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New spatial central configurations in the 5-body problem

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ABSTRACT

In this paper we show the existence of new families of convex and concave spatial central configurations for the 5-body problem. The bodies studied here are arranged as follows: three bodies are at the vertices of an equilateral triangle $\mathcal{T}$, and the other two bodies are on the line passing through the barycenter of $\mathcal{T}$ that is perpendicular to the plane that contains $\mathcal{T}$.

Key words: central configurations, spatial configurations, 5-body problem, convex configuration, concave configuration.

INTRODUCTION

Consider n punctual positive masses $m_1, \ldots, m_n$ with position vectors $r_1, \ldots, r_n$. Usually $r_i \in \mathbb{R}^d$, $d = 2, 3$. The *Newtonian n-body problem* in celestial mechanics consists in studying the motion of theses masses interacting amongst themselves through no other forces than their mutual gravitational attraction according to Newton's gravitational law (Newton 1687).

In this paper we denote the Euclidean distance between the bodies of masses m_i and m_j by $r_{ij} = |r_i - r_j|$. We take the *inertial barycentric* system, that is the origin of the inertial system is located at the center of mass of the system, which is given by $\sum_{j=1}^{n} m_j r_j / M$, where $M = m_1 + \ldots + m_n$ is the total mass. The configuration space is defined by $\{(r_1, r_2, \ldots, r_n) \in \mathbb{R}^{dn} : r_i \neq r_j, i \neq j\}$.

At a given instant $t = t_0$ the n bodies make a *central configuration* if there exists $\lambda \neq 0$ such that $\ddot{r}_i = \lambda r_i$, for all $i = 1, \ldots, n$. Two central configurations $(r_1, r_2, \ldots, r_n)$, $(\bar{r}_1, \bar{r}_2, \ldots, \bar{r}_n)$ of the n bodies are said to be *related* if we can pass from one to the other through a dilation and a rotation (centered at the center of mass). So we can study the classes of central configurations defined by the above equivalence relation.

In the 3-body problem the collinear solutions of Euler (Euler 1767) and the triangular equilateral solutions of Lagrange (Lagrange 1873) are the first examples in which the bodies are in central configuration at any instant of time. The Euler collinear central configurations were generalized by Moulton

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in (Moulton 1910) who showed that, to given n masses, the number of collinear central configurations is exactly $n!/2$. It is know that planar regular n-gons with n equal masses at the vertices are in a central configuration. This is a generalization of Lagrange's result.

The knowledge of central configurations allows us to compute homographic solutions (see Moeckel 1990); there is a relation between central configurations and the bifurcations of the hypersurfaces of constant energy and angular momentum (see Smale 1970); if the n bodies are going to a simultaneous collision, then the bodies tend to a central configuration (see Saari 1980). See also the following references (Hagihara 1970, Moulton 1910, Wintner 1941).

Some examples of spatial central configurations are a regular tetrahedron with arbitrary positive masses at the vertices (Lehmann-Filhés 1891) and a regular octahedron with six equal masses at the vertices (Wintner 1941).

There are also the *stacked spatial central configurations*, that is central configurations for the n-body problem in which a proper subset of the n bodies is already on a central configuration. Double nested spatial central configurations for $2n$ bodies were studied for two nested regular polyhedra in (Corbera and Llibre 2008). More recently, the same authors studied central configurations of three regular polyhedra for the spatial $3n$-body problem in (Corbera and Llibre 2009). See also (Zhu 2005) in which nested regular tetrahedrons are studied.

Recently Hampton and Santoprete (Hampton and Santoprete 2007) provided new examples of stacked spatial central configurations for the 7-body problem in which the bodies are arranged as concentric three and two dimensional simplexes. New classes of stacked spatial central configurations for the 6-body problem that have four bodies at the vertices of a regular tetrahedron and the other two bodies on a line connecting one vertex of the tetrahedron with the center of the opposite face are studied in (Mello et al. 2009a).

In this paper we study spatial central configurations for the 5-body problem that satisfy (see Fig. 1(a) and 1(b)):

1. The position vectors r_1, r_2 and r_3 are at the vertices of an equilateral triangle $\mathcal{T}$;

2. Let ρ be the line passing through the barycenter of $\mathcal{T}$ that is perpendicular to the plane that contains $\mathcal{T}$. The position vector $r_4 \in \rho$ is fixed and does not belong to the plane that contains $\mathcal{T}$;

3. The position vector $r_5 \in \rho$, $r_5 \neq r_4$.

This type of configuration is called *concave* if one body is located in the interior of the convex hull of the other four bodies (see Fig. 1(a)), otherwise the configuration is called *convex* (see Fig. 1(b)). We say that the configuration is *concave of type 1* if the body 5 is in the interior of the convex hull of the bodies 1, 2, 3 and 4. On the other hand, if the body 4 is in the interior of the convex hull of the bodies 1, 2, 3 and 5, we say that the configuration is *concave of type 2*.

