is taken over all compact domains $K \subset M$. It is well known that $\mathrm{ind}(M) \leq 1$, if M is weakly stable(see, for example, (Fischer-Colbrie 1985)).

In what follows we always assume that M is a hypersurface in R^{n+1} with constant mean curvature H. To study the hypersufaces with constant mean curvature, it is convenient to modify slightly the second fundamental form and to introduce a new linear map $\phi : T_pM \to T_pM$ by

$$\langle \phi X, Y \rangle = -\langle AX, Y \rangle + H\langle X, Y \rangle.$$

ϕ can also be diagonalized as:

$$\phi e_i = \mu_i e_i.$$

It is easily checked that $\mathrm{tr}\, \phi = 0$, and

$$|\phi|^2 := \sum_{i=1}^n \mu_i^2 = \frac{1}{2n} \sum_{i,j} (k_i - k_j)^2.$$

Thus $|\phi|^2$ measures how far M is from being totally umbilic. For the rest of this section we follow (Alencar & do Carmo 1994a). Choosing an orthonormal principal frame $\{e_i\}$, we can write

$$\frac{1}{2}\Delta|\phi|^2 = \sum_{i,j,l}\phi_{ijl}^2 + \sum_i \mu_i(\mathrm{tr}\,\phi)_{ii} + \frac{1}{2}\sum_{i,j} R_{ijij}(\mu_i - \mu_j)^2,$$

where ϕ_{ijl} are components of the covariant derivative of the tensor ϕ, and R_{ijij} is the sectional curvature of the plane $\{e_i, e_j\}$. By Gauss formula, we conclude that

$$\begin{aligned}\frac{1}{2}\sum_{i,j} R_{ijij}(\mu_i - \mu_j)^2 =& \frac{1}{2}\sum_{i,j}\mu_i\mu_j(\mu_i - \mu_j)^2 \\ &- \frac{H}{2}\sum_{i,j}(\mu_i + \mu_j)(\mu_i - \mu_j)^2 + \frac{H^2}{2}\sum_{i,j}(\mu_i - \mu_j)^2.\end{aligned}$$

Since $\sum \mu_i = 0$, it is easy to check that:

$$\begin{aligned}\sum_{i,j}(\mu_i - \mu_j)^2 &= 2n|\phi|^2, \\ \sum_{i,j}(\mu_i + \mu_j)(\mu_i - \mu_j)^2 &= 2n\sum_i \mu_i^3, \\ \sum_{i,j}\mu_i\mu_j(\mu_i - \mu_j)^2 &= -2|\phi|^4.\end{aligned}$$

From the above, it follows that

$$\frac{1}{2}\Delta|\phi|^2 = |\phi|\Delta|\phi| + |\nabla|\phi||^2 = \sum_{i,j,l}\phi_{ijl}^2 - |\phi|^4 - nH\sum_i \mu_i^3 + nH^2|\phi|^2.$$

In this case it follows by (do Carmo & Peng 1980 (2.3), (2.4)) that

$$\sum_{i,j,l}\phi_{ijl}^2 \geq \frac{2}{n}|\nabla|\phi||^2 + |\nabla|\phi||^2.$$

By using a lemma of Okumura (see (Alencar & do Carmo 1994b) for a proof), we have

$$\sum_i \mu_i^3 \leq \frac{n-2}{\sqrt{n(n-1)}}|\phi|^3.$$

So we have finally

$$|\phi|\Delta|\phi| + |\phi|^4 + \frac{n(n-2)}{\sqrt{n(n-1)}}H|\phi|^3 - nH^2|\phi|^2 \geq \frac{2}{n}|\nabla|\phi||^2. \tag{1.5}$$

2. A RESULT ON NODAL DOMAINS

In this section we prove a result on the nodal domains of $|\phi|$ which will be needed in our proof of main theorems. We first need to recall the definition of nodal domains.

DEFINITION. An open domain D is called the nodal domain of function f if $f(x) \neq 0$ for $x \in \operatorname{int} D$ and vanishes on the boundary of ∂D. We denote by $N(f)$ the number of disjoint *bounded* nodal domains of f.

