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Maximum principles for hypersurfaces with vanishing curvature functions in an arbitrary Riemannian manifold

FRANCISCO X. FONTENELE[1] **and SÉRGIO L. SILVA**[2]

[1]Universidade Federal Fluminense, Instituto de Matemática, Departamento de Geometria,
rua Mario Santos Braga s/no., 24020-140 Niterói, RJ, Brazil
[2]Universidade Estadual do Rio de Janeiro-UERJ, Departamento de Estruturas Matemáticas-IME
20550-013 RJ, Brazil

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ABSTRACT

In this paper we generalize and extend to any Riemannian manifold maximum principles for Euclidean hypersurfaces with vanishing curvature functions obtained by Hounie-Leite.

Key words: maximum principle, hypersurface, rth mean curvature.

1 INTRODUCTION

In this paper we generalize and extend to any Riemannian manifold maximum principles for hypersurfaces of the Euclidean space with vanishing curvature function, obtained by Hounie-Leite (1995 and 1999). In order to state our results, we need to introduce some notations and consider some facts. Given an hypersurface M^n of a Riemannian manifold N^{n+1}, denote by $k_1(p), \ldots, k_n(p)$ the principal curvatures of M^n at p with respect to a unitary vector that is normal to M^n at p. We always assume that $k_1(p) \leq k_2(p) \leq \cdots \leq k_n(p)$. The *rth mean curvature* $H_r(p)$ of M^n at p is defined by

$$H_r(p) = \frac{1}{\binom{n}{r}} \sigma_r(k_1(p), \ldots, k_n(p)), \tag{1}$$

where $\sigma_r : \mathbb{R}^n \to \mathbb{R}$ is the rth elementary symmetric function. It is easy to see that σ_r is positive on the positive cone $\mathcal{O}^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_i > 0, \forall i\}$. Denote by Γ_r the connected component

Correspondence to: Francisco X. Fontenele
E-mail: fontenele@mat.uff.br

of $\{\sigma_r > 0\}$ that contains the vector $(1, \dots, 1) \in \mathbb{R}^n$. It was proved in Gårding (1959) that Γ_r is an open convex cone and that

$$\Gamma_1 \supset \Gamma_2 \supset \cdots \supset \Gamma_n. \tag{2}$$

Moreover on $\Gamma_r,\ 1 \le r \le n$, it holds that (see Caffarelli et al. 1985, Proposition 1.1)

$$\frac{\partial \sigma_r}{\partial x_i} > 0, \quad 1 \le i \le n. \tag{3}$$

As it was observed in Hounie-Leite (1995), the subset $\{\sigma_r = 0\}$ can be decomposed as the union of r continuous leaves $Z_1, \dots, Z_r$, being Z_1 the boundary $\partial\Gamma_r$ of the cone Γ_r. Furthermore each leaf Z_j may be identified with the graph of a continuous function defined in the plane $x_1 + \cdots + x_n = 0$. Following Hounie-Leite(1995), we say that a point $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ has rank r if exactly r components of x do not vanish.

As in Fontenele-Silva (2001), given $p \in M^n$ and a unitary vector η_o that is normal to M^n at p, we can parameterize a neighborhood of M^n containing p and contained in a normal ball of N^{n+1} as

$$\varphi(x) = \exp_p(x + \mu(x)\eta_o), \tag{4}$$

where the vector x varies in a neighborhood W of zero in T_pM and $\mu : W \to \mathbb{R}$ satisfies $\mu(0) = 0$ and $\nabla\mu(0) = 0$, being ∇ the gradient operator in the Euclidean space T_pM. Choosing a local orientation $\eta : W \to T^{\perp}_{\varphi(W)}M$ of M^n with $\eta(0) = \eta_o$, we denote by $H_r(x)$ the rth mean curvature of M^n at $\varphi(x)$ with respect to $\eta(x)$.