In order to be more precise and without loss of generality, consider a coordinate system such that $r_1 = (x, 0, 0)$, $r_2 = (-x/2, -\sqrt{3}x/2, 0)$, $r_3 = (-x/2, \sqrt{3}x/2, 0)$, $r_4 = (0, 0, \sqrt{6}/3)$ and $r_5 = (0, 0, y)$ with $x > 0$ and $y \neq \sqrt{6}/3$. See Fig. 2. Thus, x is the radius of the circumscribed circle that contains r_1, r_2, r_3, and y is the signed height of the body 5 with respect to the plane that contains the triangle

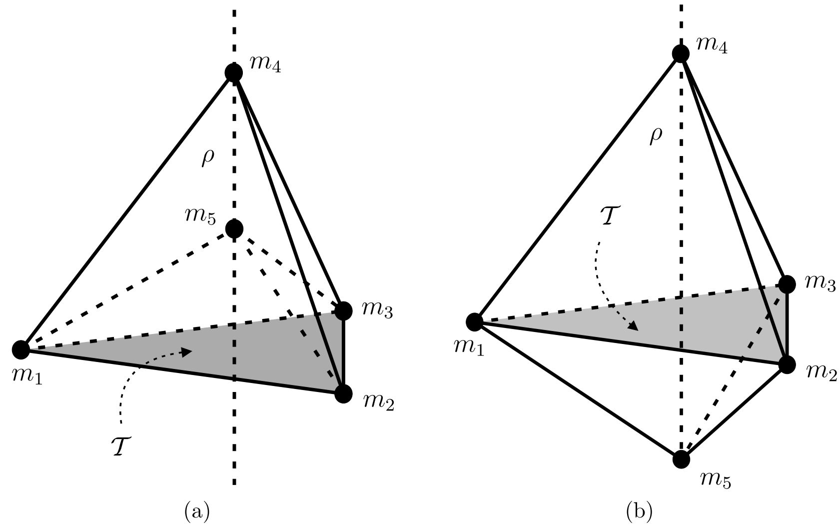

Fig. 1 – Spatial central configurations for the 5-body problem. Concave configuration (a). Convex configuration (b).

$\mathcal{T}$. There is no special reason to take r_4 at $(0, 0, \sqrt{6}/3)$. With this choice, the length of the edge of the regular tetrahedron is 1. Note that the body 5 is at the barycenter of $\mathcal{T}$ if and only if $y = 0$. Therefore, the configuration is convex if and only if $y \leq 0$, and the configuration is concave of type 1 (type 2, respectively) if and only if $0 < y < \sqrt{6}/3$ ($y > \sqrt{6}/3$, respectively).

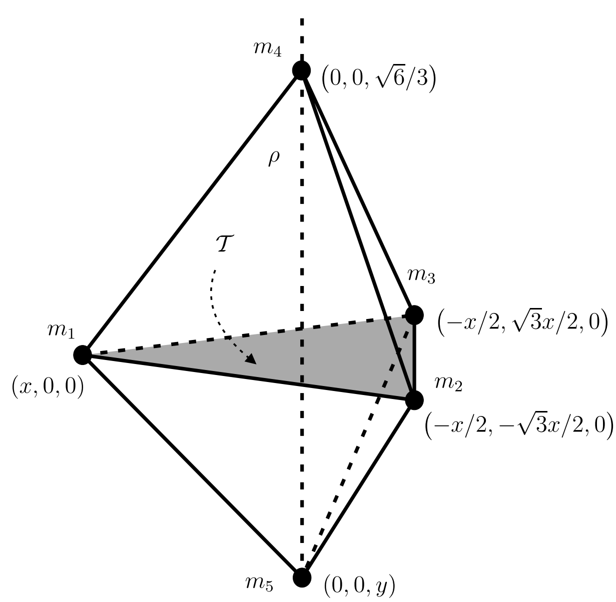

Fig. 2 – Coordinates for the five bodies.

As far as we know, the spatial central configurations studied here are new and are, in a certain sense, generalizations of the kite (planar) central configurations (Bernat et al. 2009, Yiming and Shanzhong 2002). See also (Mello et al. 2009b). Leandro in (Leandro 2003) also studied the central configurations presented here from another point of view. More precisely, Leandro studied the finiteness and bifurcations of this class of central configurations.

The main results of this paper are the following ones.

THEOREM 1. *Consider the position vectors*

$$r_1 = (x, 0, 0)\,, \quad r_2 = \left(-x/2, -\sqrt{3}x/2, 0\right), \quad r_3 = \left(-x/2, \sqrt{3}x/2, 0\right)$$

at the vertices of an equilateral triangle $\mathcal{T}$, and the position vectors $r_4 = \left(0, 0, \sqrt{6}/3\right)$, $r_5 = (0, 0, y)$ on ρ, where ρ is the line passing through the barycenter of $\mathcal{T}$ and perpendicular to the plane that contains $\mathcal{T}$, according to Fig. 2. In this way, the following statements hold.

