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A relation between the right triangle and circular tori with constant mean curvature in the unit 3-sphere

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ABSTRACT

In this note we will show that the inverse image under the stereographic projection of a circular torus of revolution in the 3-dimensional euclidean space has constant mean curvature in the unit 3-sphere if and only if their radii are the catet and the hypotenuse of an appropriate right triangle.

Key words: Flat torus, constant mean curvature, circular tori, stereographic projection.

1 INTRODUCTION

We will denote by $T(r, a)$ the standard circular torus of revolution in $\mathbb{R}^3$ obtained from the circle Γ in the $xz-plane$ centered at $(r, 0, 0)$ with radius $a < r$, i.e.

$$T(r, a) = \left\{(x, y, z) \in \mathbb{R}^3 \colon (\sqrt{x^2 + y^2} - r)^2 + z^2 = a^2\right\}.$$

Now let $\rho\colon \mathbb{S}^3 \setminus \{n\} \to \mathbb{R}^3$ be the stereographic projection of the Euclidean sphere $\mathbb{S}^3 = \{x \in \mathbb{R}^4 :\mid x \mid^2 = 1\}$, where $n = (0, 0, 0, 1)$ is its north pole. The inverse image of a circular torus in $\mathbb{R}^3$ under the stereographic projection will be called a circular torus in $\mathbb{S}^3$. We would like to know when circular tori in $\mathbb{R}^3$ comes from constant mean curvature circular tori in $\mathbb{S}^3$ under the stereographic projection. A circular torus in $\mathbb{S}^3$ meant that it is obtained from a revolution of a circle in $\mathbb{S}^3$ under a rigid motion. A general $T(r, a)$ will not satisfy the above requirement. For instance, it was proved by Montiel and Ros (Montiel and Ros 1981) that a compact embedded surface S with constant mean curvature contained in an open hemisphere of $\mathbb{S}^3$ must be a round sphere. Hence for $T(r, a)$ contained inside or outside of the unit ball $B(1) \subset \mathbb{R}^3$, $\rho^{-1}(T(r, a))$ will be contained in an open hemisphere of $\mathbb{S}^3$ and can not have constant mean curvature. Then among

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all tori $T(r, a)$ which intercept the inside and the outside of the unit ball $B(1)$ we will describe those which have the desired property. We will show that to construct such a torus we take an arbitrary point $P(\alpha) = (\cos\alpha, 0, \sin\alpha)$ on the unit circle of the $xz - plane$, $0 < \alpha < \pi/2$, draw its tangent until it meets the x axis at the point $Q(\alpha) = (\sec\alpha, 0, 0)$ which will be the center of the circle Γ whereas its radius will be $a = \tan\alpha$, i.e. the torus $T(\sec\alpha, \tan\alpha)$ will satisfy the previous requirement. We note if O denotes the origin of $\mathbb{R}^3$ then the triangle OPQ is a right triangle. This description will yield that the Clifford torus is associated to a right triangle with two equal sides. More precisely, our aim in this note is to present a proof of the following fact:

THEOREM 1. *Let $T^2 \subset \mathbb{S}^3$ be a circular torus of constant mean curvature. Then*

$$T^2 = \rho^{-1}(T(\sec\alpha, \tan\alpha)) = S^1(\cos\alpha) \times S^1(\sin\alpha).$$

Moreover, the mean curvature of T^2 is given by $\overline{H} = \dfrac{(\tan^2\alpha - 1)}{2\tan\alpha}$.

2 PRELIMINARIES

For an immersion $f\colon M \to \overline{M}$ between Riemannian manifolds we will denote by ds_f^2 the induced metric on M by f. Now let M^n, M_1^m and M_2^m be Riemannian manifolds, where the superscript denote the dimension of the manifold. Consider $\psi\colon M^n \to M_1^m$ be an immersion, $\rho\colon M_1^m \to M_2^m$ a conformal mapping and set $\varphi = \rho \circ \psi$. Let $\phi\colon M \to \mathbb{R}$ be a function verifying $ds_\varphi^2 = e^{2\phi} ds_\psi^2$. If $\bar{k}_i$ and k_i denote the principal curvatures of ψ and $\varphi = \rho \circ \psi$, respectively, then we get

$$k_i = e^{-\phi}\left(\bar{k}_i - \frac{\partial\phi}{\partial\xi}\right), \tag{1}$$

where ξ is a unit normal vector field to $\psi(M)$, see for instance (Abe 1982) or (Willmore 1982). At first we will recall the following known lemma of which we sketch the proof.

LEMMA 1. *Let $\psi = (\psi_1, \psi_2, \psi_3, \psi_4)\colon M^2 \to \mathbb{S}^3 \setminus \{n\}$ be an immersion of a surface M^2, set $\varphi = \rho \circ \psi$ and suppose $ds_\varphi^2 = e^{2\phi} ds_\psi^2$. Then we get*

$$k_i = e^{-\phi}(\bar{k}_i - g), \tag{2}$$

where $g = \langle \nu, \varphi \rangle$ denotes the support function on $M^2 \subset \mathbb{R}^3$.