Now we have the following lemma which follows directly from Proposition 2.2 below. We are indebted to the referee who provided its proof and corrected a mistake in our original version.

LEMMA 2.1. *Let M be a hypersurface in R^{n+1} with constant mean curvature H. Then*

$$N(|\phi|) = 0. \tag{2.1}$$

PROOF. Let $\varphi(u) = u^2 + \frac{n(n-2)}{\sqrt{n(n-1)}} Hu - nH^2$. Then from (1.5), with $|\phi| = u$, and Proposition 2.2 below the lemma follows. □

PROPOSITION 2.2. *Let (M, g) be Riemannian manifold and $u \geq 0$ be a continuous function satisfying the following inequality of Simons' type in the distribution sense*

$$u^2\varphi(u) \geq a|\nabla u|_g^2 - u\Delta_g u \tag{2.2}$$

where $a > 0$ is a constant and φ is a continuous function on R.

Then u has no relatively compact nodal domain.

PROOF. Suppose that u admits a relatively compact nodal domain D. Write $q := \varphi(u)$ and $v := \log u$ on D. Thus (2.2) can be written as

$$q \geq a|\nabla v|_g^2 - \Delta_g v - |\nabla v|_g^2.$$

Then for any Lipschitz function f with support in D and vanishing at ∂D, we have

$$\int_D (|\nabla f|^2 - qf^2) \leq -a \int_D f^2|\nabla v|^2 + \int_D |\nabla f - f\nabla v|^2.$$

Let $f = wu$, for some function w to be determined. We obtain

$$\int_D (|\nabla f|^2 - qf^2) \leq -a \int_D w^2|\nabla u|^2 + \int_D u^2|\nabla w|^2.$$

For all b such that $U/2 \leq b \leq U$, where $U := \sup_D u$, we set

$$w_b(x) = \begin{cases} b & \text{if } u(x) \leq b, \\ u(x), & \text{if } u(x) > b. \end{cases}$$

Denote D_+ (resp. D_-) the set of points in D with $u(x) \geq b$ (resp. $u(x) \leq b$). A simple calculation leads to

$$\int_D (|\nabla f|^2 - qf^2) \leq \int_{D_+} u^2|\nabla u|^2 - \frac{aU^2}{4}\int_D |\nabla u|^2.$$

When b goes to U, the first term of right hand side tends to 0 (because $|\nabla u|^2$ is integrable), while the second term is fixed. It follows that $\int_D (|\nabla f|^2 - qf^2) < 0$ for all functions $f = w_b u$, when b is close to U. These functions w_b form an infinite dimensional vector which leads to a contradiction to the fact that D is relatively compact and q is continuous. □

3. BERNSTEIN-TYPE THEOREMS

Before proving our main theorem, we need an auxiliary proposition. Set

$$P(r) = \int_{B(r)} |\phi|^2 dv.$$

PROPOSITION 3.1. *Let M be a complete noncompact hypersurface of R^{n+1} ($n \leq 5$) with constant mean curvature $H (H \neq 0)$ and finite index. Assume that $P(r) \leq Ce^{\alpha Hr}$ for some positive constants C, and α, where α is a constant that can expressed explicitly in terms of n. Then $\int_M |\phi|^2 < +\infty$.*

Our Theorem 1 is a corollary of the above proposition. It is a combination of the proposition and theorems in (Alencar & do Carmo 1994a) and (do Carmo & Peng 1980). Before proving Proposition 3.1 we give the proof of Theorem 1.

