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Hypersurfaces with constant mean curvature and two principal curvatures in $\mathbb{S}^{n+1}$

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ABSTRACT

In this paper we consider compact oriented hypersurfaces M with constant mean curvature and two principal curvatures immersed in the Euclidean sphere. In the minimal case, Perdomo (Perdomo 2004) and Wang (Wang 2003) obtained an integral inequality involving the square of the norm of the second fundamental form of M, where equality holds only if M is the Clifford torus. In this paper, using the traceless second fundamental form of M, we extend the above integral formula to hypersurfaces with constant mean curvature and give a new characterization of the H(r)-torus.

Key words: Hypersurfaces, constant mean curvature, Simons formula, $H(r)$-torus.

1 INTRODUCTION

Let M be a compact minimal hypersurface of the $(n+1)$-dimensional unit Euclidean sphere $\mathbb{S}^{n+1}$. As usual, let S be the square of the length of the second fundamental form A of M. If $0 \leq S \leq n$, then Simons (Simons 1968) proved that either $S = 0$ or $S = n$. On the other hand, Do Carmo et al. 1970 and Lawson (Lawson 1969) proved, independently, that the Clifford tori are the only minimal hypersurfaces with $S = n$. The particular case $n = 3$ was studied by Peng and Terng. They proved in (Peng and Terng 1983) that if $S \geq 3$ is a constant function, then $S \geq 6$. In (Otsuki 1970) Otsuki proved that minimal hypersurfaces of $\mathbb{S}^{n+1}$ having distinct principal curvatures of multiplicities k and $m = n - k \geq 2$ are locally product of spheres of the type $\mathbb{S}^m(c_1) \times \mathbb{S}^{n-m}(c_2)$, and he constructed examples of compact minimal hypersurfaces in $\mathbb{S}^{n+1}$ with two distinct principal

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curvatures and one of them being simple. Recently, Hasanis and Vlachos (Hasanis and Vlachos 2000) proved that if M is a compact minimal hypersurface with two principal curvatures, one of them with multiplicity 1 and $S \geq n$, then $S = n$ and M is a Clifford torus. Using a traceless tensor $\Phi = A - HI$, the so called traceless second fundamental form, Alencar and do Carmo (Alencar and do Carmo 1994) proved that if M^n is compact with constant mean curvature H and $|\Phi|^2 \leq B_H$, where B_H is a constant that depends only on H and n, then either $|\Phi|^2 = 0$ or $|\Phi|^2 = B_H$. They also proved that the $H(r)$-tori $\mathbb{S}^{n-1}(r) \times \mathbb{S}^1(\sqrt{1-r^2})$ with $r^2 \leq (n-1)/n$ are the only hypersurfaces with constant mean curvature H and $|\Phi|^2 = B_H$. These results do not characterize the other tori $\mathbb{S}^{n-1}(r) \times \mathbb{S}^1(\sqrt{1-r^2})$, with $r^2 > (n-1)/n$, nor $\mathbb{S}^{n-k}(r) \times \mathbb{S}^k(\sqrt{1-r^2})$ for $2 \leq k \leq n-1$. Recently, the third author jointly with Barbosa, Costa and Lazaro (Barbosa et al. 2003) obtained a generalization of the result of Hasanis and Vlachos without any additional hypothesis on the mean curvature. They obtained a characterization for the $H(r)$-tori $\mathbb{S}^{n-1}(r) \times \mathbb{S}^1(\sqrt{1-r^2})$ with $r^2 \geq (n-1)/n$. More precisely:

THEOREM 1 (Barbosa et al. 2003). *Let M be a compact oriented hypersurface immersed in the sphere $\mathbb{S}^{n+1}$, with two distinct principal curvatures λ and μ with multiplicities* 1 *and* $n-1$, *respectively. Suppose in addition that $n \geq 3$ and $|\Phi|^2 \geq C_H$, where*

$$C_H = n + \frac{n(n^2-2n+2)H^2}{2(n-1)} + \frac{n(n-2)|H|}{2(n-1)}\sqrt{n^2H^2+4(n-1)}.$$

Then H is constant, $|\Phi|^2 = C_H$ and M is isometric to an $H(r)$-torus $\mathbb{S}^{n-1}(r) \times \mathbb{S}^1(\sqrt{1-r^2})$ with $r^2 \geq (n-1)/n$.

