Anais da Academia Brasileira de Ciências (2006) 78(4): 615-621
(Annals of the Brazilian Academy of Sciences)
ISSN 0001-3765
www.scielo.br/aabc

Marcinkiewicz strong laws for linear statistics of ρ^*-mixing sequences of random variables

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Manuscript received on April 4, 2006; accepted for publication on August 1st, 2006; presented by MANFREDO DO CARMO

ABSTRACT

Strong laws are established for linear statistics that are weighted sums of a random sample. We show extensions of the Marcinkiewicz-Zygmund strong laws under certain moment conditions on both the weights and the distribution. These not only generalize the result of Bai and Cheng (2000, Statist Probab Lett 46: 105–112) to ρ^*-mixing sequences of random variables, but also improve them.

Key words: ρ^*-mixing, Marcinkiewicz-Zygmund strong laws, weighted sums.

1 INTRODUCTION

As Bai and Cheng (2000) remarked, many useful linear statistics based on a random sample are weighted sums of i.i.d. random variables. Examples include least-squares estimators, nonparametric regression function estimators and jackknife estimates, among others. In this respect, studies of strong laws for these weighted sums have demonstrated significant progress in probability theory with applications in mathematical statistics. But a random sample is often dependent. So we want to know if the results obtained for i.i.d. random variables are still true for ρ^*-mixing sequences of random variables.

Let $S, T \subset \mathcal{N}$ be nonempty and define $\mathcal{F}_S = \sigma(X_k, k \in S)$, and the maximal correlation coefficient $\rho_n^* = \sup corr(f, g)$ where the supremum is taken over all (S, T) with $dist(S, T) \geq n$ and all $f \in L_2(\mathcal{F}_S)$, $g \in L_2(\mathcal{F}_T)$ and where $dist(S, T) = \inf_{x\in S, y\in T} |x - y|$.

A sequence of random variables $\{X_n, n \geq 1\}$ on a probability space $\{\Omega, \mathcal{F}, P\}$ is called ρ^*-mixing if

$$\lim_{n\to\infty} \rho_n^* < 1. \tag{1.1}$$

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AMS Classification: 60F15; 62G05

As for ρ^*-mixing sequences of random variables, one can refer to Bryc and Smolenski (1993), who found bounds for the moments of partial sums for a sequence of random variables satisfying (1.1), and to Peligrad (1996) for CLT, Peligrad (1998) for invariance principles, Peligrad and Gut (1999) for the Rosenthal type maximal inequality, Utev and Peligrad (2003) for invariance principles of nonstationary sequences. The main purpose of this paper is to establish the Marcinkiewicz-Zygmund strong laws for linear statistics of ρ^*-mixing sequences of random variables. The results obtained (see Theorem 2.1 and Corollary 2.1) not only generalize the result of Bai and Cheng (2000) to ρ^*-mixing sequences of random variables, but also improve them. In Theorem 2.2 of Bai and Cheng (2000), they believe the choice of b_n can hardly be improved in view of Cuzick (1995, Lemma 2.1), but now we improve the choice of b_n using a new method.

2 THE MARCINKIEWICZ-ZYGMUND STRONG LAWS

Throughout this paper, C will represent a positive constant though its value may change from one appearance to the next, and $a_n = O(b_n)$ will mean $a_n \leq Cb_n$.

In order to prove our results, we need the following lemma.

LEMMA 2.1. **(Utev and Peligrad, 2003)**. *Let $\{X_i, i \geq 1\}$ be a ρ^*-mixing sequence of random variables, $EX_i = 0, E|X_i|^p < \infty$ for some $p \geq 2$ and for every $i \geq 1$. Then there exists $C = C(p)$, such that*

$$E \max_{1\leq k\leq n} |\sum_{i=1}^{k} X_i|^p \leq C\left\{ \sum_{i=1}^{n} E|X_i|^p + \left(\sum_{i=1}^{n} EX_i^2 \right)^{p/2} \right\}.$$

