On Complete Spacelike Hypersurfaces with Constant Scalar Curvature in the De Sitter Space

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ABSTRACT

Let M^n be a complete spacelike hypersurface with constant normalized scalar curvature R in the de Sitter Space S_1^{n+1}. Let H the mean curvature and suppose that $\overline{R} = (R-1) > 0$ and $\overline{R} \leq \sup H^2 \leq C_{\overline{R}}$, where $C_{\overline{R}}$ is a constant depending only on R and n. It is proved that either $\sup H^2 = \overline{R}$ and M^n is totally umbilical, or $\sup H^2 = C_{\overline{R}}$ and M^n is the hyperbolic cylinder $H^1(1-\coth^2 r) \times S^{n-1}(1-\tanh^2 r)$.

Key words: hyperbolic cylinder, spacelike hypersurfaces, de Sitter space.

1. INTRODUCTION

The study of spacelike hypersurfaces with constant scalar curvature in the de Sitter space $S_1^{n+1}$is related to an analogue of Goddard's conjecture for the second elementary symmetric polynomial in the principal curvatures; more precisely: "Let M^n be a complete spacelike hypersurface with constant scalar curvature immersed in de Sitter space S_1^{n+1}. Then M^n is totally umbilical ".

S. Montiel (Montiel 1996) described the hyperbolic cylinders $H^1(1-\coth^2 r) \times S^{n-1}(1-\tanh^2 r)$, $n \geq 3$, in S_1^{n+1} with constant mean curvature $H^2 = \frac{4(n-1)}{n^2}$ and normalized constant scalar curvature $R = 1 + \frac{1}{n}(2 + (n-2)\tanh^2 r))$.

In (Zheng 1995), (Zheng 1996) and (Cheng & Ishikawa 1998) partial results were obtained. Recently, Haizhong Li (Li 1997)(and also S. Montiel in a more general spacetime (Montiel 1999)) obtained the following result: "Let M^n be a compact spacelike hypersurface immersed into the de Sitter space S_1^{n+1} with normalized constant scalar curvature R satisfying $R \geq 0$. Then M is totally umbilical".

In this note we announce the following

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THEOREM 1.1. *Let M^n be an n-dimensional complete ($n \geq 3$) spacelike hypersurface immersed into the de Sitter space S_1^{n+1} with constant scalar curvature R such that $\overline{R} = R - 1 > 0$ and supose that $\overline{R} \leq \sup H^2 \leq C_{\overline{R}}$ where*

$$C_{\overline{R}} = \frac{1}{n}\left((n-1)^2\frac{n\bar{R}-2}{n-2} + 2(n-1) + \frac{n-2}{n\bar{R}-2}\right).$$

Then

(i) $\sup H^2 = \overline{R}$ *and M is totally umbilic or*

(ii) $\sup H^2 = C_{\overline{R}}$ *and M is isometric to* $H^1(1-\coth^2 r) \times S^{n-1}(1-\tanh^2 r)$.

2. PRELIMINARIES

Let R_1^{n+2} be the real vector space R^{n+2} endowed with the Lorentzian metric $\langle , \rangle$ given by $\langle v, w\rangle = -v_0 w_0 + v_1 w_1 + \ldots + v_{n+1} w_{n+1}$ that is, $R_1^{n+2} = L^{n+2}$ is the Lorentz-Minkowski $(n+2)$-dimensional space. We define the de Sitter space as the following hyperquadric of R_1^{n+2} : $S_1^{n+1} = \{x \epsilon R_1^{n+2}; |x|^2 = 1\}$. The induced metric $\langle , \rangle$ makes S_1^{n+1} into a Lorentz manifold with constant sectional curvature 1. Let M^n be a n-dimensional orientable manifold, complete and let $f : M^n \longrightarrow S_1^{n+1} \subset L^{n+2}$ be a spacelike immersion of M^n into the de Sitter S_1^{n+1}. Choose a unit normal η along f and denote by $A : T_pM \to T_pM$ the linear map of the tangent space T_pM at the point $p \in M$, associated to the second fundamental form of f along η,

$$\langle AX, Y\rangle = -\langle \nabla_X Y, \eta\rangle,$$

where X and Y are tangent vector fields on M and ∇ is the connection on S_1^{n+1}. Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis which diagonalizes A with eigenvalues k_i of T_pM, i.e., $Ae_i = k_i e_i$, $i = 1, \ldots, n$. We will denote by $H = \frac{1}{n}\sum k_i$ the mean curvature of f and $|A|^2 = \sum k_i^2$. In our case it is convenient to define a linear map $\phi : T_pM \to T_pM$ by

$$\langle \phi X, Y\rangle = \langle AX, Y\rangle - H\langle X, Y\rangle.$$

It is easily checked that $trace(\phi) = 0$ and that

$$|\phi|^2 = \frac{1}{2n}\sum (k_i - k_j)^2,$$

so that $|\phi|^2 = 0$ if and only if M^n is totally umbilical. Let $\mu_i = k_i - H$ be the eigenvalues of ϕ; then $\sum_i \mu_i = 0$, and

$$\begin{aligned} |\phi|^2 &= \sum_i \mu_i^2 = |A|^2 - nH^2 \\ \sum_i k_i^3 &= nH^3 + 3H\sum_i \mu_i^2 - \sum_i \mu_i^3. \end{aligned} \tag{2.1}$$

