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The Gauss Map of Complete Minimal Surfaces with Finite Total Curvature

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ABSTRACT

In this paper we are concerned with the image of the normal Gauss map of a minimal surface immersed in $\mathbb{R}^3$ with finite total curvature. We give a different proof of the following theorem of R. Osserman: *The normal Gauss map of a minimal surface immersed in* $\mathbb{R}^3$ *with finite total curvature, which is not a plane, omits at most three points o*f $\mathbb{S}^2$.
Moreover, under an additional hypothesis on the type of ends, we prove that this number is exactly 2.

Key words: Gauss map, minimal surfaces, Finite total curvature, Image of the Gauss map.

INTRODUCTION

Complete minimal surfaces immersed or embedded in $\mathbb{R}^3$ constitute a honorable field of research. In particular, many geometers studied with special emphasis the Gauss map properties of such surfaces. Several important results have been shown and many other problems have emerged, among them the following one: let $X : M \to \mathbb{R}^3$ be a complete non-planar minimal immersion and let $N : M \to \mathbb{S}^2 \cong \mathbb{C} \cup \{\infty\}$ be its Gauss map. How many points can be omitted by N?

In 1961 R. Osserman (Osserman 1961) proved that N does not omit more than one set of logarithmic capacity zero. In 1981 F. Xavier (Xavier 1981) proved that N can omit at most six points. In 1988, H. Fujimoto (Fujimoto 1988) refined Xavier's result showing that N omits at most four points. This is the best possible result since there are complete immersed minimal surfaces in $\mathbb{R}^3$ whose Gauss map omits exactly four values. An example of such a surface is the Scherk's surface and other examples can be found in the book of R. Osserman (Osserman 1986). However, these examples have infinite total curvature. This is an expected fact since R. Osserman (Osserman 1964) proved that if M has finite total curvature, then N omits at most three points. Until now examples of complete minimal surfaces with finite total curvature whose Gauss map omits three points of $\mathbb{S}^2$ are not known. Hence, the maximum number of points that can be omitted by N is either two or three. By trying to answer this question we naturally ask what are the conditions for the existence of such a surface. However little progress has been made in this direction since the question was posed by R. Osserman. We mention that a result of R. Osserman (Osserman 1961) states that if M is a complete, orientable minimal surface in $\mathbb{R}^3$; then its total curvature is a multiple of -4π: We recall that even if a minimal surface M is not orientable, its oriented double covering $\widetilde{M}$ is an orientable

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minimal surface. In that way we can study the Gauss map of $\widetilde{M}$. Here we consider only the orientable case. In 1987, Weitsman and Xavier (Weitsman and Xavier 1987) proved that if N omits three values, the total curvature of M is less than or equal to -16π:

In this paper we give a new proof of the Osserman's theorem in the case of finite total curvature (Osserman 1964). We believe that a simpler proof, using fewer arguments of complex variables, could lead to a better understanding and even to the possibility of improving the result. Our proof uses the Weierstrass representation, the fact that M is a compact surface minus a finite number of points and the Poincaré's theorem about the sum of indices of a differentiable vector field with isolated singularities on a compact surface.

We consider certain particular functions on the surface $\widehat{M}$ and we apply the Poincaré's theorem to the gradient vector field of these functions. This permits us to obtain relations between the Euler characteristic of $\widehat{M}$; the number of ends, the index of each end, the total curvature of M and the number of points omitted by the spherical image of M: Finally, under additional hypothesis on the type of ends of the surface, we obtain results that improve the Osserman's theorem.

Let $X : M \to \mathbb{R}^3$ be an isometric minimal immersion of a Riemann surface M into $\mathbb{R}^3$: Let N be Gauss map and $C(M) := \int_M KdM$ the total curvature of M. If M has finite total curvature (ie $|C(M)| < \infty$), then we know the following results (see Osserman 1986).

1.1. *The Riemann surface M is diffeomorphic to a compact surface $\widehat{M}$ of genus γ minus a finite set points, that is,*

$$M \cong \widehat{M} \setminus \{p_1, p_2, ..., p_n\};$$

1.2. *The Gauss mapping N extends continuously to a function $\widehat{N} : \widehat{M} \to \mathbb{S}^2$: So, $\widehat{N}$ is a branched covering of* $\mathbb{S}^2(1)$;

1.3. *If m represents the number of times that $\widehat{N}$ covers each point of $\mathbb{S}^2$ then the total curvature of M is given by* $-4\pi m$:

Let $\pi : \mathbb{S}^2 \to \mathbb{C} \cup \{\infty\}$ be any identication of $\mathbb{S}^2$ with $\mathbb{C} \cup \{\infty\}$. Represent by g the composition $g := \pi \circ N$ and set $\widehat{g} := \pi \circ \widehat{N}$.