Given hypersurfaces M and M' of N^{n+1} that are tangent at p and a unitary vector η_o that is normal to M at p, we parameterize M and M' as in (4), obtaining correspondent functions $\mu : W \to \mathbb{R}$ and $\mu' : W \to \mathbb{R}$, defined in a neighborhood W of zero in $T_pM = T_pM'$. As in Fontenele-Silva (2001), we say that M remains above M' in a neighborhood of p with respect to η_o if $\mu(x) \ge \mu'(x)$ for all x in a neighborhood of zero. We say that M remains on one side of M' in a neighborhood of p if either M is above M' or M' is above M in a neighborhood of p. Finally, denote by $\overrightarrow{k}(p) = (k_1(p), \dots, k_n(p))$ and by $\overrightarrow{k'}(p) = (k'_1(p), \dots, k'_n(p))$ the principal curvature vectors at p of respectively M and M'.

We can now state our results:

THEOREM 1.a. *Let M and M' be hypersurfaces of N^{n+1} that are tangent at p, with normal vectors pointing in the same direction. Suppose that M remains on one side of M' and that $H_r(x) = H'_r(x)$ in a neighborhood of zero in T_pM, for some $r,\ 1 \le r < n$. If $r \ge 2$, suppose further that $\overrightarrow{k}(p)$ and $\overrightarrow{k'}(p)$ belong to same leaf of $\{\sigma_r = 0\}$ and the rank of either $\overrightarrow{k}(p)$ or $\overrightarrow{k'}(p)$ is at least r. Then, M and M' must coincide in a neighborhood of p.*

THEOREM 1.b. *Let M and M' be hypersurfaces of N^{n+1} with boundaries ∂M and $\partial M'$, respectively, and assume that M and M', as well as ∂M and $\partial M'$, are tangent at $p \in \partial M \cap \partial M'$, with normal* *vectors pointing in the same direction. Suppose that M remains on one side of M' and that $H_r(x) = H_r'(x)$ in a neighborhood of zero in T_pM, for some $r,\ 1 \leq r < n$. If $r \geq 2$, suppose further that $\overrightarrow{k}(p)$ and $\overrightarrow{k'}(p)$ belong to same leaf of $\{\sigma_r = 0\}$ and the rank of either $\overrightarrow{k}(p)$ or $\overrightarrow{k'}(p)$ is at least r. Then, M and M' must coincide in a neighborhood of p.*

As a consequence of Theorems 1.a and 1.b, we obtain the following corollaries, that extend Theorem 0.1 in Hounie-Leite (1995) to any Riemannian manifold.

COROLLARY 1.a. *Let M and M' be hypersurfaces of N^{n+1} that are tangent at p, with normal vectors pointing in the same direction and with both having r-mean curvature equal to zero for some $r,\ 1 \leq r < n$. For $r \geq 2$, suppose further that $\overrightarrow{k}(p)$ and $\overrightarrow{k'}(p)$ belong to same leaf of $\{\sigma_r = 0\}$ and the rank of either $\overrightarrow{k}(p)$ or $\overrightarrow{k'}(p)$ is at least r. Under these conditions, if M remains on one side of M', then M and M' must coincide in a neighborhood of p.*

COROLLARY 1.b. *Let M and M' be hypersurfaces of N^{n+1} with boundaries ∂M and $\partial M'$, respectively, so that M and M', as well as ∂M and $\partial M'$, are tangent at $p \in \partial M \cap \partial M'$, with normal vectors pointing in the same direction. Assume that M and M' have r-mean curvature equal to zero for some $r,\ 1 \leq r < n$. For $r \geq 2$, suppose further that $\overrightarrow{k}(p)$ and $\overrightarrow{k'}(p)$ belong to same leaf of $\{\sigma_r = 0\}$ and the rank of either $\overrightarrow{k}(p)$ or $\overrightarrow{k'}(p)$ is at least r. Under these conditions, if M remains on one side of M', then M and M' must coincide in a neighborhood of p.*

The extension of Theorem 1.3 in Hounie-Leite (1999) is given in the following theorems.