The standard examples of spacelike umbilical hypersurfaces with constant mean curvature in the de Sitter space are given by

$$M^n = \{p \in S_1^{n+1} |\ \langle p, a\rangle = \tau\},$$

where $a \in R_1^{n+2}$, $|a|^2 = \rho = 1, 0, -1$ and $\tau^2 > \rho$. The corresponding mean curvature H of such surfaces satisfies

$$H^2 = \frac{\tau^2}{(\tau^2 - \rho)}$$

(Montiel 1988) and M^n is isometric to a hyperbolic space, an Euclidean space or a sphere according to ρ equal to $1, 0, -1$, respectively. On the other hand, hyperbolic cylinders are the hypersurfaces given by

$$M^n = \{p \in S_1^{n+1}; -p_o^2 + p_1^2 + \ldots + p_k^2 = -\sinh^2 r\},$$

with $r \in R$ and $1 \leq k \leq n$.

Such hyperbolic cylinders have constant mean curvature

$$nH = [k \coth r + (n-k)\tanh r].$$

Thus, we have

$$H^2 \geq \frac{4(n-1)}{n^2}$$

and the equality is attained for $k = 1$ and $\coth^2 r = (n-1)$. Their normalized scalar curvature is $R = 1 + \frac{1}{n}(2 + (n-2)\tanh^2 r)$.

We point out that these examples have only two different constant principal curvatures at each point and one of them has multiplicity one. Moreover, they are isometric to the Riemannian product

$$H^1(1 - \coth^2 r) \times S^{n-1}(1 - \tanh^2 r).$$

The Gauss equation relates the scalar curvature, the mean curvature and the square of the norm of the second fundamental formula as follows:

$$n(n-1)(R-1) = n^2H^2 - |A|^2. \tag{2.2}$$

Let $T = \sum_{i,j} T_{ij}\omega_i \otimes \omega_j$ be a symmetric tensor defined on M^n, where

$$T_{ij} = nH\delta_{ij} - h_{ij}.$$

Following (Cheng & Yau 1977), we introduce the operator L_1 associated to T acting on C^2 functions f on M^n by

$$L_1 f = \sum_{i,j} T_{ij} f_{ij} = \sum_{i,j} (nH\delta_{ij} - h_{ij}) f_{ij}. \tag{2.3}$$

Around a given point $p \in M$ we choose an orthonormal frame field $\{e_1, \ldots, e_n\}$ with dual frame field $\{w_1, \ldots, w_n\}$ so that $h_{ij} = k_i \delta_{ij}$ at p. We have the following computation by using (2.3) and Gauss equation (2.2) :

$$\begin{aligned} L_1(nH) &= nH\Delta(nH) - \sum_i k_i (nH)_{ii} \\ &= \frac{1}{2}\Delta(nH)^2 - \sum_i (nH)_i^2 - \sum_i k_i(nH) \\ &= \frac{1}{2}n(n-1)\Delta R + \frac{1}{2}\Delta|A|^2 - n^2\,|\nabla H|^2 - \sum_i k_i (nH)_{ii}. \end{aligned} \tag{2.4}$$

On the other hand, using Simons Formula (see Zheng 1995) we get

$$\frac{1}{2}|A|^2 = |\nabla A|^2 + n\sum_i k_i H_{ii} + \frac{1}{2}\sum_{i,j} R_{ijij}(k_i - k_j)^2. \tag{2.5}$$

From (2.4) and (2.5), we have:

$$L_1(nH) = \frac{1}{2}n(n-1)\Delta R + |\nabla A|^2 - n^2\,|\nabla H|^2 + \frac{1}{2}\sum_{i,j} R_{ijij}(k_i - k_j)^2. \tag{2.6}$$