THEOREM 2.1. *Let $\{X, X_i, i \geq 1\}$ be a ρ^*-mixing sequence of identically distributed random variables, $T_n = \sum_{i=1}^{n} a_{ni} X_i, n \geq 1$, where the weights $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ are random variables which are independent of $\{X_i, i \geq 1\}$ (the case of deterministic weights is included). Suppose that for some α with $0 < \alpha < 2$ we have that $\sum_{i=1}^{n} |a_{ni}|^\alpha = O(n)$ almost surely. If $1 < \alpha < 2$, we assume additionally that $EX = 0$. Set $b_n = n^{\frac{1}{\alpha}} (\log n)^{\frac{1}{\gamma}}$. We assume that for some $h, \gamma > 0$, we have*

$$E \exp(h|X|^\gamma) < \infty. \tag{2.0}$$

Then

$$\forall \varepsilon > 0, \sum_{n=1}^{\infty} n^{-1} P\Big(\max_{1\leq j\leq n} |T_j| > \varepsilon b_n\Big) < \infty. \tag{2.1}$$

PROOF. $\forall i \geq 1$, define $X_i^{(n)} = X_i I(|X_i| \leq b_n)$, $T_j^{(n)} = \sum_{i=1}^{j} \left(a_{ni} X_i^{(n)} - Ea_{ni} X_i^{(n)}\right)$, then $\forall \varepsilon > 0$, we have

$$
\begin{aligned}
&P\Big(\max_{1\le j\le n}|T_j| > \varepsilon b_n\Big)\\
&\le P\Big(\max_{1\le j\le n}|X_j| > b_n\Big) + P\Big(\max_{1\le j\le n}|T_j^{(n)} + \sum_{i=1}^{j} Ea_{ni}X_i^{(n)}| > \varepsilon b_n\Big)\\
&\le P\Big(\max_{1\le j\le n}|X_j| > b_n\Big) + P\Big(\max_{1\le j\le n}|T_j^{(n)}| > \varepsilon b_n - \max_{1\le j\le n}|\sum_{i=1}^{j} Ea_{ni}X_i^{(n)}|\Big).
\end{aligned}
\tag{2.2}
$$

First we show that

$$
b_n^{-1}\max_{1\le j\le n}\left|\sum_{i=1}^{j} Ea_{ni}X_i^{(n)}\right| \to 0, \quad \text{as} \quad n\to\infty. \tag{2.3}
$$

By $\sum_{i=1}^{n}|a_{ni}|^\alpha = O(n)$ and Hölder inequality, $\forall 1\le k<\alpha$, then

$$
\sum_{i=1}^{n}|a_{ni}|^k \le \left(\sum_{i=1}^{n}|a_{ni}|^{k\frac{\alpha}{k}}\right)^{\frac{k}{\alpha}}\left(\sum_{i=1}^{n}1\right)^{\frac{\alpha-k}{\alpha}} \le Cn. \tag{2.4}
$$

When $1<\alpha<2$, using $EX=0$, (2.4), Markov inequality and (2.0), when $n\to\infty$, then

$$
\begin{aligned}
&b_n^{-1}\max_{1\le j\le n}|\sum_{i=1}^{j} Ea_{ni}X_i^{(n)}|\\
&\le b_n^{-1}\sum_{i=1}^{n} E|a_{ni}X_i|I(|X_i|>b_n)\\
&= b_n^{-1}\sum_{i=1}^{n}|a_{ni}|E|X|I(|X|>b_n)\\
&\le Cb_n^{-1}nE|X|I(|X|>b_n)\\
&= Cb_n^{-1}n\sum_{k=n}^{\infty}E|X|I(b_k<|X|\le b_{k+1})\\
&\le Cb_n^{-1}n\sum_{k=n}^{\infty}b_{k+1}P(|X|>b_k)\\
&\le Cb_n^{-1}n\sum_{k=n}^{\infty}b_{k+1}\frac{E\exp(h|X|^\gamma)}{\exp(hb_k^\gamma)}\\
&\le Cb_n^{-1}n\sum_{k=n}^{\infty}(k+1)^{\frac{1}{\alpha}}(\log(k+1))^{\frac{1}{\gamma}}k^{-hk^{\gamma/\alpha}}\\
&\le Cn^{-\frac{1}{\alpha}}(\log n)^{-\frac{1}{\gamma}}nn^{-1}\\
&= Cn^{-\frac{1}{\alpha}}(\log n)^{-\frac{1}{\gamma}}\to 0.
\end{aligned}
\tag{2.5}
$$