For each choice of π there is a meromorphic one-form ω on $\widehat{M}$ satisfying:

1.4. *at each point of M where g has a pole of order v the form ω has a zero of order* $2v$;

1.5. *the form* $\alpha := \left(\frac{1}{2}(1-g^2)\,\omega, \frac{i}{2}(1+g^2)\,\omega, g\omega\right)$ *has no real periods on M*;

1.6. *the Weierstrass representation gives the following integral expression*

$$X = 2\Re\left(\int^{z} \alpha\right).$$

Since M is complete with the induced metric, then α must have poles at each point of $\widehat{M} \setminus M$. We know that (see, for example, Osserman 1986):

1.7. *the poles of α have order greater than or equal to two.*

The points of $\widehat{M} \setminus M$, or punctured neighborhoods of such points, are called *ends of M.* L. Jorge and W. Meeks (Jorge and Meeks 1983) have shown that:

1.8. for $\rho > 0$ *sufficiently large, the set* $M_\rho := \{p \in M, |X(p)| \geq \rho\}$ *is a disjoint union of closed sets* $E_1, E_2, ..., E_n$, *where each E_i is an end of M and* $V_i := E_i \cup \{p_i\}$ *is a disk neighborhood of p_i in $\widehat{M}$ for* $i = 1, ..., n$;

1.9. *if* $\gamma_\rho^i : \mathbb{S}^1 \to M$ *parametrizes* ∂V_i *and* $C_\rho^i := \frac{1}{\rho} X(\gamma_\rho^i)$, *then* C_ρ^i *converges, in the* C^∞ *topology, to the great circle of* $\mathbb{S}^2(1)$ *perpendicular to* $\widehat{N}(p_i)$ *covered* $I(p_i)$ *times;*

1.10. *the number* $I(p_i)$ *is an integer greater than or equal to one.*

If $I(p_i) = 1$ *then, for* ρ *sufficiently large,* $X(E_i)$ *is graph of a function* $f : \mathcal{U}_i \to \mathbb{R}$, *where* $\mathcal{U}_i$ *is some neighborhood of infinite in the subspace perpendicular to* $\widehat{N}(p_i)$:

HEIGHT FUNCTIONS

For each $\xi \in \mathbb{S}^2(1)$ we define:

$$f_\xi : M \to \mathbb{R}, \quad f_\xi(p) := \langle X(p), \xi \rangle. \tag{1}$$

Hence, a simple computation yields:

$$grad(f_\xi) = \xi^T, \tag{2}$$

where $\xi^T(p)$ means the orthogonal projection of ξ into T_pM. The singularities of this vector fields are the points in $N^{-1}\{\xi, -\xi\}$.

Proposition 2.1. *If* $p \in N^{-1}\{\xi, -\xi\}$ *then* $grad(f_\xi)(p) = 0$ *and the index of this singularity is* $-v(p)$, *where* $v(p)$ *is the order of* p *as a zero of* $g - g(p)$.

Proof. By a change of coordinates in $\mathbb{R}^3$; if necessary, we may suppose that $\xi = \pm(0, 0, 1)$. Consider the identication $\pi : \mathbb{S}^2(1) \to \mathbb{C} \cup \{0\}$ for which it holds that $\pi(0, 0, 1) = \infty$ and $\pi(0, 0, -1)=0$. As $N(p) = \pm\xi$ we have $g(p) = 0$ or $g(p) = \infty$. We may suppose that $g(p) = 0$. Let $z = u + iv$ be a local complex parameter of M such that p corresponds to $z = 0$. Then, in a neighborhood of p, we have:

$$g(z) = z^{v(p)}\tilde{g}(z),\ \tilde{g}(0) \neq 0 \quad \text{and} \quad \omega = f(z)dz, \quad f(0) \neq 0.$$