1. *There exists a minimum positive value $x = x_{\min} = (3\sqrt{2} - 2\sqrt{3})/3$ such that if $0 < x \leq x_{\min}$ there are no positions $r_1, \ldots, r_5$ and positive masses $m_1, \ldots, m_5$ such that these bodies are in a central configuration according to Fig. 2;*

2. *There are two open intervals $I_1 = \left((3\sqrt{2} - 2\sqrt{3})/3, \sqrt{3}/3\right)$ and $I_2 = \left(\sqrt{3}/3, \sqrt{6}/3\right)$ such that for each $x \in I_1 \cup I_2$ there is one non-empty segment of possible positions for r_5 and positive masses $m_1, \ldots, m_5$ such that these bodies form a 1-parameter family of concave central configurations of type 1;*

3. *There exists one distinguished and well-determined value $x = \bar{x}$ for which there is just one position for r_5 such that these five bodies form a 2-parameter family of concave central configurations of type 1. Indeed this concave central configuration is exactly the well-known central configuration whose four equal masses are at the vertices of a regular tetrahedron and the fifth mass is at the center of the tetrahedron;*

4. *There exists one open interval $I_3 = \left(\sqrt{3}/3, (2\sqrt{3} + 3\sqrt{2})/3\right)$ such that for each $x \in I_3$ there is one non-empty segment of possible positions for r_5 and positive masses $m_1, \ldots, m_5$ such that these bodies form a 1-parameter family of convex central configurations;*

5. *There exists one open and unbounded interval $I_4 = \left(4\sqrt{3}/3, +\infty\right)$ such that for each $x \in I_4$ there is one non-empty segment of possible positions for r_5 and positive masses $m_1, \ldots, m_5$ such that these bodies form a 1-parameter family of concave central configurations of type 2;*

6. *There exists one distinguished and well-determined value $x = \bar{\bar{x}}$ for which there is just one position for r_5 such that these five bodies form a 2-parameter family of concave central configurations of type 2. As in the above item 3, this concave central configuration is exactly the one whose four equal masses are at the vertices of a regular tetrahedron and the fifth mass is at the center of the tetrahedron.*

REMARK 2. Items 2, 3 and 4 of Theorem 1 are closely related with some results obtained by Leandro in (Leandro 2003). However, the assumptions, statements and proofs presented here are different and much simpler than those that appear in (Leandro 2003). Items 3 and 6 of Theorem 1 are not new. We have included them here for completeness.

The proof of Theorem 1 is given in the next section. Concluding comments are presented in Section 3.

PROOF OF THEOREM 1

The equations of motion of the n-body problem are given by

$$\ddot{r}_i = - \sum_{\substack{j=1\\ j\neq i}}^{n} m_j \frac{r_i - r_j}{r_{ij}^3}, \tag{1}$$

for $i = 1, 2, \ldots, n$. In (1) the gravitational constant is taken equal to one.

From the definition of central configuration, equation (1) can be written as

$$\lambda r_i = - \sum_{\substack{j=1\\ j\neq i}}^{n} m_j \frac{r_i - r_j}{r_{ij}^3}, \tag{2}$$

for $i = 1, 2, \ldots, n$. For the planar case, that is $d = 2$, simple computations allow us to write equation (2) in the following form

$$f_{ij} = \sum_{\substack{k=1\\ k\neq i,j}}^{n} m_k \left(R_{ik} - R_{jk}\right) \Delta_{ijk} = 0, \tag{3}$$

for $1 \leq i < j \leq n$, where $R_{ij} = 1/r_{ij}^3$ and $\Delta_{ijk} = (r_i - r_j) \wedge (r_i - r_k)$. In fact, Δ_{ijk} is twice the oriented area of the triangle formed by the bodies of masses m_i, m_j and m_k. See the references (Hagihara 1970, Mello et al. 2009b). These $n(n-1)/2$ equations are called Dziobek or Laura-Andoyer equations.

The computation of spatial central configuration is very difficult if we begin with equation (2). Instead of working with equation (2) we shall use another equivalent system of equations (see equation (6), p. 295 of (Hampton and Santoprete 2007) and the references therein)

$$f_{ijh} = \sum_{\substack{k=1\\ k\neq i,j,h}}^{n} m_k \left(R_{ik} - R_{jk}\right) \Delta_{ijhk} = 0, \tag{4}$$

for $1 \leq i < j \leq n$, $h = 1, \ldots, n$, $h \neq i, j$. Here, again, $R_{ij} = 1/r_{ij}^3$ and $\Delta_{ijhk} = (r_i - r_j) \wedge (r_j - r_h) \cdot (r_h - r_k)$. Thus, Δ_{ijhk} gives six times the signed volume of the tetrahedron formed by the bodies of masses m_i, m_j, m_h and m_k. These $n(n-1)(n-2)/2$ equations are also called here Dziobek equations.

For the proof of Theorem 1 we use Dziobek equations (4). For five bodies, system (4) is a set of 30 equations. From the distances between the bodies it follows that $R_{12} = R_{13} = R_{23}$, $R_{14} = R_{24} = R_{34}$ and $R_{15} = R_{25} = R_{35}$. Taking into account the symmetries, we have the following equalities among others:

$$\Delta_{1425} = -\Delta_{1435} = -\Delta_{2415} = \Delta_{2435} = \Delta_{3415} = -\Delta_{3425},$$

$$\Delta_{1523} = -\Delta_{1532} = -\Delta_{2513} = \Delta_{2531} = \Delta_{3512} = -\Delta_{3521},$$

$$\Delta_{1524} = -\Delta_{1534} = -\Delta_{2514} = \Delta_{2534} = \Delta_{3514} = -\Delta_{3524}.$$

Taking these symmetries into the Dziobek equations, the following 9 equations of (4) are trivially satisfied $f_{123} = 0$, $f_{124} = 0$, $f_{125} = 0$, $f_{132} = 0$, $f_{134} = 0$, $f_{135} = 0$, $f_{231} = 0$, $f_{234} = 0$ and $f_{235} = 0$.