PROOF. If we put $\psi = \psi(u_1, u_2)$ then a direct computation gives

$$\langle \frac{\partial\varphi}{\partial u_i}, \frac{\partial\varphi}{\partial u_j} \rangle = \lambda^2 \langle \frac{\partial\psi}{\partial u_i}, \frac{\partial\psi}{\partial u_j} \rangle, \tag{3}$$

where $\lambda = (1 - \psi_4)^{-1} = \frac{1+|\varphi|^2}{2}$. So we can write $ds_\varphi^2 = e^{2\phi} ds_\psi^2$ with $e^\phi = \frac{1+|\varphi|^2}{2}$. Thus if ν denotes a unit normal vector field to $\varphi(M^2)$ then $\nu = e^{-\phi}\xi$, where ξ stands for a unit normal vector field to $\psi(M^2)$. Hence we have from (1)

$$k_i = e^{-\phi}\bar{k}_i - \frac{\partial\phi}{\partial\nu} = e^{-\phi}(\bar{k}_i - \langle \nu, \varphi \rangle) = e^{-\phi}(\bar{k}_i - g),$$

as we wished to prove. □

3 PROOF OF THE THEOREM

PROOF. First we note that the circle $\Gamma = \left\{(x, 0, z) \in \mathbb{R}^3 : (x - r)^2 + z^2 = a^2\right\}$ can be parametrized by the map $\gamma : [0, 2\pi] \to \mathbb{R}^3$ defined by

$$\gamma(t) = \left(\frac{r^2 - a^2}{r - a\sin t}, 0, \frac{a\sqrt{r^2 - a^2}\cos t}{r - a\sin t}\right).$$

In fact, it is enough to check that

$$\left(\frac{r^2 - a^2}{r - a\sin t} - r\right)^2 + \left(\frac{a\sqrt{r^2 - a^2}\cos t}{r - a\sin t}\right)^2 = a^2.$$

Representing by R_θ a rotation on $\mathbb{R}^3$ around the $z - axis$, we see that $R_\theta(\gamma(t))$ is a circular torus $T(r, a)$ if γ is a parametrization of the circle Γ given above. We put now $\sigma = \sqrt{r^2 - a^2}$, $\theta = ru_1/\sigma^2$ and $t = ru_2/a\sigma$. We note that such a choice implies $0 \le u_1 \le (2\pi\sigma^2)/r$ and $0 \le u_2 \le (2\pi a\sigma)/r$. Let us call $R_\theta(\gamma(t))$ of $\varphi(u_1, u_2)$, i.e.

$$\varphi(u_1, u_2) = \sigma(r - a\sin t)^{-1}(\sigma\cos\theta, \sigma\sin\theta, a\cos t). \tag{4}$$

Hence we have

$$e^\phi = \frac{1+ \mid \varphi \mid^2}{2} = \frac{q(t)}{2(r - a\sin t)}, \tag{5}$$

where $q(t) = a(\sigma^2 - 1)\sin t + r(\sigma^2 + 1)$. Now a straightforward computation yields

$$\begin{cases} \dfrac{\partial\varphi}{\partial u_1} = \dfrac{r}{(r - a\sin t)}\left(-\sin\theta, \cos\theta, 0\right), \\ \dfrac{\partial\varphi}{\partial u_2} = \dfrac{r}{(r - a\sin t)^2}\left(\sigma\cos t\cos\theta, \sigma\cos t\sin\theta, a - r\sin t\right). \end{cases}$$

From that we derive that φ is a conformal parametrization of $T\,(r, a)$ satisfying

$$\left\langle\frac{\partial\varphi}{\partial u_i}, \frac{\partial\varphi}{\partial u_j}\right\rangle = \frac{r^2\delta_{ij}}{(r - a\sin t)^2}. \tag{6}$$

Moreover, a unit vector field normal to φ is given as follows:

$$\nu\,(u_1, u_2) = -\frac{1}{(r - a\sin t)}\left((a - r\sin t)\cos\theta, (a - r\sin t)\sin\theta, -\sigma\cos t\right).$$

Therefore we conclude that

$$g = \frac{\sigma^2\sin t}{(r - a\sin t)}. \tag{7}$$

On the other hand a new computation gives us

$$\begin{cases} \dfrac{\partial \nu}{\partial u_1} &= -\dfrac{(a - r\sin t)}{\sigma^2}\dfrac{\partial \varphi}{\partial u_1}, \\ \dfrac{\partial \nu}{\partial u_2} &= \dfrac{1}{a}\dfrac{\partial \varphi}{\partial u_2}. \end{cases} \tag{8}$$