It follows from (1.5) and (1.6) that:

$$\begin{aligned} \frac{\partial f_\xi}{\partial z} &= \left\langle \frac{\partial X}{\partial z}, \xi \right\rangle = \pm g(z) f(z) \\ &= \pm z^{v(p)}(\tilde{g} \circ f)(z) = (h(z))^{v(p)} \end{aligned}$$

where h is a holomorphic function such that $h(0) = 0$ and $h'(0) \neq 0$: Hence, we get:

$$\frac{\partial f_\xi}{\partial z} = 2\Re(h^v), \quad \frac{\partial f_\xi}{\partial v} = -2\Im(h^\mu)$$

and

$$grad(f_\xi) = \frac{1}{\left|\frac{\partial X}{\partial z}\right|^2}\left\{ \Re(h^v)\frac{\partial X}{\partial u} - 2\Im(h^v)\frac{\partial X}{\partial u} \right\},$$

what implies that the index of $grad(f_\xi)$ at p is $-v(p)$.

In order to determine the behavior of the vector field $grad(f_\xi)$ at the points $p_i \in \widehat{M} \setminus M$ we consider its restriction to the curves γ_ρ^i defined in (1.9).

Proposition 2.2. *For* ρ *sufficiently large, the index of grad* (f_ξ) along γ_ρ^i *is*

$$I(p_i) + 1 \quad if \quad \widehat{N}(p_i) \neq \pm\,\xi, \tag{3}$$
$$I(p_i) - \nu(p_i) + 1 \quad if \quad \widehat{N}(p_i) = \pm\,\xi. \tag{4}$$

Proof. Choose coordinates on $\mathbb{R}^3$ so that $\xi = \pm\,(0,0,1)$ and $\widehat{g}\,(p_i) \neq \infty$. With respect to the local parameter $z = u + iv$ around p_i we have:

$$\widehat{g}(z) = a_0 + z^{\nu(p_i)}\,\widetilde{g}(z), \quad \text{where } \widetilde{g}(0) \neq 0 \text{ and}$$
$$\omega(z) = f(z) = dz = \frac{\widetilde{f}(z)}{z^{\mu(p_i)}}\,dz, \quad \text{where } \widetilde{f}(0) \neq 0.$$

It follows from (1.7) that $\mu(p_i) \geq 2$. As in the proof of the Proposition 2.1 we have:

$$\frac{\partial f_\xi}{\partial z} = \left\langle \frac{\partial X}{\partial z}, \xi \right\rangle = \pm f(z)g(z).$$

It follows that one of the following cases can occur:

(A) $\widehat{N}(p_i) \neq \pm\xi$. In this case, $\widehat{g}(0) \neq 0$ and we can write $\frac{\partial f_\xi}{\partial z} = \pm z^{-\mu}h_1(z)$ where $h_1 = \widetilde{f}\,\widehat{g}$ and $h_1(0) \neq 0$.

(B) $\widehat{N}(p_i) = \pm\xi$. Now $\widehat{g}(0) = 0$ and $\frac{\partial f_\xi}{\partial z} = \pm z^{\nu-\mu}h_2(z)$ where $h_2 = \widetilde{f}\,\widetilde{g}$ and $h_2(0) \neq 0$.

These cases correspond respectively to:

(A') $\frac{\partial f_\xi}{\partial z} = W_1(z)^{-\mu} = W_1(z)^{n_1}$ and

(B') $\frac{\partial f_\xi}{\partial z} = W_2(z)^{\nu-\mu} = W_2(z)^{n_2}$, where $n_1 = -\mu$, $n_2 = \nu - \mu$, the functions W_i, $i = 1{,}2$ are holomorphic and $W_i'(0) \neq 0$.

As before, consider the gradient vector field

$$grad(f_\xi) = \frac{1}{\left|\frac{\partial X}{\partial z}\right|}\left\{ \Re(W_i^{n_i})\frac{\partial X}{\partial u} - \Im(W_i^{n_i})\frac{\partial X}{\partial u} \right\}.$$

Then, take large enough so that $\gamma_\rho^i\,(\mathbb{S}^1)$ is contained in the neighborhood of p_i we are considering. It follows that the index of $grad(f_\xi)$ along γ_ρ^i is μ in the case (A) and $\nu - \mu$ in the case (B).

The following lemma completes the proof of Proposition 2.2.