Again looking for symmetries, we have the following equivalence between the equations:

$$f_{145} = 0 \Leftrightarrow (R_{12} - R_{24})\,\Delta_{1452}(m_2 - m_3) = 0, \tag{5}$$

$$f_{245} = 0 \Leftrightarrow (R_{12} - R_{14})\,\Delta_{2451}(m_1 - m_3) = 0, \tag{6}$$

$$f_{345} = 0 \Leftrightarrow (R_{13} - R_{14})\,\Delta_{3451}(m_1 - m_2) = 0. \tag{7}$$

As our classes of central configurations satisfy $r_4 \neq r_5$, we have $\Delta_{1452} \neq 0$, $\Delta_{2451} \neq 0$ and $\Delta_{3451} \neq 0$. From equations (5), (6) and (7), we have two cases to analyze: the masses $m_1 = m_2 = m_3$ and r_1, r_2, r_3 and r_4 are at the vertices of a regular tetrahedron.

CASE 1. Consider r_1, r_2, r_3 and r_4 at the vertices of a regular tetrahedron. We have the following lemma.

LEMMA 3. *Consider r_1, r_2, r_3 and r_4 at the vertices of a regular tetrahedron. Then, there exists just one position for r_5 at the center of the tetrahedron and positive masses $m_1 = m_2 = m_3 = m_4 = m$ and m_5 such that these bodies form a 2-parameter family of concave central configurations of type 1.*

PROOF. Due to symmetries, we have the following equivalences between equations of (4):

$$f_{142} = 0 \Leftrightarrow f_{241} = 0, \quad f_{143} = 0 \Leftrightarrow f_{341} = 0, \quad f_{243} = 0 \Leftrightarrow f_{342} = 0,$$

$$f_{152} = 0 \Leftrightarrow f_{251} = 0, \quad f_{153} = 0 \Leftrightarrow f_{351} = 0, \quad f_{253} = 0 \Leftrightarrow f_{352} = 0.$$

Hence the remaining 21 equations are reduced to 15 equations. From $f_{142} = 0$, $f_{143} = 0$ and $f_{243} = 0$ we have $m_5\,(R_{15} - R_{45})\,\Delta_{1425} = 0$, $m_5\,(R_{15} - R_{45})\,\Delta_{1435} = 0$ and $m_5\,(R_{25} - R_{45})\,\Delta_{2435} = 0$, respectively. As $\Delta_{1425} \neq 0$, $\Delta_{1435} \neq 0$ and $\Delta_{2435} \neq 0$, it follows that $R_{15} = R_{45} = R_{25}$. This implies that r_5 must be at the center of the tetrahedron. Adding this information into $f_{152} = 0$, $f_{153} = 0$ and $f_{253} = 0$, we have $(m_3 - m_4)\,(R_{13} - R_{35})\,\Delta_{1523} = 0$, $(m_2 - m_4)\,(R_{12} - R_{25})\,\Delta_{1532} = 0$ and $(m_1 - m_4)\,(R_{12} - R_{15})\,\Delta_{2531} = 0$, respectively. These last three equations are verified just when $m_1 = m_2 = m_3 = m_4$. By another side m_5 can assume any positive value. The remaining equations of (4) are trivially satisfied. The lemma is proved. □

Define $\bar{x} = \sqrt{3}/3$. Thus, $r_1 = (\sqrt{3}/3, 0, 0)$, $r_2 = (-\sqrt{3}/6, -1/2, 0)$, $r_3 = (-\sqrt{3}/6, 1/2, 0)$, $r_4 = (0, 0, \sqrt{6}/3)$ and $r_5 = (0, 0, \sqrt{6}/12)$. From Lemma 3, item 3 of Theorem 1 is proved. In the remainder of the statements we omit this type of concave central configuration.

CASE 2. Consider $m_1 = m_2 = m_3 = m$. It follows that equations (5), (6), (7), $f_{154} = 0$, $f_{451} = 0$, $f_{254} = 0$, $f_{452} = 0$, $f_{354} = 0$ and $f_{453} = 0$ are satisfied, and

$$f_{142} = 0 \Leftrightarrow f_{143} = 0 \Leftrightarrow f_{241} = 0 \Leftrightarrow f_{243} = 0 \Leftrightarrow f_{341} = 0 \Leftrightarrow f_{342} = 0,$$

$$f_{152} = 0 \Leftrightarrow f_{153} = 0 \Leftrightarrow f_{251} = 0 \Leftrightarrow f_{253} = 0 \Leftrightarrow f_{351} = 0 \Leftrightarrow f_{352} = 0.$$

In other words, the initial 30 equations were reduced to 2 equations, which are the following:

$$f_{142} = m\,(R_{13} - R_{34})\,\Delta_{1423} + m_5\,(R_{15} - R_{45})\,\Delta_{1425} = 0, \tag{8}$$

$$f_{152} = m\,(R_{13} - R_{35})\,\Delta_{1523} + m_4\,(R_{14} - R_{45})\,\Delta_{1524} = 0. \tag{9}$$

Equations (8) and (9) can be explicitly solved in the form $m_4 = m_4(x, y, m)$ and $m_5 = m_5(x, y, m)$, respectively. In these equations, m can be understood as a parameter for the central configurations.