$\sum_{i=1}^{n}|a_{ni}|^{\alpha}=O(n)$ implies that $\max_{1\le i\le n}|a_{ni}|=O(n^{\frac{1}{\alpha}})$. By this and Hölder inequality, $\forall k\ge\alpha$, then

$$\sum_{i=1}^{n}|a_{ni}|^{k}=\sum_{i=1}^{n}|a_{ni}|^{\alpha}|a_{ni}|^{k-\alpha}\le Cnn^{\frac{k-\alpha}{\alpha}}=Cn^{\frac{k}{\alpha}}. \tag{2.6}$$

When $0<\alpha\le 1$, using (2.6), Markov inequality and (2.0), when $n\to\infty$, then

$$\begin{aligned}
&b_n^{-1}\max_{1\le j\le n}\Big|\sum_{i=1}^{j}Ea_{ni}X_i^{(n)}\Big|\\
&\le b_n^{-1}\sum_{i=1}^{n}E|a_{ni}X_i|I(|X_i|\le b_n)\\
&=b_n^{-1}\sum_{i=1}^{n}|a_{ni}|E|X|I(|X|\le b_n)\\
&\le Cb_n^{-1}n^{\frac{1}{\alpha}}E|X|I(|X|\le b_n)\\
&=Cb_n^{-1}n^{\frac{1}{\alpha}}\sum_{k=2}^{n}E|X|I(b_{k-1}<|X|\le b_k)\\
&\le Cb_n^{-1}n^{\frac{1}{\alpha}}\sum_{k=2}^{n}b_kP(|X|>b_{k-1})\\
&\le Cb_n^{-1}n^{\frac{1}{\alpha}}\sum_{k=2}^{n}b_k\frac{E\exp(h|X|^{\gamma})}{\exp(hb_{k-1}^{\gamma})}\\
&\le Cb_n^{-1}n^{\frac{1}{\alpha}}\sum_{k=2}^{n}k^{\frac{1}{\alpha}}(\log k)^{\frac{1}{\gamma}}(k-1)^{-h(k-1)^{\gamma/\alpha}}\\
&\le Cn^{-\frac{1}{\alpha}}(\log n)^{-\frac{1}{\gamma}}n^{\frac{1}{\alpha}}\\
&=C(\log n)^{-\frac{1}{\gamma}}\to 0.
\end{aligned} \tag{2.7}$$

From (2.5) and (2.7), Hence (2.3) is true.

From (2.2) and (2.3), it follows that for n large enough

$$P\Big(\max_{1\le j\le n}|T_j|>\varepsilon b_n\Big)\le\sum_{j=1}^{n}P\big(|X_j|>b_n\big)+P\Big(\max_{1\le j\le n}|T_j^{(n)}|>\frac{\varepsilon}{2}b_n\Big).$$

Hence we need only to prove that

$$\begin{aligned}
I&=:\sum_{n=1}^{\infty}n^{-1}\sum_{j=1}^{n}P\big(|X_j|>b_n\big)<\infty,\\
II&=:\sum_{n=1}^{\infty}n^{-1}P\Big(\max_{1\le j\le n}|T_j^{(n)}|>\frac{\varepsilon}{2}b_n\Big)<\infty.
\end{aligned} \tag{2.8}$$

From the fact that $E\exp(h|X|^\gamma) < \infty$, it follows easily that

$$\begin{aligned} I &= \sum_{n=1}^{\infty} n^{-1} n P(|X| > b_n) \\ &= \sum_{n=1}^{\infty} P(|X| > b_n) \\ &\leq \sum_{n=1}^{\infty} \frac{E\exp(h|X|^\gamma)}{\exp(hb_n^\gamma)} \\ &\leq C \sum_{n=1}^{\infty} \frac{1}{n^{hn^{\gamma/\alpha}}} < \infty. \end{aligned} \tag{2.9}$$