Lemma 2.3. *If $p \in \widehat{M} \setminus M$, $\widehat{g}(p) \neq \infty$ and ω has a pole of order μ at p, then $I(p) = \mu - 1$.*

Proof. We may choose coordinates in $\mathbb{S}^2(1)$ in such a way that $\widehat{g}(p) = 0$. By using (1.6) we have:

$$\begin{aligned}
X_1 + iX_2 &= \Re\left(\int^{z}(1-g^2)fdz\right) - i\Im\left(\int^{z}(1-g^2)fdz\right)\\
&= \Re\left(\int^{z}(1-z^{2\nu}\widetilde{g}^2)\frac{\widetilde{f}}{z^\mu}\,dz\right) - i\Im\left(\int^{z}(1+z^{2\nu}\widetilde{g}^2)\frac{\widetilde{f}}{z^\mu}\,dz\right)\\
&= \Re\left(\frac{b_0}{z^{\mu-1}} + f_1(z)\right) - i\Im\left(\frac{b_0}{z^{\mu-1}} + f_2(z)\right)
\end{aligned}$$

where f_1 and f_2 are power series with at most terms in $\frac{1}{z^{\mu-2}}$.

Setting $z = re^{i\theta}$ and $b_0 = r_0e^{i\theta_0}$ we obtain

$$\lim_{r\to 0} r^{\mu-1}(X_1 + iX_2) = r_0e^{i((1-\mu)\theta+\theta_0)}.$$

Therefore, for ρ sufficiently large, $X_1 + iX_2$ takes the circle of radius r centered at p into a closed curve that rotates $(\mu - 1)$-times around the origin, that is $I(p) = \mu - 1$.

Corollary 2.4. *Let* $X : \widehat{M}' \setminus \{p_1, p_2, ..., p_n\} \to \mathbb{R}^3$ *be a complete minimal isometric immersion with finite total curvature. If m is the number of times that* $\widehat{N}$ *covers* $\mathbb{S}^2(1)$ *then:*

$$\chi(\widehat{M}) = \sum_{i=1}^{n} (\mathrm{I}(p_i)+1) \ -2m.$$

Choose ξ such that $\{-\xi, \xi\}$ does not intersect $\{\widehat{N}(p_i)\text{: } i = 1, ..., n\}$. We have:

$$\begin{aligned}
\chi(\widehat{M}) &= \sum_{p\in N^{-1}\{-\xi,\,\xi\}} index_p\,(grad(f_\xi)) \ \sum_{i=1}^{n} (I(p_i)+1) \\
&= -\sum_{p\in N^{-1}\{-\xi,\,\xi\}} v(p) + \sum_{i=1}^{n} (I(p)+1) \\
&= -2m + \sum_{i=1}^{n} (I(p)+1)
\end{aligned}$$

If the genus of $\widehat{M}$ is γ, then $2(1-\gamma) = \sum_{i=1}^{n} (I(p)+1) -2m$. Hence, we obtain $2m = n + \sum_{i=1}^{n} I(p) - 2(1-\gamma)$. Since $I(p_i) \geq 1$ for all $i \in \{1, 2, ..., n\}$ it follows that:

$$\chi(\widehat{M}) \geq 2(n-m). \tag{5}$$

Note that the equality holds if and only if $I(p_i) = 1$ for all $i \in \{1, 2, ..., n\}$. In this case every end of M are embedded. Moreover, (5) is equivalent to:

$$\gamma \leq m - n + 1. \tag{6}$$

SUPPORT TYPE FUNCTIONS

Now we consider, for each $\xi \in \mathbb{S}^2$, the support type function

$$\mathfrak{s}_\xi : M \to \mathbb{R}, \quad \mathfrak{s}_\xi\,(p) := \left\langle N(p), \xi \right\rangle \tag{7}$$

A simple computation shows that the singular points of $grad(\mathfrak{s}_\xi)$ are the points where $\xi^T = 0$ or $K = 0$ (branch points of g). By a change of coordinates in $\mathbb{R}^3$ we may always assume that $\xi = (0, 0, 1)$.

Since:

$$N = \left(\frac{2\Re(g)}{1+|g|^2}, \frac{2\Im(g)}{1+|g|^2}, \frac{|g|^2-1}{1+|g|^2}\right)$$

(see, for example, Barbosa and Colares 1986), we have:

$$\mathfrak{s}_\xi = \frac{|g|^2-1}{1+|g|^2}. \tag{8}$$

Consider a point p such that $\widehat{N}(p) = \pm\xi$. In terms of local coordinates $z = u+iv$; with p corresponding to $z = 0$; we have $g(z) = z^v\widetilde{g}(z)$; $\widetilde{g}(0) \neq 0$; when v is a nonzero integer.