From equations (8) and (9) we have

$$\frac{m_4}{m} = \frac{(R_{35} - R_{13})\,\Delta_{1523}}{(R_{14} - R_{45})\,\Delta_{1524}}, \tag{10}$$

$$\frac{m_5}{m} = \frac{(R_{34} - R_{13})\,\Delta_{1423}}{(R_{15} - R_{45})\,\Delta_{1425}}. \tag{11}$$

We wish to find subsets of $\mathcal{D} = \{x > 0, y \in \mathbb{R}, y \neq \sqrt{6}/3\}$ whose ratios of the masses m_4/m and m_5/m are positive. For the study of the signs of the terms that appear in equations (10) and (11), we have $R_{35} - R_{13} = 0$ if and only if $(x, y) \in \{x > 0, y = -\sqrt{2}x\} \cup \{x > 0, y = \sqrt{2}x\}$ (straight lines), $R_{14} - R_{45} = 0$ if and only if $(x, y) \in \{x > 0, y = \sqrt{6}/3 - \sqrt{9x^2 + 6}/3\} \cup \{x > 0, y = \sqrt{6}/3 + \sqrt{9x^2 + 6}/3\}$ (hyperbolas), $R_{34} - R_{13} = 0$ if and only if $(x, y) \in \{x = \sqrt{3}/3\}$, $R_{15} - R_{45} = 0$ if and only if $(x, y) \in \{x > 0, y = \sqrt{6}(2 - 3x^2)/12\}$ (parabola), $\Delta_{1523} = 0$ if and only if $(x, y) \in \{x > 0, y = 0\}$, $\Delta_{1524} = 0$ if and only if $(x, y) \in \{x > 0, y = \sqrt{6}/3\}$, $\Delta_{1425} = 0$ if and only if $(x, y) \in \{x > 0, y = \sqrt{6}/3\}$. See Figs. 3 and 4.

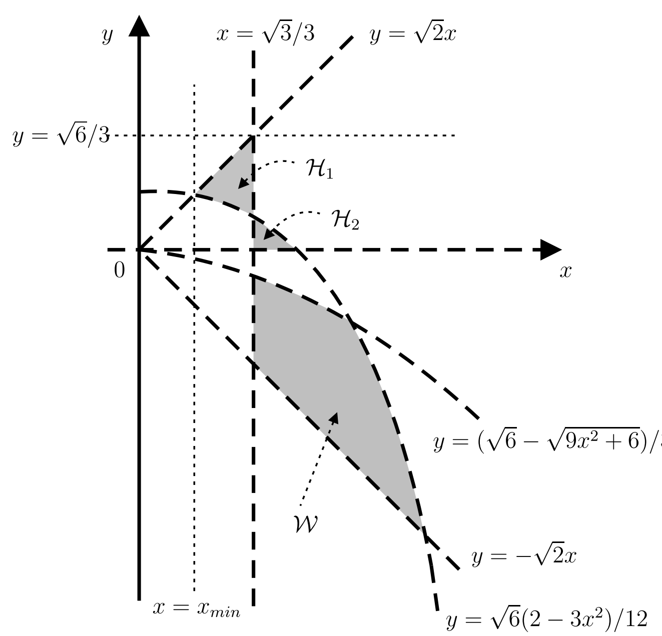

Fig. 3 – The open regions $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{W}$.

CASE 2.1. Consider $x > 0$ and $y = 0$.

LEMMA 4. *Consider* $r_5 = (0, 0, 0)$*, that is* $y = 0$*. Then, there is no value* $x > 0$ *such that 5 bodies with positive masses form a central configuration according to Fig. 2.*

PROOF. From equation (9) we have

$$m\,(R_{13} - R_{35})\,\Delta_{1523} + m_4\,(R_{14} - R_{45})\,\Delta_{1524} = 0.$$

By assumption, $\Delta_{1523} = 0$. As $(R_{14} - R_{45}) \neq 0$ and $\Delta_{1524} \neq 0$; then, $m_4 = 0$. This is a contradiction. □

CASE 2.2. Consider $0 < x \leq (3\sqrt{2} - 2\sqrt{3})/3$ and $y \neq \sqrt{6}/3$. We have the following lemma.

LEMMA 5. *Consider* $0 < x \leq (3\sqrt{2} - 2\sqrt{3})/3$*. Then, there is no position for the body 5 on the line* ρ *and positive masses* m_i*,* $i = 1, \ldots, 5$ *such that these bodies form a central configuration according to Fig. 2.*

PROOF. There are three cases to analyze: $y < 0$, $0 < y < \sqrt{6}/3$ and $y > \sqrt{6}/3$.