Recently, Perdomo (Perdomo 2004) and Wang (Wang 2003) independently obtained the following integral formula for compact minimal hypersurfaces with two principal curvatures immersed in $\mathbb{S}^{n+1}$.

THEOREM 2 (Perdomo 2004, Wang 2003). *Let M^n be a compact minimal hypersurface immersed in $\mathbb{S}^{n+1}$. If M has two principal curvatures everywhere, then*

$$\int_M |A|^2 = n|M| - \frac{n-2}{n}\int_M |\nabla \ln \lambda|^2, \tag{1}$$

if λ is a principal curvature with multiplicity $n-1$.

A natural consequence of the integral formula above is that if M is a compact minimal hypersurface with two principal curvatures immersed in $\mathbb{S}^{n+1}$, then

$$\int_M |A|^2 \leq n|M|,$$

with equality if and only if M is a Clifford hypersurface. In this paper we will extend the integral formula (1) for compact hypersurfaces with constant mean curvature and obtain a new characterization of the $H(r)$-torus $\mathbb{S}^{n-1}(r) \times \mathbb{S}^1(\sqrt{1-r^2})$. Explicitly, we have the following result.

THEOREM 3. *Let M be a compact oriented hypersurface immersed in the sphere $\mathbb{S}^{n+1}$ with constant mean curvature H. Suppose in addition that M has two distinct principal curvatures λ and μ with multiplicities $(n-1)$ and 1, respectively. If $\Phi = A - HI$ is the traceless second fundamental form of M and P_H is the Alencar-do Carmo polynomial*

$$P_H(x) = x^2 + \frac{n(n-2)}{\sqrt{n(n-1)}}Hx - n(1+H^2),$$

then

$$\int_M P_H(c|\Phi|) = -\frac{(n-2)}{n}\int_M |\nabla \ln|\Phi||^2, \tag{2}$$

where $c = \pm 1$ is the sign of the difference $\lambda - \mu$.

The polynomial P_H was first introduced by Alencar and do Carmo in (Alencar and do Carmo 1994) in their study on hypersurfaces with constant mean curvature in the sphere. Actually, the sharp positive constant B_H found by them in that paper is given precisely as the square of the positive root of P_H (a constant that depends only on H and n). For that reason we will refer to P_H as the Alencar-do Carmo polynomial.

In the minimal case, we make $H = 0$ in the integral formula (2) and retrieve Perdomo' s and Wang' s integral formula (1). From the above result we obtain.

COROLLARY 4. *Let M be a compact oriented hypersurface immersed in the sphere $\mathbb{S}^{n+1}$ with constant mean curvature H. Suppose in addition that M has two distinct principal curvatures λ and μ with multiplicities $(n-1)$ and 1, respectively. Let $c = \pm 1$ be the sign of the difference $\lambda - \mu$. Then*

$$\int_M P_H(c|\Phi|) \leq 0,$$

with equality only if M is an $H(r)$-torus $\mathbb{S}^{n-1}(r) \times \mathbb{S}^1(\sqrt{1-r^2})$.

2 PRELIMINARIES

Let M be a compact hypersurface with constant mean curvature H immersed in an $(n+1)$-dimensional unit sphere $\mathbb{S}^{n+1}$. Choose a local orthornormal frame field E_1, ..., E_n in a neighborhood U of M and let ω_1, ..., ω_n be its dual coframe. As is well known, there are smooth 1-forms ω_{ij} on U uniquely determined by the equations

$$d\omega_i = -\sum_{j=1}^{n} \omega_{ij} \wedge \omega_j, \quad \omega_{ij} + \omega_{ji} = 0. \tag{3}$$

The square of the length of the second fundamental form

$$A = \sum_{i,j=1}^{n} h_{ij}\omega_i \otimes \omega_j,$$

is given by $S = \sum_{i,j} h_{ij}^2$. Note that $h_{ij} = h_{ji}$ and

$$H = \frac{1}{n}\sum_{i=1}^{n} h_{ii}. \tag{4}$$

The covariant derivative of A is given by

$$\nabla A = \sum_{i,j,k=1}^{n} h_{ijk}\omega_i \otimes \omega_j \otimes \omega_k, \tag{5}$$

where

$$\sum_k h_{ijk}\omega_k = dh_{ij} - \sum_m h_{im}\omega_{mj} - \sum_m h_{mj}\omega_{mi}. \tag{6}$$

It is well known that h_{ijk} is symmetric in all indices.