PROOF OF THEOREM 1. To prove the conclusion of Theorem 1 we only need to show that $H = 0$ by Theorem A. Otherwise $H \neq 0$, and by Proposition 3.1 we know that $\int_M |\phi|^2 < +\infty$. This is impossible by Theorem B. Thus the proof is complete. □

We now prove the proposition:

PROOF OF PROPOSITION 3.1. Introduce $f|\phi|^{q+1}$ in the stability inequality (1.1). It has been shown in (Alencar & do Carmo 1994a) that for all $\epsilon > 0$,

$$\int_M f^2|\phi|^{2+2q}[A|\phi|^2 - B|\phi| + C] \leq D\int_M |\phi|^{2q+2}|\nabla f|^2, \tag{3.1}$$

where

$$A = 1 - (1 + q + \epsilon)(\frac{2}{n} + q)^{-1}q,$$
$$B = (1 + q + \epsilon)(\frac{2}{n} + q)^{-1}(1 + q)\frac{n(n-2)}{\sqrt{n(n-1)}}H,$$
$$C = \left(1 + (1 + q + \epsilon)(\frac{2}{n} + q)^{-1}(2 + q)\right)nH^2,$$
$$D = (1 + q + \epsilon)(\frac{2}{n} + q)^{-1} + 1 + \frac{1 + q}{\epsilon}.$$

If M has finite index then it is stable outside some ball $B(R)$. In (3.1), we choose $q = 0$; then $A = 1$ and

$$B = (1+\epsilon)\frac{n}{2}\frac{n(n-2)}{\sqrt{n(n-1)}}H,$$
$$C = [1+n(1+\epsilon)]nH^2,$$
$$D = \frac{n(1+\epsilon)}{2} + 1 + \frac{1}{\epsilon}.$$

So in this case we have

$$\begin{aligned} A|\phi|^2 - B|\phi| + C =&(|\phi| - \frac{B}{2})^2 + \frac{4C - B^2}{4} \\ \geq&\frac{4C - B^2}{4} \\ =&-[n^2(n-2)^2\epsilon^2 + 2n(n^3 - 4n^2 - 4n + 8)\epsilon \\ &+ n^4 - 4n^3 - 12n^2 + 16]H^2. \end{aligned}$$

It can be checked that when $n \leq 5$, we can find sufficiently small $\epsilon > 0$ such that $4C - B^2 > 0$. So there exists a constant β which can expressed in terms of n such that

$$H^2 \int_{M\backslash B(R)} f^2|\phi|^2 \leq \beta^{-2} \int_{M\backslash B(R)} |\phi|^2|\nabla f|^2, \tag{3.2}$$

for any piecewise smooth function f with compact support in $M\backslash B(R)$. Then

$$\beta^2 H^2 \leq \frac{\int_{M\backslash B(R)} |\phi|^2|\nabla f|^2}{\int_{M\backslash B(R)} f^2|\phi|^2}\ . \tag{3.3}$$

We claim that we can choose R large enough such that $P'(r) > 0$ for all $r > R$. Otherwise we can find two positive constants $r_1 < r_2$ such that $|\phi|(x) = 0$ when $x \in \partial B(r_i)$. Thus $B(r_2)\backslash\overline{B(r_1)}$ contains a nodal domain and this contradicts Lemma 2.1.

Assume for the sake of the contradiction that $P(+\infty) = +\infty$. Then from our oscillation theorem in (do Carmo & Zhou 1999 Theorem 2.1) we have that for any $\lambda > \frac{\alpha^2 H^2}{4}$ we can find $x(t)$ which is not identically zero and is an oscillatory solution of

$$[P'(t)x'(t)]' + \lambda P'(t)x(t) = 0.$$

Choose $f(x) = x(r(x))$ where $r(x)$ is the distance function to some fixed point in M. We can find T_1 and T_2, such that $T_2 > T_1 > R$ and $x(T_1) = x(T_2) = 0$, $x(t) > 0$ for all $t \in (T_1, T_2)$. Now choose $\lambda = (\frac{\alpha^2}{4} + \delta)H^2$, where $\delta > 0$ is a constant such that $\beta^2 - \delta > 0$ and set $\alpha < 2\sqrt{\beta^2 - \delta}$. It follows that

$$\beta^2 H^2 \leq \frac{-\int_{T_1}^{T_2}[P'(r)x'(r)]'x(r)dr}{\int_{T_1}^{T_2} P'(r)x^2(r)dr} = \lambda < \beta^2 H^2.$$

This is a contradiction which shows our conclusion.