If $v > 0$ then $\mathfrak{s}_\xi(0) = -1$ and

$$\mathfrak{s}_\xi\,(z) + 1 = \frac{2|g(z)|^2}{1+|g(z)|^2} = \frac{2|z|^{2v}|\widetilde{g}(z)|^2}{1+2|z|^{2v}\,|\widetilde{g}(z)|^2}\,.$$

The point p is then a local minimum for $\mathfrak{s}_\xi$ with index + 1.

If $v < 0$ then $\mathfrak{s}_\xi(0) = 1$ and

$$\mathfrak{s}_\xi(z) - 1 = \frac{-2}{1 + |g(z)|^2} = -\frac{2|z|^{-2v}}{|z|^{-2v} + |\tilde{g}(z)|^2}$$

The point p is then a local maximum and so its index is also +1.

When $\widehat{N}(p) \neq \pm\xi$, then $g(z) = a_0 + z^v\tilde{g}(z)$, $\tilde{g}(0) \neq 0$, where $v \geq 1$. It follows that $\mathfrak{s}_\xi(0) = \dfrac{|a_0|^2-1}{|a_0|^2+1}$ and

$$\mathfrak{s}_\xi(z) - \mathfrak{s}_\xi(0) = \frac{2(|g(z)|^2 - |a_0|)^2}{(|g(z)|^2+1)(|a_0|^2+1)} = \frac{4\Re(\overline{a}_0 z^v \tilde{g}(z)) + 2|z|^{2v}|\tilde{g}(z)|^2}{(|g(z)|^2+1)(|a_0|^2+1)}.$$

Considering $z = re^{i\theta}$ one obtains

$$\Psi(\theta) := \lim_{r\to 0} \frac{1}{r^v}(\mathfrak{s}_\xi(z) - \mathfrak{s}_\xi(0)) = \frac{4\Re(e^{iv\theta}\overline{a}_0 b_0)}{(|a_0|^2+1)^2},$$

where $b_0 := \tilde{g}(0)$. Setting $\overline{a}_0 b_0 = Re^{i\alpha}$, we have:

$$\Psi(\theta) = 4R\frac{\cos(v\theta + \alpha)}{(|a_0|^2+1)^2}.$$

From this expression it follows that the difference $\mathfrak{s}_\xi(z) - \mathfrak{s}_\xi(0)$ changes sign in any neighborhood of $z = 0$. It follows easily that the index of $grad(\mathfrak{s}_\xi)$ at p is $1 - v(p)$.

Therefore we have proved:

Proposition 3.1. *If $\widehat{N}(p) = \pm\xi$ then p is a critical point of $\mathfrak{s}_\xi$ of index 1. If $\widehat{N}(p) \neq \pm\xi$ and p is a critical point of $\mathfrak{s}_\xi$ then g has a branch point of order v at p and the index of grad $(\mathfrak{s}_\xi)$ at p is $1 - v$.*

THE OSSERMAN'S THEOREM

The following theorem was proved by Osserman (Osserman 1964) using a different method.

Theorem 4.1. *Let $X : M = \widehat{M} \setminus \{p_1, p_2, ..., p_n\} \to \mathbb{R}^3$ be a complete minimal isometric immersion with finite total curvature, $-4\pi m$. Suppose that $\{\xi_1, \xi_2, ..., \xi_k\} = \mathbb{S}^2(1) \setminus g(M)$. If $k \geq 4$, then M is flat.*

Proof. Set:

$$A_i := \{p \in \widehat{M} \setminus M : \widehat{g}(p) = \xi_i\}, \quad i \in \{1, 2, ..., k\},$$

$$B := \{p \in \widehat{M} \setminus M : \widehat{g}(p) \in g(M)\}$$

and

$$C := \{p \in M : v(p) > 1\}.$$

Assume M is not flat and let $\xi \in g(M)$ be a regular value of g. Counting indices of the singularities of $grad(\mathfrak{s}_\xi)$ we obtain:

$$\begin{aligned}\chi(\widehat{M}) &= \sum_{\widehat{N}(p)=\pm\xi} index_p(grad(\mathfrak{s}_\xi)) + \sum_{\widehat{N}(p)\neq\pm\xi} index_p(grad(\mathfrak{s}_\xi)) \\ &= 2m + \sum_{i=1}^{k}\sum_{p\in A_i}(1 - v(p)) + \sum_{p\in B}(1 - v(p)) + \sum_{p\in C}(1 - v(p)).\end{aligned}$$