Consider $y < 0$. From equation (8) we have

$$m\,(R_{13} - R_{34})\,\Delta_{1423} + m_5\,(R_{15} - R_{45})\,\Delta_{1425} = 0.$$

By assumption $R_{13} - R_{34} < 0$, $R_{15} - R_{45} < 0$, $\Delta_{1423} > 0$ and $\Delta_{1425} > 0$. Therefore, the coefficients of the above equation have the same sign. This implies that m and m_5 must have opposite signs.

Consider $0 < y < \sqrt{6}/3$. By assumption $R_{13} - R_{34} < 0$, $R_{13} - R_{35} > 0$, $R_{14} - R_{45} < 0$, $\Delta_{1423} > 0$, $\Delta_{1425} > 0$, $\Delta_{1523} > 0$, $\Delta_{1524} < 0$, $R_{15} - R_{45} < 0$, if $0 < y < \sqrt{6}(2 - 3x^2)/12$ and $R_{15} - R_{45} > 0$, if $\sqrt{6}(2 - 3x^2)/12 < y < \sqrt{6}/3$.

From equation (8) we have

$$m\,(R_{13} - R_{34})\,\Delta_{1423} + m_5\,(R_{15} - R_{45})\,\Delta_{1425} = 0.$$

If $0 < y < \sqrt{6}(2 - 3x^2)/12$, then the coefficients of the above equation have the same sign. Therefore, the masses m and m_5 have opposite signs. Now, from equation (9), we have

$$m\,(R_{13} - R_{35})\,\Delta_{1523} + m_4\,(R_{14} - R_{45})\,\Delta_{1524} = 0.$$

If $\sqrt{6}(2 - 3x^2)/12 < y < \sqrt{6}/3$, then the coefficients of the above equation have the same sign. Therefore, the masses m and m_4 have opposite signs. If $y = \sqrt{6}(2 - 3x^2)/12$, then $R_{15} - R_{45} = 0$ and this implies that the mass m must be zero in equation (8).

Consider $y > \sqrt{6}/3$. From equation (8) we have

$$m\,(R_{13} - R_{34})\,\Delta_{1423} + m_5\,(R_{15} - R_{45})\,\Delta_{1425} = 0.$$

By assumption $R_{13} - R_{34} < 0$, $R_{15} - R_{45} > 0$, $\Delta_{1423} > 0$ and $\Delta_{1425} < 0$. Therefore, the coefficients of the above equation have the same sign. This implies that m and m_5 must have opposite signs. □

From Lemma 5 item 1 of Theorem 1 is proved. The value $x_{\min} = (3\sqrt{2} - 2\sqrt{3})/3$ is defined as the x-coordinate of an intersection of the curves $R_{35} - R_{13} = 0$ and $R_{15} - R_{45} = 0$ (see Fig. 3).

CASE 2.3. Consider $x > 0$ and $0 < y < \sqrt{6}/3$.

Define $\mathcal{H} = \mathcal{H}_1 \cup \mathcal{H}_2$ (see Fig. 3) where

$$\mathcal{H}_1 = \left\{ \frac{3\sqrt{2} - 2\sqrt{3}}{3} < x < \frac{\sqrt{3}}{3}, \frac{\sqrt{6}(2 - 3x^2)}{12} < y < \sqrt{2}x \right\},$$

$$\mathcal{H}_2 = \left\{ \frac{\sqrt{3}}{3} < x < \frac{\sqrt{6}}{3}, 0 < y < \frac{\sqrt{6}(2 - 3x^2)}{12} \right\}.$$

For $(x, y) \in \mathcal{H}_1$ we have $x > 0$, $0 < y < \sqrt{6}/3$, $R_{35} - R_{13} > 0$, $R_{34} - R_{13} < 0$, $R_{15} - R_{45} < 0$, $R_{14} - R_{45} < 0$, $\Delta_{1523} > 0$, $\Delta_{1524} < 0$, $\Delta_{1423} > 0$ and $\Delta_{1425} > 0$. For $(x, y) \in \mathcal{H}_2$ we have $x > 0$, $0 < y < \sqrt{6}/3$, $R_{35} - R_{13} > 0$, $R_{34} - R_{13} > 0$, $R_{15} - R_{45} > 0$, $R_{14} - R_{45} < 0$, $\Delta_{1523} > 0$, $\Delta_{1524} < 0$, $\Delta_{1423} > 0$ and $\Delta_{1425} > 0$.

It is simple to see that for $(x, y) \in \mathcal{H}$ the right-hand sides of equations (10) and (11) are positive and, therefore, we have concave central configurations of type 1.

The orthogonal projections of the open sets $\mathcal{H}_1$ and $\mathcal{H}_2$ onto the x-axis give two open intervals by $I_1 = \big((3\sqrt{2} - 2\sqrt{3})/3, \sqrt{3}/3\big)$ and $I_2 = \big(\sqrt{3}/3, \sqrt{6}/3\big)$, respectively. For each $x_* \in I_1 \cup I_2$ the straight line $x = x_*$ intersects $\mathcal{H}_1 \cup \mathcal{H}_2$ in a non-empty segment. This proves item 2 of Theorem 1.