We now give the proof of Theorem 2:

PROOF OF THEOREM 2. We can assume that $H \neq 0$; otherwise from (do Carmo & Peng 1980) the theorem holds. Notice that in (3.1)

$$
\begin{aligned}
B^2 - 4AC =& \frac{nH^2}{(n-1)(2+nq)^2}\{n^4q^4 + 2n^4(\epsilon+2)q^3 \\
&+ n^2(n^2\epsilon^2 + 6n^2\epsilon + 6n^2 - 16n + 16)q^2 \\
&+ 2n[n^3\epsilon^2 + (3n^2 - 8n + 8)n\epsilon + 2(n^3 - 4n^2 - 4n + 8)]q \\
&+ [n^2(n-2)^2\epsilon^2 + 2n(n^3 - 4n^2 - 4n + 8)\epsilon + n^4 - 4n^3 - 12n^2 + 16]\}.
\end{aligned} \tag{3.4}
$$

Consider the terms without ϵ in the large bracket:

$$
\begin{aligned}
g(n,q) :=& n^4q^4 + 4n^4q^3 + n^2(6n^2 - 16n + 16)q^2 \\
&+ 4n(n^3 - 4n^2 - 4n + 8)q + n^4 - 4n^3 - 12n^2 + 16.
\end{aligned} \tag{3.5}
$$

Then, by choosing $q = -\frac{1}{n}$,

$$
\begin{aligned}
g(n, -\frac{1}{n}) &= 1 - 4n + 6n^2 - 16n + 16 - 4n^3 + 16n^2 \\
&\quad + 16n - 32 + n^4 - 4n^3 - 12n^2 + 16 \\
&= n^4 - 8n^3 + 10n^2 - 4n + 1 \\
&= (n-1)(n^3 - 7n^2 + 3n - 1).
\end{aligned} \tag{3.6}
$$

It is easy to see that $g(n, -\frac{1}{n}) < 0$ when $n \leq 6$. Thus we can always choose ϵ sufficient small such that $B^2 - 4AC < 0$. Notice that our choice of $q = -\frac{1}{n}$ makes $A > 0$. By using Young's inequality in (3.1)

$$
\int_M f^2|\phi|^{2+2q}[A|\phi|^2 - B|\phi| + C] \leq \delta \int_M |\phi|^{2q+4} f^2 + \beta_1 \int_M \frac{|\phi|^2|\nabla f|^{2(1+q)}}{f^{2q}}, \tag{3.7}
$$

where $\beta_1 > 0$ is a constant (depending on $n, \epsilon,$ and q) and $\delta > 0$ can be chosen arbitrarily small. Now set $\bar{A} = A - \delta$ and choose δ small enough so that $B^2 - 4\bar{A}C > 0$ and $\bar{A} > 0$. It follows from (3.7) that

$$
\int_M f^2|\phi|^{2+2q} \leq \beta_2 \int_M \frac{|\phi|^2|\nabla f|^{2(1+q)}}{f^{2q}}.
$$

Writing $f = h^{1+q}$, we have

$$
\int_M f^{2+2q}|\phi|^{2+2q} \leq \beta_3 \int_M |\phi|^2|\nabla f|^{2+2q}.
$$

where β_3 is a constant depending only on n, ϵ. The rest of the proof follows exactly as in (do Carmo & Peng 1980), and we find that $H = 0$, a contradiction.

4. SOME FURTHER RESULTS

In this section we want to give some further related results. Using the eigenvalue estimate in (do Carmo & Zhou 1999) we can get an index estimate for hypersurfaces with nonzero constant mean curvature.

Define $\alpha(M) := \limsup_{r\to+\infty} \frac{\log V(r)}{r}$ where $V(r)$ is the volume of geodesic ball $B(r)$. It is easy to see that $\alpha(M) = 0$ if M has polynomial volume growth.

THEOREM 4.1. *If M is complete noncompact hypersurface in R^{n+1} with nonzero constant mean curvature H and $\alpha(M) < 2\sqrt{n}H$, then* $\operatorname{ind}(M) = +\infty$.

In order to prove this Theorem we need to use the eigenvalue estimate theorem proved by the authors in (do Carmo & Zhou 1999) which is now restated as follows.