If $\Phi = A - HI$ is the traceless second fundamental form, then $|\Phi|^2 = |A|^2 - nH^2$. Observe that $|\Phi|^2 \geq 0$ and equality holds precisely at the umbilic points of M. For that reason, Φ is also called the total umbilicity tensor of M. We also have the Simons formula (see for instance (Alencar and do Carmo 1994), taking into account the different choice of sign in their definition of Φ)

$$\frac{1}{2}\Delta|\Phi|^2 = |\nabla\Phi|^2 + |\Phi|^2\left(n(1+H^2) - |\Phi|^2\right) + nHtr(\Phi^3). \tag{7}$$

From now we will assume that M^n is a compact hypersurface with constant mean curvature having everywhere two distinct principal curvatures λ and μ with multiplicities $n-k$ and k, respectively. First, we need a classical result of Otsuki (Otsuki 1970).

Proposition 5. *Let M be a hypersurface in $\mathbb{S}^{n+1}$ such that the multiplicities of its principal curvatures are constant. Then the distribution D_λ of the space of principal vectors corresponding to each principal curvature λ is completely integrable. In particular, if the multiplicity of a principal curvature is greater than* 1*, then this principal curvature is constant on each of the integral leaves of the corresponding distribution of the space of its principal vectors.*

A consequence of this result is the following lemma:

Lemma 6. *Let M^n be a compact oriented hypersurface in $\mathbb{S}^{n+1}$ with constant mean curvature and two principal curvatures λ and μ, with multiplicities $n-k$ and k respectively. If $1 < k < n-1$, then M is isometric to $\mathbb{S}^{n-k}(r) \times \mathbb{S}^k(\sqrt{1-r^2})$.*

Proof. We may choose a local orthonormal basis $E_1, \dots, E_k, E_{k+1}, \dots, E_n$ such that for $1 \leq i \leq k$ and $k+1 \leq j \leq n$,

$$E_i \in D_\lambda = \left\{v \in T_pM\colon\ p \in M,\quad Av = \lambda v\right\}$$

$$E_j \in D_\mu = \left\{v \in T_pM\colon\ p \in M,\quad Av = \mu v\right\}.$$

By Proposition 5 above, we have $E_i(\lambda) = E_j(\mu) = 0$ for $i \leq k$ and $k+1 \leq j$. As

$$H = \frac{k\lambda + (n-k)\mu}{n}$$

is a constant function, it follows that

$$kE_j(\lambda) = (k-n)E_j(\mu) = 0.$$

Therefore μ and λ are constant and M is an isoparametric hypersurface. Note that since M is compact, then M is isometric to $\mathbb{S}^{n-k}(r) \times \mathbb{S}^k(\sqrt{1-r^2})$. □

We should mention that if $H = (1/n)trA$ is the mean curvature of a hypersurface M in $\mathbb{S}^{n+1}$ with second fundamental form A, then λ is an eigenvalue of A if and only if $\widetilde{\lambda} = \lambda - H$ is an eigenvalue of the traceless second fundamental form $\Phi = A - HI$. Those eigenvalues have the same multiplicities. In the following lemma we are going to evaluate the Laplacian of $\ln|\Phi|$. It turns out that $\Delta \ln|\Phi|$ depends on the Alencar-do Carmo polynomial

$$P_H(x) = x^2 + \frac{n\,(n-2)\,H}{\sqrt{n\,(n-1)}}x - n\left(1 + H^2\right).$$

LEMMA 7. *Let M be a compact oriented hypersurface with constant mean curvature H immersed in $\mathbb{S}^{n+1}$ having two principal curvatures λ and μ, with multiplicities $n-1$ and 1 respectively. Then*

$$\Delta \ln|\Phi| = \frac{1}{|\Phi|^2}\left(|\nabla\Phi|^2 - 2|\nabla|\Phi||^2\right) - P_H(c|\Phi|), \tag{8}$$

where $c = \pm 1$ is the sign of the difference $\lambda - \mu$.