CASE 2.4. Consider $x > 0$ and $y < 0$.

In order to have convex central configurations, it is necessary that $y < 0$. Define the open set $\mathcal{W} = \mathcal{W}_1 \cup \mathcal{W}_2$ (see Fig. 3), where

$$\mathcal{W}_1 = \left\{ \frac{\sqrt{3}}{3} < x \leq \sqrt{2}, -\sqrt{2}x < y < \frac{\sqrt{6}}{3} - \frac{\sqrt{9x^2 + 6}}{3} \right\},$$

$$\mathcal{W}_2 = \left\{ \sqrt{2} \leq x < \frac{2\sqrt{3} + 3\sqrt{2}}{3}, -\sqrt{2}x < y < \frac{\sqrt{6}(2 - 3x^2)}{12} \right\}.$$

For $(x, y) \in \mathcal{W}$ we have $x > 0$, $y < 0$, $R_{35} - R_{13} > 0$, $R_{34} - R_{13} > 0$, $R_{15} - R_{45} > 0$, $R_{14} - R_{45} > 0$, $\Delta_{1523} < 0$, $\Delta_{1524} < 0$, $\Delta_{1423} > 0$ and $\Delta_{1425} > 0$.

It is simple to see that, for $(x, y) \in \mathcal{W}$, the right-hand sides of equations (10) and (11) are positive and, therefore, we have convex central configurations.

The orthogonal projection of the open set $\mathcal{W}$ onto the x-axis gives one open interval

$$I_3 = \left(\sqrt{3}/3, (2\sqrt{3} + 3\sqrt{2})/3\right).$$

For each $x_* \in I_3$ the straight line $x = x_*$ intersects $\mathcal{W}$ in a non-empty segment. This proves item 4 of Theorem 1.

CASE 2.5. Consider $x > 0$ and $y > \sqrt{6}/3$.

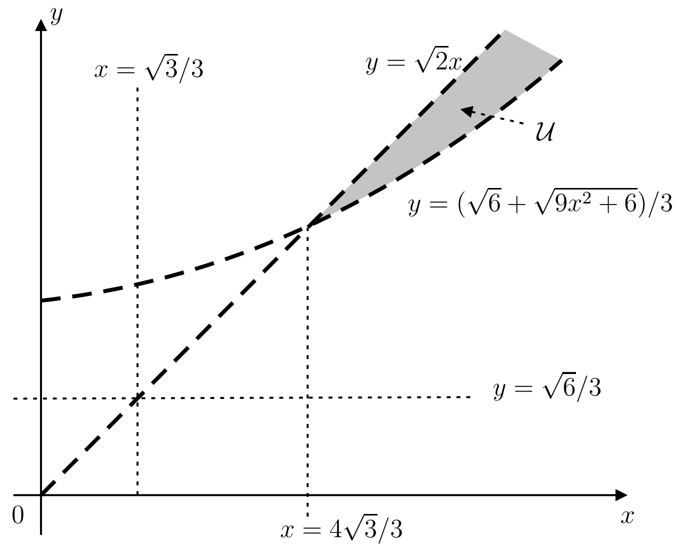

Fig. 4 – The open region $\mathcal{U}$.

Define the unbounded open set (see Fig. 4)

$$\mathcal{U} = \left\{ \frac{4\sqrt{3}}{3} < x, \frac{\sqrt{6} + \sqrt{9x^2+6}}{3} < y < \sqrt{2}x \right\}.$$

For $(x, y) \in \mathcal{U}$ we have $x > 0$, $y > \sqrt{6}/3$, $R_{35} - R_{13} > 0$, $R_{34} - R_{13} > 0$, $R_{15} - R_{45} < 0$, $R_{14} - R_{45} > 0$, $\Delta_{1523} > 0$, $\Delta_{1524} > 0$, $\Delta_{1423} > 0$ and $\Delta_{1425} < 0$.

It is simple to see that, for $(x, y) \in \mathcal{U}$, the right-hand sides of equations (10) and (11) are positive and, therefore, we have concave central configurations of type 2.

The orthogonal projection of the open set $\mathcal{U}$ onto the x-axis gives one open and unbounded interval $I_4 = (4\sqrt{3}/3, +\infty)$. For each $x_* \in I_4$ the straight line $x = x_*$ intersects $\mathcal{U}$ in a non-empty segment. This proves item 5 of Theorem 1.

CASE 2.6. Consider r_1, r_2, r_3 and r_5 at the vertices of a regular tetrahedron. We have the following lemma.