THEOREM. *Let M be a complete noncompact Riemannian manifold with infinite volume and Ω be an arbitrary compact subset of M. Then*

$$\lambda_1(M\backslash\Omega) \le \frac{\alpha^2}{4}.$$

PROOF OF THEOREM 4.1. It suffices to prove that for any natural number N we can find piecewise smooth functions $f_1, f_2, \cdots, f_N$ with compact supports such that $\operatorname{supp}(f_i)$ are disjoint and $I(f_i) < 0$.

Note that from (Frensel, 1996) the volume of M is infinite, so from the Theorem we have:

$$\lambda_1(M\backslash\Omega) \le \frac{\alpha^2}{4} < nH^2, \tag{4.1}$$

for any compact set Ω in M. So we can find a compact domain D_1 such that $\lambda_1(D_1) \le \frac{\alpha^2}{4} < nH^2$. We also have $\lambda_1(M\backslash D_1) \le \frac{\alpha^2}{4} < nH^2$. So we can find again a compact domain $D_2 \subset M\backslash D_1$ such that $\lambda_1(D_2) \le \frac{\alpha^2}{4} < nH^2$. and $\lambda_1(M\backslash(D_1 \cup D_2)) \le \frac{\alpha^2}{4} < nH^2$. Repeating this procedure, we can find disjoint compact domains $D_1, D_2, \cdots, D_N$, such that $\lambda_1(D_i) < nH^2$.

Let φ_i be the positive first eigenfunction of Δ_M on D_i, i.e.: $\Delta\varphi_i = \lambda_i(D_i)\varphi_i$ in D_i and $\varphi_i = 0$ on ∂D_i. We now define $f_i(x) := \varphi_i(x)$ for $x \in D_i$ and $f_i(x) \equiv 0$ for $x \in M\backslash D_i$. So

$$\int_M |\nabla f_i|^2 = \lambda_1(D_i)\int_M f_i^2 < nH^2 \int_M f_i^2 \le \int_M |A|^2 f_i^2. \tag{4.2}$$

Thus $I(f_i, f_i) < 0$ for $i = 1, 2, \cdots, N$. This shows that $\operatorname{ind}(M) \ge N$, for any N. So $\operatorname{ind}(M) = +\infty$.

The following is an easy consequence of Theorem 4.1.

COROLLARY 4.2. *If M is complete noncompact hypersurface with nonzero constant mean curvature H and polynomial volume growth, then* $\operatorname{ind}(M) = +\infty$. *In particular,* $\operatorname{ind}(M) = +\infty$, *when $M = S^k \times R^{n-k}$ with the standard metric; here S^k is a k-dimensional sphere in R^{k+1}.*

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REFERENCES

Alencar H & do Carmo MP. 1994a. Hypersurfaces with constant mean curvature in space forms. *An Acad Bras Ci* **66:** 265-274.

Alencar H & do Carmo MP. 1994b. Hypersurfaces with constant mean curvature in spheres. *Proc Amer Math Soc* **120:** 1223-1229.

do Carmo MP & Peng CK. 1979. Stable complete minimal surfaces in R^3 are planes. *Bull Amer Math Soc N.S.* **1:** 903-906.

do Carmo MP & Peng CK. 1980. *Stable complete minimal hypersurfaces.* Proc of the 1980 Beijing Symp. Chern SS & Wentsun W (Eds.), Gordan and Breach Science Pub., 1349-1358.

do Carmo MP & Zhou D. 1999. Eigenvalue Estimate on Complete Noncompact Riemannian Manifolds and Application. *Trans Amer Math Soc* **351:** 1391-1401.

Fisher-Colbrie D. 1985. On complete minimal surfaces with finite Morse index in three-manifolds. *Invent Math* **82:** 121-132.

Fisher-Colbrie D & Schoen R. 1980. The structure of complete stable minimal surfaces in 3-manifolds of nonnegative curvature. *Comm Pure Appl Math* **33:** 199-211.

Frensel KR. 1996. Stable complete surfaces with constant mean curvature. *Bol Soc Bras Mat* **27:** 129-144.