THEOREM 2.a. *Let M and M' be hypersurfaces of N^{n+1} that are tangent at p, with normal vectors pointing in the same direction. Suppose that M remains above M' and that $H_r' \geq 0 \geq H_r$, for some $r,\ 2 \leq r < n$. Suppose further that $H_j'(p) \geq 0,\ 1 \leq j \leq r-1$, and either $H_{r+1}(p) \neq 0$ or $H_{r+1}'(p) \neq 0$. Then, M and M' must coincide in a neighborhood of p.*

THEOREM 2.b. *Let M and M' be hypersurfaces of N^{n+1} with boundaries ∂M and $\partial M'$, respectively, and assume that M and M', as well as ∂M and $\partial M'$, are tangent at $p \in \partial M \cap \partial M'$ with normal vectors pointing in the same direction. Suppose that M remains above M' and that $H_r' \geq 0 \geq H_r$, for some $r,\ 2 \leq r < n$. Suppose further that $H_j'(p) \geq 0,\ 1 \leq j \leq r-1$, and either $H_{r+1}(p) \neq 0$ or $H_{r+1}'(p) \neq 0$. Then M and M' must coincide in a neighborhood of p.*

It will be clear from the proofs that in Theorems 2.a and 2.b we only need to require $H_r'(x) \geq H_r(x)$, in a neighborhood of zero in T_pM, and $H_r'(p) \geq 0 \geq H_r(p)$ instead of $H_r' \geq 0 \geq H_r$ everywhere. For $r = 1$, it must be observed that, in Theorems 2.a and 2.b, we can assume only that $H_r'(x) \geq H_r(x)$ and that M remains above M' in a neighborhood of zero in T_pM (see Theorems 1.1 and 1.2 in Fontenele-Silva (2001)).

2 PRELIMINARIES

In this section we will present the necessary material for our proofs.

Following Hounie-Leite (1995), we say that $x \in \mathbb{R}^n$ is an elliptic root of σ_r if $\sigma_r(x) = 0$ and either $\frac{\partial \sigma_r}{\partial x_j}(x) > 0,\ \ j = 1, \ldots, n$, or $\frac{\partial \sigma_r}{\partial x_j}(x) < 0,\ \ j = 1, \ldots, n$. It is easy to see that any root of

$\sigma_1 = 0$ is elliptic. For $2 \leq r < n$, we have the following criterion of ellipticity (see Corollary 2.3 in Hounie-Leite (1995) and Lemma 1.1 in Hounie-Leite (1999)):

LEMMA 1. *Let $x \in \mathbb{R}^n$ and assume that $\sigma_r(x) = 0$ for some $2 \leq r < n$. Then, the following conditions are equivalent*

(i) *x is elliptic.*

(ii) *the rank of x is at least r.*

(iii) $\sigma_{r+1}(x) \neq 0$.

For the proofs of our results, we will also need of the following lemmas:

LEMMA 2. *If y, w belong to a leaf Z_j of $\sigma_r = 0$, $w - y$ belongs to the closure $\overline{\mathcal{O}^n}$ of $\mathcal{O}^n$ and either y or w is an elliptic root, then $y = w$.*

LEMMA 3. *For $1 \leq r \leq n$, if $x \in \mathbb{R}^n$ satisfies $\sigma_j(x) \geq 0$, $1 \leq j \leq r$, then $x \in \overline{\Gamma_r}$.*

Lemma 2 is a particular case of Lemma 1.3 in Hounie-Leite (1995) and Lemma 3 follows from the proof of Lemma 1.2 in Hounie-Leite (1999).

For $d = (n(n+1)/2) + 2n + 1$, write an arbitrary point p at $\mathbb{R}^d$ as

$$p = (r_{11}, \ldots, r_{1n}, r_{22}, \ldots r_{2n}, \ldots, r_{(n-1)n}, r_{nn}, r_1, \ldots, r_n, z, x_1, \ldots, x_n)$$

or, in short, as $p = (r_{ij}, r_i, z, x)$ with $1 \leq i \leq j \leq n$, and $x = (x_1, \ldots, x_n)$. A C^1-function $\Phi : \Gamma \to \mathbb{R}$ defined in an open set Γ of $\mathbb{R}^d$ is said to be elliptic in $p \in \Gamma$ if

$$\sum_{i \leq j = 1}^{n} \frac{\partial \Phi}{\partial r_{ij}}(p)\xi_i \xi_j > 0 \quad \text{for all nonzero} \quad (\xi_1, \xi_2, \ldots, \xi_n) \in \mathbb{R}^n. \tag{5}$$