By Lemma 2.1, it follows that for $q \geq 2$

$$\begin{aligned} II &\leq C \sum_{n=2}^{\infty} n^{-1} b_n^{-q} E \max_{1\leq j\leq n} |T_j^{(n)}|^q \\ &\leq C \sum_{n=2}^{\infty} n^{-1} b_n^{-q} \left\{ \sum_{j=1}^{n} E|a_{nj} X_j^{(n)}|^q + \left(\sum_{j=1}^{n} E|a_{nj} X_j^{(n)}|^2 \right)^{q/2} \right\} \\ &=: II_1 + II_2. \end{aligned} \tag{2.10}$$

Let $\max(2, \alpha, \gamma + 1) \leq q$, using (2.6), we have

$$\begin{aligned} II_1 &= C \sum_{n=2}^{\infty} n^{-1} b_n^{-q} \sum_{i=1}^{n} |a_{ni}|^q E|X|^q I(|X| \leq b_n) \\ &\leq C \sum_{n=2}^{\infty} n^{-1} b_n^{-q} n^{\frac{q}{\alpha}} E|X|^q I(|X| \leq b_n) \\ &= C \sum_{n=2}^{\infty} n^{-1} b_n^{-q} n^{\frac{q}{\alpha}} \sum_{k=2}^{n} E|X|^q I(b_{k-1} < |X| \leq b_k) \\ &\leq C \sum_{k=2}^{\infty} \sum_{n=k}^{\infty} n^{-1+\frac{q}{\alpha}} n^{-q/\alpha} (\log n)^{-q/\gamma} b_k^q P(|X| > b_{k-1}) \\ &\leq C \sum_{k=2}^{\infty} b_k^q \frac{E\exp(h|X|^\gamma)}{\exp(hb_{k-1}^\gamma)} \\ &\leq C \sum_{k=2}^{\infty} k^{\frac{q}{\alpha}} (\log k)^{\frac{q}{\gamma}} (k-1)^{-h(k-1)^{\gamma/\alpha}} < \infty. \end{aligned} \tag{2.11}$$

By $0 < \alpha < 2$, (2.6) and $q \geq (\gamma + 1)$, we have

$$\begin{aligned}
II_2 &= C\sum_{n=2}^{\infty} n^{-1} b_n^{-q}\Big(\sum_{i=1}^{n}|a_{ni}|^2\Big)^{q/2}\big(E|X|^2 I(|X| \leq b_n)\big)^{q/2}\\
&\leq C\sum_{n=2}^{\infty} n^{-1} b_n^{-q}\big(n^{\frac{2}{\alpha}}\big)^{q/2}\big(E|X|^2 I(|X| \leq b_n)\big)^{q/2}\\
&= C\sum_{n=2}^{\infty} n^{-1}(\log n)^{-q/\gamma}\Big(\sum_{k=2}^{n} E|X|^2 I(b_{k-1} < |X| \leq b_k)\Big)^{q/2}\\
&\leq C\sum_{n=2}^{\infty} n^{-1}(\log n)^{-q/\gamma}\Big(\sum_{k=2}^{n} b_k^2 P(|X| > b_{k-1})\Big)^{q/2}\\
&\leq C\sum_{n=2}^{\infty} n^{-1}(\log n)^{-q/\gamma}\Big(\sum_{k=2}^{n} b_k^2 \frac{E\exp(h|X|^{\gamma})}{\exp(hb_{k-1}^{\gamma})}\Big)^{q/2} \qquad (2.12)\\
&\leq C\sum_{n=2}^{\infty} n^{-1}(\log n)^{-q/\gamma}\Big(\sum_{k=2}^{n} \frac{k^{\frac{2}{\alpha}}(\log k)^{\frac{2}{\gamma}}}{\exp(h(k-1)^{\gamma/\alpha}\log(k-1))}\Big)^{q/2}\\
&= C\sum_{n=2}^{\infty} n^{-1}(\log n)^{-q/\gamma}\Big(\sum_{k=2}^{n} k^{\frac{2}{\alpha}}(\log k)^{\frac{2}{\gamma}}(k-1)^{-h(k-1)^{\gamma/\alpha}}\Big)^{q/2}\\
&\leq C\sum_{n=2}^{\infty} n^{-1}(\log n)^{-q/\gamma}\Big(\sum_{k=2}^{n} k^{-2}\Big)^{q/2}\\
&\leq C\sum_{n=2}^{\infty} n^{-1}(\log n)^{-q/\gamma} < \infty.
\end{aligned}$$