Observe that $\sum_{p\in A_i} v(p) = m$ and $\sum_{i=1}^{k} \#(A_i) + \#(B) = n$. Hence,

$$\chi(\widehat{M}) = 2m - km + n - \sum_{p\in B} v(p) - \sum_{p\in C}(v(p) - 1). \tag{9}$$

Now using 2.4, we have $\chi(\widehat{M}) \sum_{i=1}^{n} I(p_i) + 1) - 2m$. Using (9) it follows that:

$$\begin{aligned} 0 < \sum_{i=1}^{n} I(p_i) &= 4m - km - \sum_{p\in B} v(p) - \sum_{p\in C}(v(p) - 1) \\ &\leq (4-k)m. \end{aligned}$$

Thus $(4 - k)m > 0$. Therefore $k < 4$.

Corollary 4.2. *Let* $X : M = \widehat{M} \setminus \{p_1, p_2, ..., p_n\} \to \mathbb{R}^3$ *be a complete minimal isometric immersion with finite total curvature* $-4\pi m$. *If* $N : M \to \mathbb{S}^2(1)$ *omits 3 points then* $\chi(\widehat{M}) \leq 0$. *Moreover, if* $\chi(\widehat{M}) = 0$ *we have:*

(a) $m = n$;

(b) $B = C = \phi$;

(c) The ends of M *are embedded.*

Proof. It follows from equation (9) that if $k = 3$ then:

$$\chi(\widehat{M}) = (n - m) - \sum_{p\in B} v(p) - \sum_{p\in C} (v(p) - 1) \leq n - m.$$

On the other hand, from (5) we have: $\chi(\widehat{M}) \geq 2(n - m)$. Therefore $n - m \leq 0$, where $\chi(\widehat{M}) \leq 0$. Moreover, if $\chi(\widehat{M}) = 0$ then $m = n$ and $B = C = \phi$. It is consequence of (5) that the ends M are embedded.

Corollary 4.3. *Under the same hypothesis of the previous corollary it holds that if* $N : M \to \mathbb{S}^2$ *omits 3 points then* $m \geq 3$:

Proof. From the proof of the previous corollary we have $0 \geq \chi(\widehat{M}) \geq 2(n - m)$. Hence $m \geq n$: However, for each point $q \in \mathbb{S}^2 \setminus N(M)$ there is at least one point $p \in \widehat{M} \setminus M$ such that $\widehat{N}(p) = q$. Hence, we have $n \geq 3$.

RESULTS

Now we study the behavior of M at infinity. For this purpose we consider $X : M = \widehat{M} \setminus \{p_1, p_2, ..., p_n\} \to \mathbb{R}^3$ as before and let (g, ω) be the the corresponding Weierstrass data. After a rotation we may assume, for j fixed, that $\widehat{g}(p_j) = 0$. Let z be a local parameter in an open subset $D \subset \widehat{M}$ with $p_j \in D$ such that $(p_j) = 0$. Hence, locally we have:

$$g(z) = z^{v} \sum_{n=0}^{\infty} b_n z^n, \quad v \geq 1,\ b_0 \neq 0. \tag{10}$$

Moreover, as the immersion is complete it follows that f has pole at p_j whose order is greater than or equal to 2. Hence we have near p_j that:

$$f(z)\, z^{-\mu} \sum_{n=0}^{\infty} a_n z^n, \quad \mu \geq 2, a_o \neq 0. \tag{11}$$

By Lemma (2.3) we have $I(p_j) = \mu - 1$: Thus, p_j is an embedded end if and only if $\mu(p_j) = 2$. Suppose that p_j is an embedded end. Then $\alpha_3 := g\omega = gfdz$ and

$$\begin{aligned} gf(z) &= z^{\nu\mu} \sum_{n=0}^{\infty} \left(\sum_{n=0}^{\infty} a_i b_{n-i} \right) \quad z^n, \quad (\mu = 2) \\ &= a_0 b_0 z^{\nu-2} + (a_o b_1 + a_1 b_0) z^{\nu-1} + ... \end{aligned}$$