From this we have $k_1 = \frac{(a - r\sin t)}{(r^2 - a^2)}$ and $k_2 = -\frac{1}{a}$. Taking into account (5), (7) and (8) we conclude from Lemma 1 that

$$\overline{H} = \frac{1}{4a\sigma^2}\left(ra\left(\sigma^2 - 1\right)\sin t + \left(\sigma^2 + 1\right)\left(2a^2 - r^2\right)\right).$$

Now we have that $\overline{H}$ is constant if and only if $\sigma^2 = 1$. Moreover, $\sigma^2 = 1$ yields $\overline{H} = \frac{1}{2a}\left(a^2 - 1\right)$. Since $a < r$ we put $a = r\sin\alpha$, $r = \sec\alpha$ and this completes the proof of the theorem. □

We point out that $\overline{H} = 0$ if and only if $a = 1$ and $r = \sqrt{2}$ which corresponds to the right triangle with two equal sides.

4 THE WILLMORE MEASURE ON $T(r, a)$

In this section we will present a simple way to compute $\int_{T(r,a)} H^2 dA$ by using the parametrization of $T(a, r)$ given by (4). We observe that if dA denotes the element of area of $T(r, a)$ then its Willmore measure is given by

$$\left(H^2 - K\right) dA = \frac{r^4}{4a^2\sigma^4} du_1 du_2.$$

Hence, using Gauss-Bonnet theorem, we easily conclude that

$$\int_{T(r,a)} H^2 dA = \frac{r^4}{4a^2\sigma^4}\int_0^{\frac{2\pi a\sigma}{r}}\int_0^{\frac{2\pi\sigma^2}{r}} du_1 du_2 = \frac{r^2}{a\sqrt{r^2 - a^2}}\pi^2. \tag{9}$$

Therefore the family of tori $T\left(\sqrt{2}a, a\right)$, which corresponds to the family of right triangles with two equal sides, yields the minimum for $\int_{T(r,a)} H^2 dA$ among all circular tori. Moreover, from (9) its value is (see also Willmore 1982)

$$\int_{T(\sqrt{2}a,a)} H^2 dA = 2\pi^2.$$

Since $a < r$, if we choose α such that $\sin\alpha = \frac{a}{r}$, we conclude from (9) the following corollary.

COROLLARY 1. *Given a circular torus $T(r, a) \subset \mathbb{R}^3$ we have a circular torus $T(\sec\alpha, \tan\alpha) \subset \mathbb{R}^3$ such that $\int_{T(r,a)} H^2 dA = \int_{T(\sec\alpha,\tan\alpha)} H_\alpha^2 dA_\alpha$. In other words, the family of circular tori with constant mean curvature in $\mathbb{S}^3$ cover all values of $\int_{T(r,a)} H^2 dA$.*

5 CONCLUDING REMARKS

We point out that Theorem 2 of K. Nomizu and B. Smyth (Nomizu and Smyth 1969) guarantees that a flat torus of constant mean curvature in $\mathbb{S}^3$ is isometric to a product of circles. Then $\rho^{-1}T(a, r)$ is flat if and only if it has constant mean curvature. We notice if we set $\psi = \rho^{-1}\varphi$ where φ was given by (4) then we have

$$\psi(u_1, u_2) = \frac{1}{q(t)}\big(2\sigma^2\cos\theta, 2\sigma^2\sin\theta, 2a\sigma\cos t, r(\sigma^2 - 1) + a(\sigma^2 + 1)\sin t\big),$$

where $q(t) = a(\sigma^2 - 1)\sin t + r(\sigma^2 + 1)$, (see(5)). Hence by using (3), (5), (6) and putting $z = u_1 + iu_2$ we conclude that

$$ds^2_\psi = e^{-2\phi}ds^2_\varphi = \frac{4r^2}{q^2(t)} \mid dz \mid^2 .$$

According to our theorem the metric ds^2_ψ is flat if and only if $\rho^{-1}T\ (r, a)$ has constant mean curvature in $\mathbb{S}^3$. In this case we have

$$\psi\ (u_1, u_2) = \frac{1}{\sqrt{a^2 + 1}}\ (\cos\theta, \sin\theta, a\cos t, a\sin t)\ ,$$

i.e. $\rho^{-1}T(r, a)$ is isometric to the product of circles $S^1(\frac{1}{\sqrt{a^2+1}}) \times S^1(\frac{a}{\sqrt{a^2+1}})$. We note that this yields $\cos\alpha = \frac{1}{\sqrt{a^2+1}}$ and $\sin\alpha = \frac{a}{\sqrt{a^2+1}}$, i.e. $\rho(S^1(\cos\alpha) \times S^1(\sin\alpha)) = T(\sec\alpha, \tan\alpha)$.

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RESUMO

Neste artigo mostraremos que a imagem inversa pela projeção estereográfica de um toro circular de revolução no espaço euclidiano de dimensão 3 tem curvatura média constante se e somente se os seus raios são o cateto e a hipotenusa de um triângulo retângulo apropriado.

Palavras-chave: Toro plano, Curvatura média constante, Toro circular, Projeção estereográfica.

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