We say that Φ is elliptic in Γ if Φ is elliptic in p for all $p \in \Gamma$. Given a function $f : U \to \mathbb{R}$ of class C^2, defined in an open set $U \subset \mathbb{R}^n$, and $x \in U$, we associate a point $\Lambda(f)(x)$ in $\mathbb{R}^d$ setting

$$\Lambda(f)(x) = (f_{ij}(x), f_i(x), f(x), x), \tag{6}$$

where $f_{ij}(x)$ and $f_i(x)$ stand for $\dfrac{\partial^2 f}{\partial x_i \partial x_j}(x)$ and $\dfrac{\partial f}{\partial x_i}(x)$, respectively. Saying that the function Φ is elliptic with respect to f means that $\Lambda(f)(x)$ belongs to Γ and Φ is elliptic in $\Lambda(f)(x)$ for all $x \in U$. For elliptic functions it holds the following maximum principle(see Alexandrov 1962):

MAXIMUM PRINCIPLE. *Let $f, g : U \to \mathbb{R}$ be C^2-functions defined in an open set U of $\mathbb{R}^n$ and let $\Phi : \Gamma \subset \mathbb{R}^d \to \mathbb{R}$ be a function of class C^1. Suppose that Φ is elliptic with respect to the functions $(1-t)f + tg$, $t \in [0, 1]$. Assume also that*

$$\Phi(\Lambda(f)(x)) \geq \Phi(\Lambda(g)(x)) \quad , \forall x \in U, \tag{7}$$

and that $f \leq g$ on U. Then, $f < g$ on U unless f and g coincide in a neighborhood of any point $x_o \in U$ such that $f(x_o) = g(x_o)$.

Consider now a hypersurface $M^n \subset N^{n+1}$, a point $p \in M$ and a unitary vector η_o that is normal to M^n at p. Fix an orthonormal basis $e_1, \ldots, e_n$ in T_pM and introduce coordinates in T_pM by setting $x = \sum_{i=1}^n x_i e_i$ for all $x \in T_pM$. Parameterize a neighborhood of p in M as in (4), obtaining a function $\mu : W \subset T_pM \to \mathbb{R}$. Recall that $\mu(0) = 0$ and $\frac{\partial \mu}{\partial x_i}(0) = 0$, for all i, $1 \leq i \leq n$. Choose a local orientation $\eta : W \to T^{\perp}_{\varphi(W)}M$ of M^n with $\eta(0) = \eta_o$ and denote by $A_{\eta(x)}$ the second fundamental form of M^n in the direction $\eta(x)$. Denote by $\varphi_i(x)$ the vector $\frac{\partial \varphi}{\partial x_i}(x)$ and by $A(x) = (a_{ij}(x))$ the matrix of $A_{\eta(x)}$ in the basis $\varphi_i(x)$. In Fontenele-Silva (2001), it is proved the existence of an $n \times n$-matrix valued function $\tilde{A}$ defined in an open set $\mathbb{R}^{(n(n+1)/2)+n} \times \mathcal{N} \subset \mathbb{R}^d$, being $\mathcal{N}$ an open set of $\mathbb{R}^{n+1}$, containing the origin of $\mathbb{R}^d$ such that

$$\tilde{A}(\Lambda(\mu)(x)) = A(x) \ , \quad x \in W. \tag{8}$$

Moreover, we have $\tilde{A}(r_{ij}, r_i, z, x)$ diagonalizable for all $(r_{ij}, r_i, z, x) \in \mathbb{R}^{(n(n+1)/2)+n} \times \mathcal{N}$. Consider the function $\Phi_r : \mathbb{R}^{(n(n+1)/2)+n} \times \mathcal{N} \to \mathbb{R}$ defined by