Hence, if $\nu > 1$ then gf is holomorphic and $X_3(p) = \Re(\int^{\mathrm{P}} f\omega)$ satisfies

$$\lim_{P \to P_3} X_3(p) = L \in \mathbb{R}.$$

Thus, the end F_j is asymptotic to the plane $\{X_3 = L\}$. In this case, we say that F_j is a planar end of order $\nu - 1$:

If $\nu = 1$, then $gf(z) = a_0 b_0 z^{-1}$+ (holomorphic function). Since:

$$\Re\left(\oint_C gfdz \right) = 0$$

for any closed curve $\mathcal{C} \subset M$, we have $\Re(2\pi i a_0 b_0) = 0$ and then $a_0 b_0 \in \mathbb{R}$. Therefore:

$$X_3 = a_0 b_0 \log |z| + R(z),$$

where R is an harmonic function in a neighborhood of $0 \in \mathbb{C}$ and $R(0) = 0$. The number $a_0 b_0$ is the logarithmic growth of the end F_j. In this case the end F_j is of catenoidal type. This means geometrically that F_j is asymptotic to a catenoid in $\mathbb{R}^3$.

In the general case we have $fg(z) = \sum_{n=0}^{\infty} \left(\sum_{n=0}^{\infty} a_i b_{n-i} \right) z^{n+\nu-\mu}$. Now consider $N_0 := \nu - \mu$. If $N_0 \geq 0$ then fg is holomorphic. Otherwise, if $N_0 < 0$ then $fg(z) = a_0 b_0{}^{\nu-\mu} + ... + \left(\sum_{i=0}^{N_0-1} a_i b_{N_0-(i+1)z^{-1}} \right)$ + holomorphic function.

Denition 5.1. *Let $p \in \widehat{M} \setminus M$ be a end of M. We say that $p \in \widehat{M} \setminus M$ is a non-degenerate end if, and only if, $N_0(p) := \nu(p) - \mu(p) \leq 0$.*

For non-degenerate ends we have:

Theorem 5.2. *Let $X : M = \widehat{M} \setminus \{p_1, p_2, ..., p_n\} \to \mathbb{R}^3$ be a complete non-flat minimal immersion with finite total curvature and non-degenerate ends. If the Gauss map omit k points, then $k \leq 2$:*

Proof. By the Osserman's theorem (4.1) we know that $k \leq 3$: Suppose that $k = 3$ and let ξ_1, ξ_2 and ξ_3 be the omitted points. As in the proof of the Theorem 4.1 consider the sets:

$$\begin{aligned} &A_i := \{p \in \widehat{M} \setminus M : \widehat{g}(p) = \xi_i\} \quad i = 1, 2, 3 \quad \text{and} \\ &B := \{p \in \widehat{M} \setminus M : \widehat{g}(p) \in g(M)\}. \end{aligned}$$

Counting the indices of the singularities of $grad(f_\xi)$ we obtain:

(a) in the case $\xi_2 = -\xi_1$:

$$\begin{aligned}\chi(\widehat{M}) &= \sum_{p\in A_1\cup A_2}(I(p) - v(p)+1) + \sum_{p\in A_3\cup B}(I(p)+1)\\ &\geq \sum_{p\in A_3\cup B}(I(p)+1)\\ &> 0 \quad (A_3 \neq \phi).\end{aligned}$$

(b) in the case $\xi_2 \neq -\xi_1$ and $\xi_3 \neq -\xi_1$:

$$\begin{aligned}\chi(\widehat{M}) &= \sum_{p\in A_1}(I(p) - v(p)+1) + \sum_{p\in \widehat{N}^{-1}(-\xi_1)}(-v(p)) + \sum_{p\in A_2\cup A_3\cup B}(I(p)+1)\\ &\geq -m + \sum_{p\in A_2\cup A_3\cup B}(I(p)+1)\end{aligned}$$

Moreover, since $N_0 \leq 0$ we have:

$$\sum_{p\in A_2}(I(p)+1) \geq \sum_{p\in A_2} v(p) = m.$$

In both cases, we conclude $\chi(\widehat{M}) > 0$ what contradicts Corollary 4.2. Therefore $k = 2$.