Putting (2.11) and (2.12) into (2.10) yields $II < \infty$. Now we complete the prove of Theorem 2.1.

COROLLARY 2.1. *Under the conditions of Theorem* 2.1, *then* $\lim_{n\to\infty}\frac{|T_n|}{b_n} = 0 \quad a.s.$

PROOF. By (2.1), we have

$$\begin{aligned}
\infty &> \sum_{n=1}^{\infty} n^{-1} P\big(\max_{1\leq j\leq n}|T_j| > \varepsilon b_n\big)\\
&= \sum_{i=0}^{\infty}\sum_{n=2^i}^{2^{i+1}-1} n^{-1} P\big(\max_{1\leq j\leq n}|T_j| > \varepsilon n^{\frac{1}{\alpha}}(\log n)^{\frac{1}{\gamma}}\big)\\
&\geq \frac{1}{2}\sum_{i=1}^{\infty} P\big(\max_{1\leq j\leq 2^i}|T_j| > \varepsilon 2^{\frac{i+1}{\alpha}}(\log 2^{i+1})^{\frac{1}{\gamma}}\big).
\end{aligned}$$

By Borel-Cantelli Lemma, we have

$$P\Big(\max_{1\leq j\leq 2^i}|T_j| > \varepsilon 2^{\frac{i+1}{\alpha}}(\log 2^{i+1})^{\frac{1}{\gamma}},\ \ i.o.\Big) = 0.$$

Hence

$$\lim_{i\to\infty} \frac{\max\limits_{1\le j\le 2^i} |T_j|}{2^{\frac{i+1}{\alpha}}(\log 2^{i+1})^{\frac{1}{\gamma}}} = 0 \ \ a.s.$$

and using

$$\max_{2^{i-1}\le n<2^i} \frac{|T_n|}{b_n} \le 2^{\frac{2}{\alpha}} \frac{\max\limits_{1\le j\le 2^i} |T_j|}{2^{\frac{i+1}{\alpha}}(\log 2^{i+1})^{\frac{1}{\gamma}}} \left(\frac{i+1}{i-1}\right)^{\frac{1}{\gamma}},$$

We have

$$\lim_{n\to\infty} \frac{|T_n|}{b_n} = 0 \ \ a.s.$$

REMARK 2.1. Corollary 2.1 generalizes the Theorem 2.2 of Bai and Cheng (2000) to ρ^*-mixing sequences of random variables and the restricton of b_n in Corollary 2.1 is weaker than the restricton of b_n in Theorem 2.2 of Bai and Cheng (2000).

ACKNOWLEDGMENTS

The author would like to thank an anonymous referee for his/her valuable comments. Research supported by National Natural Science Foundation of China.

RESUMO

Leis fortes são estabelecidas para estatísticas lineares que são somas ponderadas de uma amostra aleatória. Mostramos extensões das leis fortes de Marcinkiewicz-Zygmund sob certas condições tanto nos pesos quanto na distribuição. Estas últimas não só generalizam o resultado de Bai e Cheng (2000, Statist Probab Lett 46: 105-112) para sequências aleatórias "ρ^*-mixing" como também o melhoram.

Palavras-chave: "ρ^*-mixing", Marcinkiewicz-Zygmund leis fortes, somas ponderadas.

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