PROOF. Since M has two distinct principal curvatures at each point, it follows that M has no umbilical points. In particular $|\Phi| > 0$. Note that

$$\Delta \ln|\Phi| = \frac{1}{|\Phi|}\Delta|\Phi| - \frac{1}{|\Phi|^2}|\nabla|\Phi||^2$$

and

$$\frac{1}{2}\Delta|\Phi|^2 = |\Phi|\Delta|\Phi| + |\nabla|\Phi||^2,$$

which implies that

$$|\Phi|^2\Delta \ln|\Phi| = \frac{1}{2}\Delta|\Phi|^2 - 2|\nabla|\Phi||^2.$$

Using now the Simons formula (7), one gets

$$|\Phi|^2\Delta \ln|\Phi| = |\nabla\Phi|^2 - 2|\nabla|\Phi||^2 - |\Phi|^2(|\Phi|^2 - n(1+H^2)) + nHtr(\Phi^3).$$

That is,

$$\Delta \ln |\Phi| = \frac{1}{|\Phi|^2}\left(|\nabla\Phi|^2 - 2|\nabla|\Phi||^2\right) - (|\Phi|^2 - n(1+H^2)) + \frac{nHtr(\Phi^3)}{|\Phi|^2}. \tag{9}$$

Note that the eigenvalues $\widetilde{\lambda} = \lambda - H$ and $\widetilde{\mu} = \mu - H$ of Φ have multiplicities $n-1$ and 1, respectively, and are related by $\widetilde{\mu} = -(n-1)\widetilde{\lambda}$. Observe that

$$n\widetilde{\lambda} = \widetilde{\lambda} - \widetilde{\mu} = \lambda - \mu \neq 0$$

and

$$|\Phi|^2 = tr(\Phi^2) = (n-1)\widetilde{\lambda}^2 + \widetilde{\mu}^2 = n(n-1)\widetilde{\lambda}^2,$$

so that

$$\widetilde{\lambda} = \frac{c}{\sqrt{n(n-1)}}|\Phi|$$

where $c = \pm 1$ is the sign of the difference $\lambda - \mu$. Moreover,

$$tr(\Phi^3) = (n-1)\widetilde{\lambda}^3 + \widetilde{\mu}^3 = -n(n-1)(n-2)\widetilde{\lambda}^3 = -\frac{(n-2)}{\sqrt{n(n-1)}}c|\Phi|^3.$$

Therefore, using this into (9), one has

$$\Delta \ln |\Phi| = \frac{1}{|\Phi|^2}\left(|\nabla\Phi|^2 - 2|\nabla|\Phi||^2\right) - P_H(c|\Phi|). \qquad \square$$

LEMMA 8. *Let M^n, $n \geq 3$, be a compact oriented hypersurface with constant mean curvature H immersed in $\mathbb{S}^{n+1}$ having two principal curvatures λ and μ with multiplicities $n-1$ and 1 respectively, then*

$$|\nabla\Phi|^2 = \frac{n+2}{n}|\nabla|\Phi||^2 \tag{10}$$

PROOF. Let $E_1, \ldots, E_n$ be a local orthornomal frame such that $\Phi(E_n) = \tilde{\mu}E_n$ and

$$\Phi(E_i) = \tilde{\lambda}E_i,$$

for $1 \leq i \leq n-1$. Since H is a constant function and $\Phi = A - HI$, then

$$|\nabla\Phi|^2 = |\nabla A|^2 = \sum_{i,j,k} h_{ijk}^2.$$

From

$$\sum_k h_{ijk}\omega_k = dh_{ij} - \sum_m h_{im}\omega_{mj} - \sum_m h_{mj}\omega_{mi}$$

and Proposition 5, it follows that for all $1 \leq i \leq n-1$. In particular,

$$h_{iii} = d\lambda(E_i) = 0, \quad 1 \leq i \leq n-1,$$

$$h_{iij} = d\lambda(E_j) = 0, \quad 1 \leq i, j \leq n-1, i \neq j,$$

and

$$h_{iin} = d\lambda(E_n), \quad 1 \leq i \leq n-1.$$

By choosing $j \neq k$, $1 \leq j \leq n-1$ we also have a direct proof that

$$h_{iik} = h_{jjk} = h_{jkj} = (h_{jj} - h_{kk})\omega_{ij}(E_k) = 0$$

for all $k \neq n$. We also note that

$$h_{nni} = d\mu(E_i) = -(n-1)d\lambda(E_i) = h_{iii} = 0, \quad 1 \leq i \leq n-1,$$

and

$$h_{nnn} = d\mu(E_n) = -(n-1)d\lambda(E_n).$$

Finally,

$$h_{ijk} = h_{ijn} = 0, \quad 1 \leq i, j, k \leq n-1, i \neq j, i \neq k, j \neq k.$$