In the case $k = 2$ with non-degenerate ends we have:

Theorem 5.3. *Let $X : M = \widehat{M} \setminus \{p_1; p_2, ..., p_n\} \to \mathbb{R}^3$ be a complete non-at minimal immersion with finite total curvature and non-degenerate ends. Suppose that the Gauss map omits two points. Then M is topologically a sphere minus two points.*

Proof. Let $\xi_1, \xi_2 \in \mathbb{S}^2$ be the two points omitted. Similarly to the proof of Theorem 5.2 we have:

(a) in the case $\xi_2 = -\xi_1$:

$$\chi(\widehat{M}) = \sum_{p\in A_1\cup A_2}(I(p) - v(p)+1) + \sum_{p\in B}(I(p)+1) > 0.$$

(b) in the case $\xi_2 \neq -\xi_1$:

$$\begin{aligned}\chi(\widehat{M}) &= \sum_{p\in A_1}(I(p) - v(p)+1) - m + \sum_{p\in A_2\cup B}(I(p)+1)\\ &= \sum_{p\in A_1}(I(p) - v(p)+1) + \sum_{p\in B}(I(p)+1)\\ &> 0 \quad \left(\sum_{p\in A_2}(I(p)+1) = m.\right)\end{aligned}$$

In both cases, $\chi(\widehat{M}) > 0$ what implies that $\widehat{M}$ is homeomorphic to $\mathbb{S}^2$ and therefore:

$$M = \mathbb{S}^2 \setminus \{p_1; p_2, ..., p_n\}$$

We claim that $n \leq 3$.

By contradiction, suppose that $n > 3$ and let p_1, p_2, p_3, p_4 four distinct points in $\widehat{M} \setminus M$. Consider two simple and differentiable curves α_1 and α_2 joining p_1 to p_2 and p_3 to p_4 respectively. Assume also that the curves α_1 and α_2 do not intersect each other and are contained in M except for its endpoints.

Cutting M along the curves α_1 and α_2 we obtain another surface topologically equivalent to a cylinder. Joining two copies of the later surface along the curves α_1 and α_2 we obtain another compact surface of genus 1 and $2(n-4)+4$ points removed.

Denote by M' this new surface and let $R : M' \to M$ the canonical covering map. Now $X \circ R : M' \to \mathbb{R}^3$ is a minimal immersion with the same image of X. The Gauss map of $X \circ R$ is given by $N' = N \circ R$ and has the same image as the Gauss map of X. For the ends of M' we still have $N_0 < 0$. In fact, the ends that are not endpoints of the curves α_i, $i = 1, 2$, remain with the same local behavior and hence the same $I(p)$ and $\nu(p)$. Thus the value $N_0 = \nu - \mu = \nu - I - 1$ remains unaltereted. However when the ends are endpoints of the curves α_i, $i = 1, 2$, some changes occur. For instance, a neighborhood of p_1 now covers twice a old neighborhood. It follows that $I(p_1)$ is changed for $2I(p_1)$ and $g(z)$ is changed for $g(z^2) = z^{2\nu}\tilde{g}(z^2)$. Therefore, $\nu(p_1)$ becomes $2\nu(p_1)$. Since $I(p_1) - \nu(p_1)$ is changed for $2I(p_1) - 2\nu(p_1)$ the inequality $I(p_1) - \mu(p_1) + 1 > 0$. is preserved. Thus $X \circ R : M' \to \mathbb{R}^3$ is an immersion whose ends are not degenerate. Moreover, this immersion satisfies the hypothesis of the theorem. From the first part of the proof we have $\chi(\widehat{M'}) > 0$. This is a contradiction because $\chi(\widehat{M'}) = 0$. Hence $n \leq 3$ as we claimed.

Now we claim that $n = 2$.

Clearly $n \geq 2$ ($n \geq k = 2$). Assume $n = 2$ and let p_1, p_2 and p_3 be the ends of M. Consider a curve α joining p_1 to p_2 as before.

The previous construction of M' now leads to a surface with $n = 4$ which contradicts $n \leq 3$. Therefore, $n = 2$.

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RESUMO

Neste trabalho, estuda-se a imagem pela aplicação normal de Gauss de uma superfície mínima imersa em $\mathbb{R}^3$ com curvatura total finita. Apresentamos uma prova diferente do seguinte Teorema de Osserman: *A aplicação normal de Gauss de uma superfície mínima imersa em* $\mathbb{R}^3$ *com curvatura total finita, que não é um plano, omite no maximo três pontos de* $\mathbb{S}^2$: A seguir, usando uma hipótese adicional sobre o tipo de fins, provamos que este numero é exatamente dois.

Palavras-chave: aplicação de Gauss, superfícies mínimas, curvatura total finita, Imagem da aplicação de Gauss.

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