3. SKETCH OF THE PROOF OF THE THEOREM

Since R is constant, by (2.6) we obtain

$$\begin{aligned} L_1(nH) &= |\nabla A|^2 - n^2\,|\nabla H|^2 + \frac{1}{2}\sum_{i,j}(1 + k_i k_j)(k_i - k_j)^2 \\ &= |\nabla A|^2 - n^2\,|\nabla H|^2 + \frac{1}{2}n\sum_i k_i^2 + \frac{1}{2}n\sum_j k_j^2 \\ &- \sum_{i,j} k_j k_i + \frac{1}{2}\sum_{i,j} k_i^3 k_j + \frac{1}{2}\sum_{i,j} k_i k_j^3 - \sum_{i,j} k_i^2 k_j^2. \end{aligned}$$

Making $i = j$, we then have

$$L_1(nH) = |\nabla A|^2 - n^2\,|\nabla H|^2 + n|A|^2 - n^2H^2 + |A|^4 - nH\sum_i k_i^3. \tag{3.1}$$

Using (2.1) in (3.1) we obtain

$$L_1(nH) = |\nabla A|^2 - n^2\,|\nabla H|^2 + n\,|\phi|^2 + |\phi|^4 - nH\sum_i \mu_i^3 - 3nH^2|\phi|^2 - n^2H^4. \tag{3.2}$$

Then,

$$\begin{aligned} L_1(nH) = {} & |\nabla A|^2 - n^2\,|\nabla H|^2 + n\,|\phi|^2 + |\phi|^4 + n^2H^4 + 2nH^2\,|\phi|^2 \\ & - 3nH^2\,|\phi|^2 - n^2H^4 - nH\sum_i \mu_i^3. \end{aligned} \tag{3.3}$$

This yields

$$L_1(nH) = |\nabla A|^2 - n^2\,|\nabla H|^2 + |\phi|^2\,(n - nH^2 + |\phi|^2) - nH\sum_i \mu_i^3. \tag{3.4}$$

We need to estimate $tr(\phi^3)$ in (3.4). First we recall an algebraic lemma (Okumura 1974) which asserts

$$-\frac{(n-2)}{\sqrt{n(n-1)}}(|\phi|^2)^{3/2} \leq \sum_i \mu_i^3 = tr(\phi^3) \leq \frac{(n-2)}{\sqrt{n(n-1)}}(|\phi|^2)^{3/2} \tag{3.5}$$

and the equality holds on the right hand side if and only if

$$\mu_1 = ... = \mu_{n-1} = -\sqrt{\frac{1}{n(n-1)}}|\phi| \quad \text{and} \quad \mu_n = \sqrt{\frac{n-1}{n}}|\phi|.$$

Using (3.5) in (3.4), we obtain

$$L_1(nH) \geq |\nabla A|^2 - n^2\,|\nabla H|^2 + |\phi|^2\,(n - nH^2 + |\phi|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}|H||\phi|). \tag{3.6}$$

Since R is constant, by (Alencar et al. 1993)

$$|A|^2 - n^2|\nabla H|^2 \geq 0. \tag{3.7}$$

By Gauss equation (2.1) we know that

$$|\phi|^2 = |A|^2 - nH^2 = \frac{n-1}{n}(|A|^2 - n\overline{R}). \tag{3.8}$$

Using (3.8) and (3.7) in (3.6) we obtain

$$L_1(nH) \geq \frac{n-1}{n}(|A|^2 - n\overline{R})P_H(|\phi|), \tag{3.9}$$

where P_H is a polynomial given by

$$P_H(|\phi|) = (n - nH^2 + |\phi|^2 - \frac{(n-2)}{\sqrt{n(n-1)}}|H||\phi|). \tag{3.10}$$

By (3.8) we may write the above polynomial as

$$\begin{aligned} P_{\overline{R}}(|A|) &= n - 2(n-1)\overline{R} + \frac{n-2}{n}|A|^2 \\ &\quad - \frac{(n-2)}{n}\sqrt{(n(n-1)\overline{R} + |A|^2)(|A|^2 - n\overline{R})}. \end{aligned} \tag{3.11}$$

Therefore (3.9) becomes

$$L_1(nH) \geq \frac{n-1}{n}(|A|^2 - n\overline{R})P_{\overline{R}}(|A|). \tag{3.12}$$

Using the hypothesis that $\overline{R} \leq \sup H^2 \leq C_{\overline{R}}$, one proves that

$$P_{\overline{R}}(\sqrt{\sup|A|^2}) \geq 0. \tag{3.13}$$

On the other hand,

$$\begin{aligned} L_1(nH) &= \sum_{i,j}(nH\delta_{ij} - nh_{ij})(nH)_{ij} = \sum_i (nH - nh_{ii})(nH)_{ii} \\ &= n\sum_i H(nH)_{ii} - n\sum_i k_i(nH)_{ii} \leq n(|H|_{\max} - C)\Delta(nH), \end{aligned} \tag{3.14}$$

where $|H|_{\max}$ is the maximum of the mean curvature H in M and $C = \min k_i$ is the minimum of the principal curvatures in M.