LEMMA 6. *If $m_1 = m_2 = m_3 = m_5 = m$ and m_4 are at $r_1 = \left(4\sqrt{3}/3, 0, 0\right)$, $r_2 = \left(-2\sqrt{3}/3, -2, 0\right)$, $r_3 = \left(-2\sqrt{3}/3, 2, 0\right)$, $r_4 = \left(0, 0, \sqrt{6}/3\right)$ and $r_5 = \left(0, 0, 4\sqrt{6}/3\right)$, then these bodies form a 2-parameter family of concave central configurations of type 2.*

PROOF. By assumption, $x = 4\sqrt{3}/3$ and $y = 4\sqrt{6}/3$. For these values, we have $R_{13} - R_{35} = 0$ and $R_{14} - R_{45} = 0$. Therefore, equation (9) is satisfied for all $m_4 > 0$. As $m_5 = m$, equation (8) can be written as

$$m\big[(R_{13} - R_{34})\Delta_{1423} + (R_{15} - R_{45})\Delta_{1425}\big] = 0.$$

With the above values of x and y, we have $r_{13} = r_{15} = 4$, $r_{34} = r_{45} = \sqrt{6}$, $\Delta_{1423} = -\Delta_{1425} = 8\sqrt{2}$. By a simple calculation, $(R_{13} - R_{34})\Delta_{1423} + (R_{15} - R_{45})\Delta_{1425} = 0$. It is simple to see that r_1, r_2, r_3 and r_5 are at the vertices of a regular tetrahedron, and r_4 is at the center of this tetrahedron. □

The proof of item 6 of Theorem 1 follows Lemma 6 in which $\bar{\bar{x}} = 4\sqrt{3}/3$. In short we have proved Theorem 1.

CONCLUDING COMMENTS

In this paper it was shown the existence (and the nonexistence) of concave/convex spatial central configurations in the 5-body problem. The spatial central configurations studied here are generalizations of the kite planar central configurations.

Some of the results presented here (see Remark 2) are closely related with the results presented by Leandro in (Leandro 2003). However, the methods used in the two articles are different. The techniques used here are very simple, and the use of the computer is not necessary in the proofs of the results presented (see Theorem 1 and its proof). In this sense, the two articles are complementary.

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RESUMO

Neste artigo estudamos a existência de novas famílias de configurações centrais espaciais dos tipos côncavas ou convexas para o problema de 5 corpos. Os corpos estudados aqui estão dispostos da seguinte maneira: três corpos estão sobre os vértices de um triângulo equilátero $\mathcal{T}$ e os outros dois corpos estão sobre a reta que passa pelo baricentro de $\mathcal{T}$ e é perpendicular ao plano que contém $\mathcal{T}$.

Palavras-chave: configurações centrais, configurações espaciais, problema de 5 corpos, configuração convexa, configuração côncava.

REFERENCES

BERNAT J, LLIBRE J AND PÉREZ-CHAVELA E. 2009. On the planar central configurations of the 4-body problem with three equal masses. Dynam Contin Discrete Impuls Systems 16: 1–13.

CORBERA M AND LLIBRE J. 2008. Central configurations of nested regular polyhedra for the spatial $2n$-body problem. J Geom Phys 58: 1241–1252.

CORBERA M AND LLIBRE J. 2009. Central configurations of three regular polyhedra for the spatial $3n$-body problem. J Geom Phys 59: 321–339.

EULER L. 1767. De moto rectilineo trium corporum se mutuo attahentium. Novi Comm Acad Sci Imp Petrop 11: 144–151.

HAGIHARA Y. 1970. Celestial Mechanics, vol. 1, Massachusetts: MIT Press, 720 p.

HAMPTON M AND SANTOPRETE M. 2007. Seven-body central configurations: a family of central configurations in the spatial seven-body problem. Celestial Mech Dyn Astr 99: 293–305.

LAGRANGE JL. 1873. Essai sur le problème de trois corps, Oeuvres, vol. 6, Paris: Gauthier-Villars, 832 p.

LEANDRO ESG. 2003. Finiteness and bifurcations of some symmetrical classes of central configurations. Arch Rational Mech Anal 167: 147–177.

LEHMANN-FILHÉS R. 1891. Ueber zwei Fälle des Vielkörperproblems. Astr Naschr 127: 137–144.

MELLO LF, CHAVES FE AND FERNANDES AC. 2009b. Configurações centrais planares do tipo pipa. Rev Bras Ens Fis 31: 1–7, (in Portuguese).

MELLO LF, CHAVES FE, FERNANDES AC AND GARCIA BA. 2009a. Stacked central configurations for the spatial 6-body problem. J Geom Phys 59: 1216–1226.

MOECKEL R. 1990. On central configurations. Math Z 205: 499–517.

MOULTON FR. 1910. The straight line solutions of n bodies. Ann of Math 12: 1–17.

NEWTON I. 1687. Philosophi Naturalis Principia Mathematica. London: Royal Society, 512 p.

SAARI D. 1980. On the role and properties of central configurations. Celestial Mech 21: 9–20.

SMALE S. 1970. Topology and mechanics II: The planar n-body problem. Invent Math 11: 45–64.

WINTNER A. 1941. The Analytical Foundations of Celestial Mechanics. Princeton: Princeton University Press, 448 p.

YIMING L AND SHANZHONG S. 2002. Four-body central configurations with some equal masses. Arch Rational Mech Anal 162: 25–44.

ZHU C. 2005. Central configurations of nested regular tetrahedrons. J Math Anal Appl 312: 83–92.