$$\Phi_r = \frac{1}{\binom{n}{r}} \sigma_r \circ \lambda \circ \tilde{A}, \tag{9}$$

where $\lambda(\tilde{A}(w)) = (\lambda_1(\tilde{A}(w)), \ldots, \lambda_n(\tilde{A}(w))$ for all $w \in \mathbb{R}^{(n(n+1)/2)+n} \times \mathcal{N}$. Here $\lambda_1(\tilde{A}(w)) \leq \cdots \leq \lambda_n(\tilde{A}(w))$ are the eigenvalues of $\tilde{A}(w)$. It follows from (1), (8) and (9) that

$$H_r(x) = \Phi_r(\Lambda(\mu)(x)) \quad , \quad x \in W. \tag{10}$$

The proof of Proposition 3.4 in Fontenele-Silva (2001) gives

$$\sum_{k \leq \ell = 1}^{n} \frac{\partial \Phi_r}{\partial r_{k\ell}} \, (r_{ij}, 0, 0, 0) \, \xi_k \xi_\ell = \frac{1}{\binom{n}{r}} \sum_{k,\ell=1}^{n} \frac{\partial(\sigma_r \circ \lambda)}{\partial A_{k\ell}} \, (\tilde{A}((r_{ij}, 0, 0, 0))) \, \xi_k \xi_\ell, \tag{11}$$

for all $(r_{ij}, 0, 0, 0) \in \mathbb{R}^d$.

We also make use of the following lemma

LEMMA 4. *If $A_o \in \mathcal{M}^n(\mathbb{R})$ is symmetric and $\frac{\partial \sigma_r}{\partial \lambda_i}(\lambda(A_o)) > 0 \ \ (< 0)$ for all $1 \leq i \leq n$, then*

$$\sum_{i,j=1}^{n} \frac{\partial(\sigma_r \circ \lambda)}{\partial A_{ij}}(A_o) \, \xi_i \xi_j > 0 \ \ (< 0) \, , \quad \forall \ \xi = (\xi_1, \ldots, \xi_n) \neq 0. \tag{12}$$

The proof of Lemma 4 follows from the proof of Lemma 3.3 in Fontenele-Silva (2001).

3 PROOFS OF OUR RESULTS

We will prove only Theorems 1.a and 2.a, since the proofs of Theorems 1.b and 2.b are analogous.

PROOF OF THEOREM 1.a. If $r = 1$, the theorem follows from Theorem 1.1 in Fontenele-Silva (2001). Thus, we assume that $2 \leq r < n$. The assumption $H_r(x) = H'_r(x)$ in a neighborhood W of zero in T_pM and (10) imply that

$$\Phi_r(\Lambda(\mu)(x)) = \Phi_r(\Lambda(\mu')(x)) \ , \quad x \in W. \tag{13}$$

On the other hand, $\overrightarrow{k}(p)$ and $\overrightarrow{k'}(p)$ are both roots of $\sigma_r = 0$ and one of them is elliptic by our hypothesis and Lemma 1. The fact that M remains on one side of M' implies that either $\overrightarrow{k}(p) - \overrightarrow{k'}(p)$ or $\overrightarrow{k'}(p) - \overrightarrow{k}(p)$ belongs to $\overline{\mathcal{O}^n}$. Since $\overrightarrow{k}(p)$ and $\overrightarrow{k'}(p)$ belong to same leaf of $\{\sigma_r = 0\}$ by assumption, it follows from Lemma 2 that

$$\overrightarrow{k}(p) = \overrightarrow{k'}(p). \tag{14}$$

For each $t \in [0, 1]$, if we consider the hypersurface M_t parameterized by

$$\varphi(x) = \exp_p(x + ((1-t)\mu + t\mu')(x)\eta_o) \ , \quad x \in W, \tag{15}$$

we have that M_t is tangent to both M and M' in p and that M_t is between M and M' in a neighborhood of p. Using (14), we conclude that the principal curvature vector of M_t at p is equal to $\overrightarrow{k}(p) = \overrightarrow{k'}(p)$, for all $t \in [0, 1]$. This implies, by (8), that