Therefore, since $\nabla\lambda = d\lambda(E_n)E_n$, one gets

$$\begin{aligned} |\nabla\Phi|^2 &= h_{nnn}^2 + 3\sum_{i=1}^{n-1} h_{iin}^2 = (n-1)(n+2)d\lambda(E_n)^2 \\ &= (n-1)(n+2)|\nabla\lambda|^2. \end{aligned}$$

Recall that $|\Phi|^2 = n(n-1)\widetilde{\lambda}^2$ with $\widetilde{\lambda} = \lambda - H$. Since H is constant it then follows that

$$|\nabla\lambda|^2 = |\nabla\widetilde{\lambda}|^2 = \frac{|\nabla|\Phi||^2}{n(n-1)}.$$

Therefore

$$|\nabla\Phi|^2 = (n-1)(n+2)\frac{|\nabla|\Phi||^2}{n(n-1)} = \frac{(n+2)}{n}|\nabla|\Phi||^2. \qquad \square$$

3 PROOF OF THE MAIN RESULTS

PROOF OF THEOREM 3: Theorem 3 easily follows from Lemma 7 and Lemma 8. In fact, we get

$$\begin{aligned}\Delta \ln|\Phi| &= \frac{1}{|\Phi|^2}\left(|\nabla\Phi|^2 - 2|\nabla|\Phi||^2\right) - P_H(c|\Phi|)\\ &= -\frac{(n-2)}{n}\frac{1}{|\Phi|^2}|\nabla|\Phi||^2 - P_H(c|\Phi|)\\ &= -\frac{(n-2)}{n}|\nabla \ln|\Phi||^2 - P_H(c|\Phi|),\end{aligned}$$

Integrating now over M we conclude that

$$\int_M P_H(c|\Phi|) = -\frac{(n-2)}{n}\int_M |\nabla \ln|\Phi||^2.$$

This finishes the proof of Theorem 3. □

As a consequence of this result, we obtain a new characterization of the $H(r)$-tori $\mathbb{S}^{n-1}(r) \times \mathbb{S}^1(\sqrt{1-r^2})$.

PROOF OF COROLLARY 4: By the integral formula in Theorem 3 we have that

$$\int_M P_H(c|\Phi|) = -\frac{n-2}{n}\int_M |\nabla \ln|\Phi||^2 \leq 0.$$

In the equality, we have

$$\int_M |\nabla \ln|\Phi||^2 = 0.$$

Then $|\nabla \ln|\Phi||^2 = 0$ and $|\Phi|$ is a constant function. Therefore

$$\int_M P_H(c|\Phi|) = P_H(c|\Phi|)|M| = 0,$$

and $c|\Phi|$ is a root of the equation $P_H(x) = 0$. Even more, by Lemma 8 we also know that $\nabla\Phi = \nabla A = 0$, which means that M is isoparametric with two constant distinct principal curvatures. In fact, as in the proof of Lemma 8, consider $E_1, \ldots, E_n$ a local orthornomal frame such that $A(E_n) = \mu E_n$ and $A(E_i) = \lambda E_i$ for $1 \leq i \leq n-1$. Then

$$\nabla A(E_n, E_n) = E_n(\lambda)E_n = 0$$

implies that λ, and hence μ, is constant. Finally, M being compact, it follows that M is isometric to an $H(r)$-torus $\mathbb{S}^{n-1}(r) \times \mathbb{S}^1(\sqrt{1-r^2})$. □

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RESUMO

Neste trabalho tratamos das hipersuperfícies fechadas com curvatura média constante e duas curvaturas principais imersas na esfera euclidiana. No caso de hipersuperfícies mínimas, Perdomo (Perdomo 2004) e Wang (Wang 2003) obtiveram uma desigualdade integral envolvendo o quadrado da norma da segunda forma fundamental de M, onde ocorre a igualdade se e somente se M é o toro de Clifford. Neste trabalho, usando o segunda forma fundamental modificada com traço nulo de M, obtemos uma generalização da fórmula integral acima e uma nova caracterização dos H(r)-toros.

Palavras-chave: Hipersuperfícies, curvatura média constante, Fórmula de Simons, $H(r)$-toros.

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