Now, we need the maximum principle at infinity for complete manifolds by Omori and Yau(Omori 1967):
"Let M^n be an n-dimensional complete Riemannian manifold whose Ricci curvature is bounded from below. Let f be a C^2-function bounded from below on M^n. Then for each $\varepsilon > 0$ there exists a point $p_\varepsilon \in M$ such that

$$|\nabla f|(p_\varepsilon) < \varepsilon,\ \Delta f(p_\varepsilon) > -\varepsilon \quad \text{and} \quad \inf f \leq f(p_\varepsilon) < \inf f + \varepsilon." \tag{3.15}$$

The hypothesis $\overline{R} \leq \sup H^2 \leq C_{\overline{R}}$ together with Gauss equation implies that $Ric_M \geq (n-1) - \frac{nH^2}{4}$, so the Ricci curvature is bounded below. Thus we may apply Omori and Yau's result to the function

$$f = \frac{1}{\sqrt{1+(nH)^2}}.$$

We have

$$|\nabla f|^2 = \frac{1}{4}\frac{|\nabla(nH)^2|^2}{(1+(nH))^2} \tag{3.16}$$

and

$$\Delta f = -\frac{1}{2}\frac{\Delta(nH)}{(1+(nH))^{\frac{3}{2}}} + \frac{3|\nabla(nH)|^2}{4(1+nH)^{\frac{5}{2}}}. \tag{3.17}$$

Let $\{p_k\}$, $k \in N$, be a sequence of points in M given by (3.15) such that

$$\lim_{k\to\infty} f(p_k) = \inf f,\ \Delta f(p_k) > -\frac{1}{k} \text{ and } |\nabla f|^2(p_k) < \frac{1}{k^2}. \tag{3.18}$$

Using (3.18) in the two equations (3.16) and (3.17) and the fact that

$$\lim_{k\to\infty}(nH)(p_k) = \sup_{p\in M}(nH)(p),$$

we obtain

$$-\frac{1}{k} < -\frac{1}{2}\frac{\Delta(nH)}{(1+(nH))^{\frac{3}{2}}}(p_k) + \frac{3}{k^2}(1+nH(p_k))^{\frac{3}{2}}. \tag{3.19}$$

Hence

$$\frac{\Delta(nH)}{(1+nH)^{\frac{3}{2}}}(p_k) < \frac{2}{k}(\frac{1}{\sqrt{1+(nH)(p_k)}} + \frac{3}{k}). \tag{3.20}$$

On the other hand, by (3.12) and (3.14), we have

$$\frac{n-1}{n}(|A|^2 - n\overline{R})P_{\overline{R}}(|A|) \leq L_1(nH) \leq n(|H|_{\max} - C)\Delta(nH). \tag{3.21}$$

At points p_k of the sequence given in (3.18), this becomes

$$\begin{aligned}\frac{n-1}{n}(|A|^2(p_k) - n\overline{R})P_{\overline{R}}(|A|(p_k)) &\leq L_1(nH(p_k))\\ &\leq n(|H|_{\max} - C)\Delta(nH)(p_k).\end{aligned} \tag{3.22}$$

Making $k \to \infty$ and using (3.20) we have that the right hand side of (3.22) goes to zero, so by (3.15) either $\frac{n-1}{n}(\sup|A|^2 - n\overline{R}) = 0$ or $P_{\overline{R}}(\sqrt{\sup|A|^2}) = 0$. But by (3.8) $|\phi|^2 = \frac{n-1}{n}(|A|^2 - n\overline{R})$ and so $\sup|\phi|^2 = \frac{n-1}{n}(\sup|A|^2 - n\overline{R}) = 0$, then $|\phi|^2 = 0$, proving that M^n is totally umbilical.

If $P_{\overline{R}}(\sqrt{\sup|A|^2}) = 0$, it can be proved that $\sup H^2 = C_{\overline{R}}$ and so the equality holds on the right hand side of (3.9) and we obtain

$$L_1(n\sup H) = \frac{n-1}{n}(\sup|A|^2 - n\overline{R})P_{\overline{R}}(\sqrt{\sup|A|^2}).$$

One proves that the equality also holds in Okumura's lemma (3.5). After reenumeration, we finally have

$$k_1 = k_2 = \ldots = k_{n-1},\ k_1 \neq k_n, \text{ where } k_1 = \tanh r \text{ and } k_n = \coth r.$$

Therefore, M^n is isometric to $H^1(1-\coth^2 r) \times S^{n-1}(1-\tanh^2 r)$, finishing the proof.

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