$$\lambda \circ \tilde{A}((1-t)\Lambda(\mu)(0) + t\Lambda(\mu')(0)) = \overrightarrow{k}(p) = \overrightarrow{k'}(p) \ , \tag{16}$$

for all $t \in [0, 1]$. Since $\overrightarrow{k}(p) = \overrightarrow{k'}(p)$ is elliptic, it follows from (11) and Lemma 4 that either Φ_r or $-\Phi_r$ is elliptic along the line segment $(1-t)\Lambda(\mu)(0) + t\Lambda(\mu')(0) \subset \mathbb{R}^{(n(n+1)/2)+n} \times \mathcal{N} \subset \mathbb{R}^d$. Since ellipticity is an open condition, restricting W if necessary, we conclude by continuity and by the compactness of [0,1] that either Φ_r or $-\Phi_r$ is elliptic in $(1-t)\Lambda(\mu)(x) + t\Lambda(\mu')(x)$, for all $t \in [0, 1]$ and $x \in W$. Consequently either Φ_r or $-\Phi_r$ is elliptic with respect to the functions $(1-t)\mu + t\mu'$, $t \in [0, 1]$. So, by (13), we can apply the maximum principle to conclude that μ and μ' coincide in a neighborhood of zero. Therefore, M and M' coincide in a neighborhood of p. □

PROOF OF THEOREM 2.a. By our assumptions it holds that $H'_r(x) \geq H_r(x)$ for $x \in W$. This and (10) imply that

$$\Phi_r(\Lambda(\mu')(x)) - \Phi_r(\Lambda(\mu)(x)) \geq 0 \ , \quad x \in W. \tag{17}$$

Since M remains above M', we have $\overrightarrow{k}(p) - \overrightarrow{k'}(p) \in \overline{\mathcal{O}^n}$. It follows from our assumptions and Lemma 3 that $\overrightarrow{k'}(p) \in \overline{\Gamma_r}$. We claim that $\overrightarrow{k'}(p) \in \partial\Gamma_r$. Otherwise, by Lemma 4.1 in Fontenele-Silva (2001), we would have that $\overrightarrow{k}(p) \in \Gamma_r$, which is a contradiction since $H_r(p) \leq 0$. So $\overrightarrow{k'}(p) \in Z_1 = \partial\Gamma_r$. We can use Lemma 4.1 in Fontenele-Silva (2001) to conclude that $\overrightarrow{k}(p) \in Z_1 = \partial\Gamma_r$. As in the proof of Theorem 1.a, we can use Lemmas 1 and 2 to obtain that

$\overrightarrow{k}(p) = \overrightarrow{k'}(p)$. Since $\dfrac{\partial \sigma_r}{\partial x_i} > 0$ on Γ_r, $1 \leq i \leq n$, $\overrightarrow{k}(p) = \overrightarrow{k'}(p)$ is an elliptic root of $\sigma_r = 0$ and $\overrightarrow{k}(p) = \overrightarrow{k'}(p) \in \partial\Gamma_r$, we deduce that

$$\frac{\partial \sigma_r}{\partial x_i}(\overrightarrow{k}(p)) > 0 \quad , \quad \forall i = 1, \ldots, n. \tag{18}$$

Now, proceeding as in the proof of Theorem 1.a, we conclude that Φ_r is elliptic with respect to the functions $(1-t)\mu + t\mu'$, $t \in [0, 1]$. It follows from (17) and the maximum principle that μ and μ' coincide in a neighborhood of zero. Therefore, M and M' coincide in a neighborhood of p. $\square$

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RESUMO

Neste trabalho nós generalizamos e estendemos para uma variedade Riemanniana arbitrária princípios do máximo para hipersuperfícies com r-ésima curvatura média zero no espaço Euclidiano, obtidos por Hounie-Leite.

Palavras-chave: princípio do máximo, hipersuperfície, r-ésima curvatura média.

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